# Pinning Models and Renewal Theory

Giambattista Giacomin

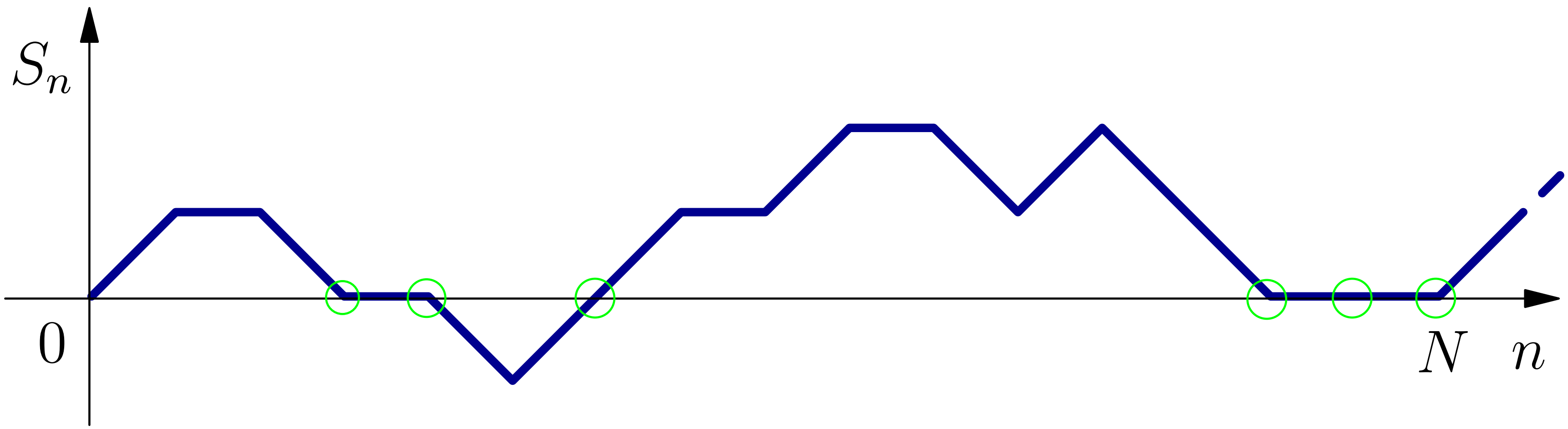
University of Paris 7 – Denis Diderot

+ Probability Lab CNRS U.M.A. 7599

# Overview

1. The model (again)
2. Homogeneous case $\leftrightarrow$ renewal process
3. Two issues:
   (a) Correlation length
   (b) Estimates in the irrelevant disorder regime

# The Pinning Model

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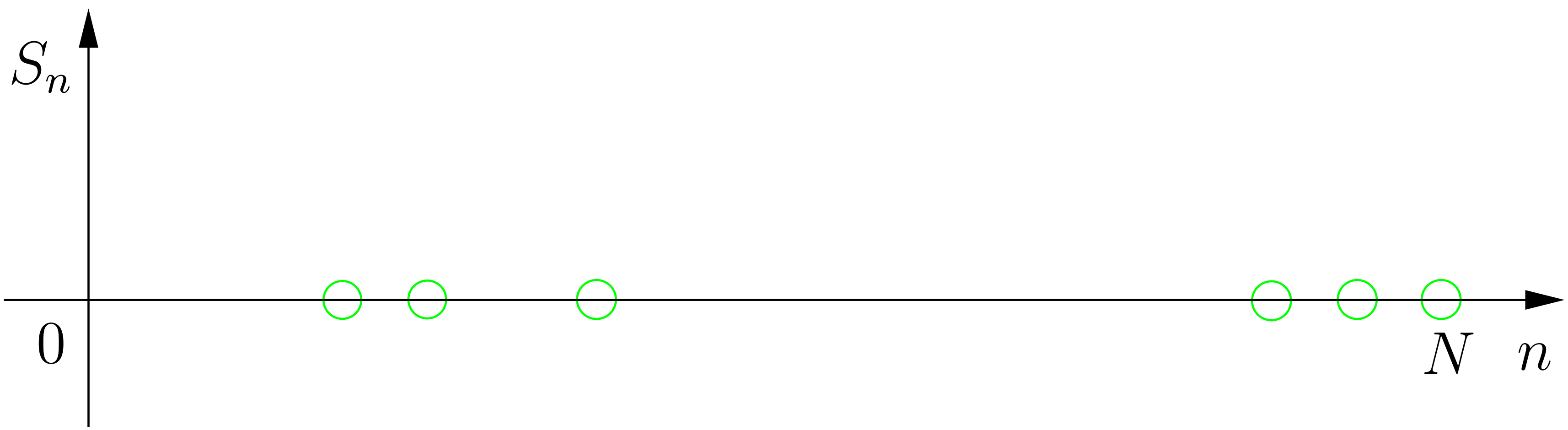


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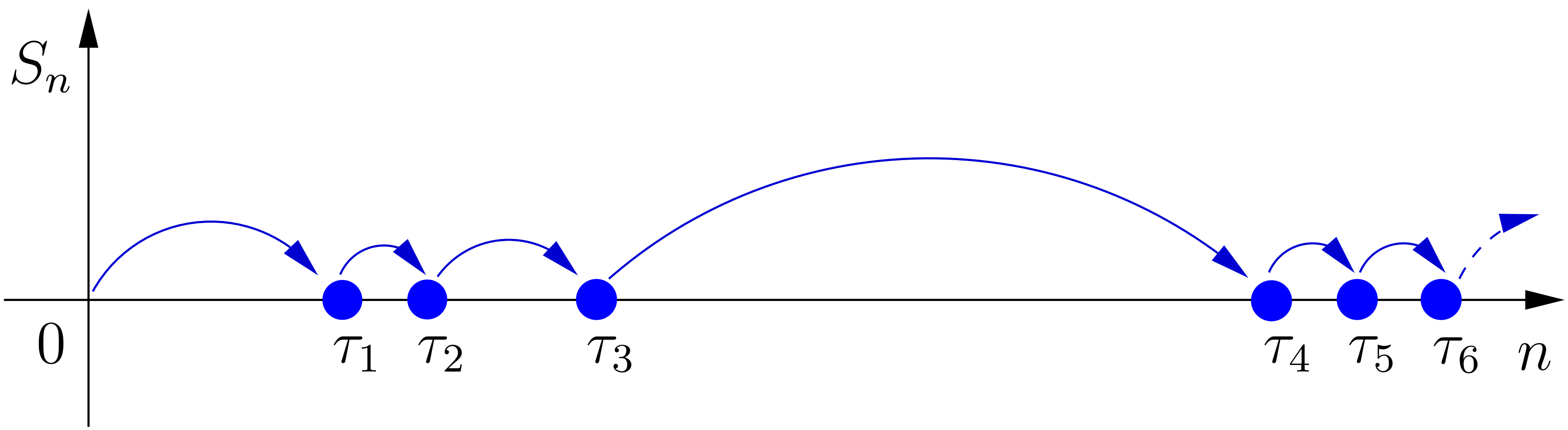


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# More details on the model

- The fundamental quantity $\tau = \{\tau_j\}_{j=0,1,...}$ is a renewal process (or a RW with positive increments): $\tau_0 = 0$ and $\{\tau_{j+1} - \tau_j\}_j$ is IID.

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- $\tau$ can be viewed as a subset of $\mathbb{N} \cup \{0\}$, therefore $S_n = 0 \Longleftrightarrow n \in \tau$.
- The disorder $\omega = \{\omega_n\}_{n=1,2,\ldots}$ is IID, centered and $\mathbb{E}[\omega_1^2] = 1$. Examples to keep in mind:
  1. $\mathbb{P}(\omega_1 = +1) = \mathbb{P}(\omega_1 = -1) = 1/2$
  2. $\omega_1$ standard Gaussian.

# The homogeneous case

If $\beta = 0$ (no disorder) we have the formula (notation: $Z_N := Z_{N,\omega}$)

$$Z_N \;=\; \exp(\mathsf{F} N)\, \mathbf{P}_{\mathsf{F}}(N \in \tau)$$

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where $\mathsf{F}(= \mathsf{F}(h))$ is the unique solution of

$$\sum_{n=1}^{\infty} K(n) \exp(-\mathsf{F} n) = \exp(-h)$$

if $\sum_n K(n) \geq \exp(-h)$. Otherwise $\mathsf{F} = 0$. Moreover:

$$\mathbf{P}_{\mathsf{F}}(\tau_1 = n) = \exp(h) K(n) \exp(-\mathsf{F} n)$$

# The renewal function

$n \mapsto \mathbf{P}_{\mathsf{F}}(n \in \tau)$ is a well known (or much studied) quantity.

- Renewal Theorem:

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- If $\mathsf{F} = 0$:
  - if $\sum_n e^h K(n) < 1$ then

$$\mathbf{P}_{\mathsf{F}}(n \in \tau) \overset{n\to\infty}{\sim} c_1(\alpha) n^{-(1+\alpha)}$$

  - if $\sum_n e^h K(n) = 1$ and $\sum_n nK(n) = \infty$ then

$$\mathbf{P}_{\mathsf{F}}(n \in \tau) \overset{n\to\infty}{\sim} c_2(\alpha) n^{\alpha-1}$$

# Free Energy and Sharp Estimates

Therefore

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but also that if $\mathsf{F} > 0$

$$Z_N \overset{N\to\infty}{\sim} \frac{1}{\mathbf{E}_{\mathsf{F}}[\tau_1]} \exp\left(N\mathsf{F}\right)$$

from which we deduce that $\mathbf{P}_N \overset{N\to\infty}{\Longrightarrow} \mathbf{P}_{\mathsf{F}}$.

# Infinite Volume Limit

Punchline:
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and such a process is

- Positive recurrent if $\mathsf{F} > 0$ (localization)
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Full solution of the model without passing to transforms [M. Fisher 84 $\rightarrow$ queuing ] (space of Gibbs measures)

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If $\omega$ is periodic (and non-trivial) the limit process is not a renewal, but it is a Markov renewal process [Bolthausen, G.]-[Caravenna, G., Zambotti] $\longrightarrow$ surprisingly rich structure (new examples of exactly solvable models exhibiting phase transitions)

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For disordered systems:

- the renewal structure reappears repeatedly in the proofs (challenging renewal questions)
- exact solution for homogeneous models sets a target

# Correlation Length ($\beta = 0$)

A definition of correlation length for $\mathbf{P}_{\mathsf{F}}$ ($\mathsf{F} > 0$):

$$C_{\mathsf{F}}(n) := \lim_{m \to \infty} \text{corr}\left(m \in \tau, m+n \in \tau\right)$$

One expects $C_{\mathsf{F}}(n) \overset{n \to \infty}{\asymp} \exp\left(-n/\ell\right)$ for some $\ell = \ell(\mathsf{F}) > 0$.

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Plenty of reasons [M. Fisher 84] to believe that $\ell(\mathsf{F}) \approx 1/\mathsf{F}$ for $\mathsf{F}$ small (close to criticality). Possibly:

$$\log \ell(\mathsf{F}) \overset{\mathsf{F} \searrow 0}{\sim} -\log \mathsf{F}$$

In short: *the correlations decay like the inter-arrival probability*

# Renewal Convergence Rates

This is a renewal theory question, in fact:

$$C_{\mathsf{F}}(n) = \frac{\mu(\mathsf{F})}{\mu(\mathsf{F})-1}\left[\mathbf{P}_{\mathsf{F}}\left(n \in \tau\right) - \frac{1}{\mu(\mathsf{F})}\right]$$

with $\mu(\mathsf{F}) := \mathbf{E}_{\mathsf{F}}\left[\tau_1\right] \in (1, \infty)$.

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Renewal Theorem:

$$\lim_{n\to\infty} \mathbf{P}_{\mathsf{F}}\left(n \in \tau\right) = \frac{1}{\mu(\mathsf{F})}$$

So $1/\ell(\mathsf{F})$ is the renewal convergence rate.

# Renewal Convergence Rates

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Examples: If $\mathbf{P}(\tau_1 = n) = 1/2$ for $n = 1, 2$ and $\mathbf{P}(\tau_1 = n) = 0$ for $n \geq 3$ then

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If $\mathbf{P}(\tau_1 = n) = p^n((1/p) - 1)$ then for $n = 1, 2, \ldots$

$$\mathbf{P}(n \in \tau) - \frac{1}{p} = 0$$

# Renewal Convergence Rates

The reason is *easily* understood in $z$-transform:
if $F(n) := \mathbf{P}(\tau_1 = n)$ and $\Delta(n) = \mathbf{P}(n \in \tau) - 1/\mathbf{E}[\tau_1]$

$$\widehat{\Delta}(z) = \frac{1}{1 - \widehat{F}(z)} - \frac{1}{\mathbf{E}[\tau_1](1 - z)}$$

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The logarithm of the radius of convergence $R_\Delta$ of $\widehat{\Delta}$ is the renewal converge rate (if the latter exists).

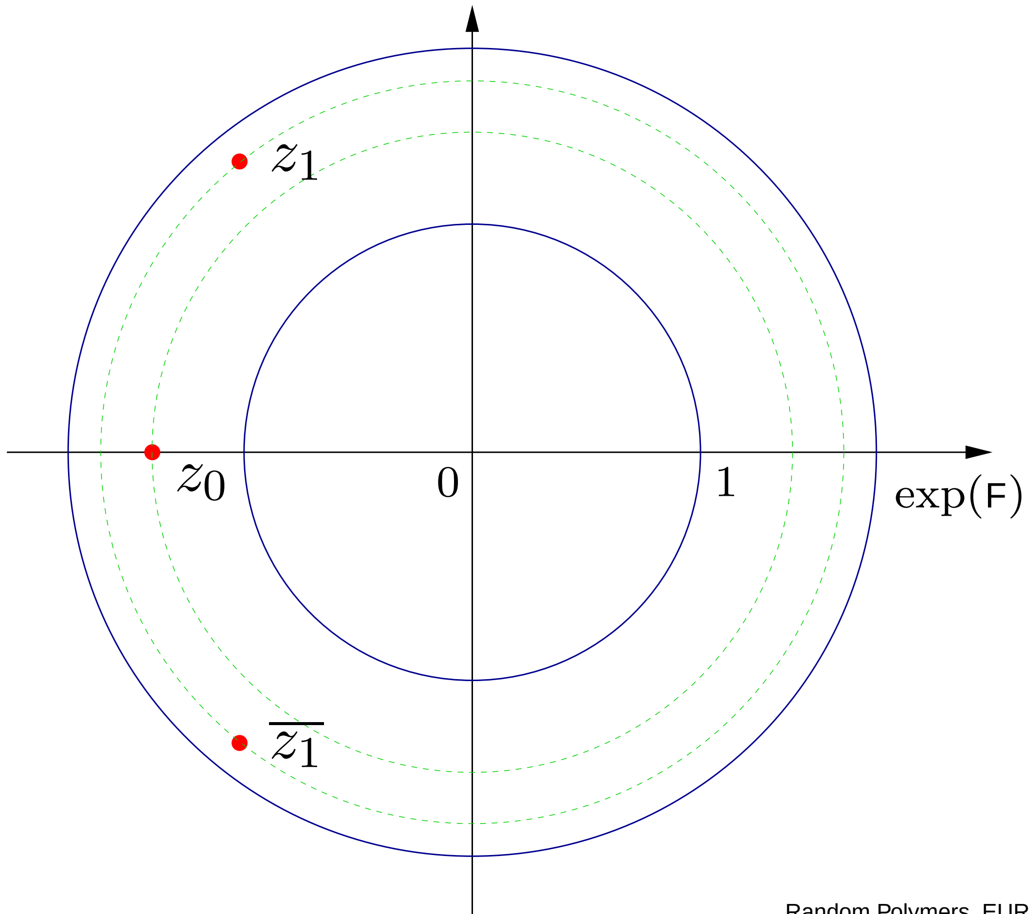
Not at all clear (in fact: false) that $R_\Delta = R_F$.

Watch out for: poles of $\widehat{F}$, solutions to $\widehat{F}(z) = 1$.

# What could happen

Back to the pinning model: $K(n) \sim const./n^{1+\alpha}$ ($\alpha > 0$) and $K_{\mathsf{F}}(n) = \exp(h)K(n)\exp(-\mathsf{F}n)$.

$$\widehat{K_{\mathsf{F}}}(z_i) = 1$$



# Correlation Decay: a Theorem

Back to the pinning model: $K(n) \sim const./n^{1+\alpha}$ ($\alpha > 0$) and $K_{\mathsf{F}}(n) = \exp(h)K(n)\exp(-\mathsf{F}n)$.

Set $\mathsf{F}_0 := \inf\{\mathsf{F} \geq 0 : \widehat{K_{\mathsf{F}}}(z) = 1$ has a solution for $1 < |z| \leq \exp(\mathsf{F})\}$. Note: $\mathsf{F}_0 \in [0, \infty]$.

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**Theorem 2.** *For every $K(\cdot)$ as above, $\mathsf{F}_0 > 0$ and if $\mathsf{F} \in (0, \mathsf{F}_0)$ then*

$$C_{\mathsf{F}}(n) \overset{n\to\infty}{\sim} \left( \frac{\mu(\mathsf{F})e^h}{(\mu(\mathsf{F})-1)(e^h-1)^2} \right) K(n)\exp(-\mathsf{F}n).$$

*in particular $\ell(\mathsf{F}) = 1/\mathsf{F}$ if $\mathsf{F} \in (0, \mathsf{F}_0)$.*

**Obs.:** $\mathsf{F}_0$ can take the value $\infty$, but for every $\varepsilon > 0$ and every $\alpha$ one can find $K(\cdot)$ such that $\mathsf{F}_0 < \varepsilon$.

# Back to Disordered Cases: $\beta > 0$

$$\frac{\mathrm{d}\mathbf{P}_{N,\omega}}{\mathrm{d}\mathbf{P}}(\tau) = \frac{1}{Z_{N,\omega}} \exp\left(\sum_{n=1}^{N} (\beta\omega_n + h)\,\mathbf{1}_{n\in\tau}\right)\mathbf{1}_{N\in\tau}.$$

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Free energy:

$$\mathsf{F}(\beta, h) := \lim_{N \to \infty} \frac{1}{N} \mathbb{E} \log Z_{N,\omega}$$

- $\mathsf{F}(\beta, h) \geq 0$ and $\mathsf{F}(\beta, h) > 0$ characterizes the localized regime;
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Another, a priori *unrelated* quantity:

$$\mu(\beta, h) := -\lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}\left[\frac{1}{Z_{N,\omega}}\right]$$

# $\mu$ versus $F$

[Albeverio, Zhou 96], [G., F. Toninelli 06]:
set gap$_N(\tau) = \max_{i:\tau_i \le N} (\tau_i - \tau_{i-1})$.

**Proposition 3.** *The following holds:*

- *For every* $(\beta, h)$, $0 \le \mu(\beta, h) \le \mathsf{F}(\beta, h)$. *Moreover* $\mu(\beta, h) \ge const.\mathsf{F}(\beta, h)^2$ *if* $\mathsf{F}(\beta, h) \le 1$.
- *If* $\mathsf{F}(\beta, h) > 0$ *for every* $\varepsilon > 0$

$$\lim_{N \to \infty} \mathbf{P}_{N,\omega} \left( \left| \frac{\textit{gap}_N(\tau)}{\log N} - \frac{1}{\mu(\beta, \mathsf{F})} \right| > \varepsilon \right) = 0$$

*in* $\mathbb{P}(\mathrm{d}\omega)$*-probability.*

# $\mu$ versus $F$

[G., F. Toninelli 06]: if $\mathsf{F}(\beta, h) > 0$, for every $\varepsilon > 0$ there exists $C > 0$ such that

- for every $N$ and every $n$

$$\mathbb{E}\mathbf{P}_{N,\omega}\big(\text{dist}(N/2, \tau) > n\big) \leq C \exp\big(-n(\mu(\beta, h) - \varepsilon)\big)$$

- for every $N$ and every $n < N/2$

$$\mathbb{E}\mathbf{P}_{N,\omega}\big(\text{dist}(N/2, \tau) > n\big) \geq \frac{1}{C} \exp\big(-n(\mu(\beta, h) + \varepsilon)\big)$$

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*Analogous statement* if the quenched set-up:

but $\mathsf{F}$ replaces $\mu$!

# Correlation Length for $\beta > 0$

Expected that

- the correlation length exponent of $\mathbf{P}_{\infty,\omega}$ is given by the critical behavior of $1/\mathsf{F}(\beta, h)$
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In [F. Toninelli 07] it is shown that

- the correlation length of $\mathbf{P}_{\infty,\omega}$ is $1/\mathsf{F}(\beta, h)$
- the correlation length of $\mathbb{E}\mathbf{P}_{\infty,\omega}$ is $1/\mu(\beta, h)$

for the wetting case (SRW $\Longrightarrow \alpha = 1/2$).

# Role of Disorder

Known: $\mu(\beta, h) < \mathsf{F}(\beta, h)$ in the localized regime.

Still compatible with

$$\log \mu(\beta, h_c(\beta) + \Delta) \overset{\Delta \searrow 0}{\sim} \log \mathsf{F}(\beta, h_c(\beta) + \Delta) \tag{1}$$

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In our case [G., F. Toninelli, in preparation] $(1)$ holds when the disorder is irrelevant ($\alpha < 1/2$).

# Irrelevant Disorder Regime

Sharpening of the result in [K. Alexander 07], following the approach of [F. Toninelli 07]: for $\beta$ and $\Delta$ small

$$\mathsf{F}(\beta, h_c(\beta)+\Delta) = \mathsf{F}_{\mathsf{ann}}(\beta, h_c(\beta)+\Delta) - \frac{\beta^2}{2}\left(\partial_\Delta \mathsf{F}_{\mathsf{ann}}(\beta, h_c(\beta)+\Delta)\right)^2 + \ldots$$

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The analysis based on interpolation leads to estimating

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in terms of

$$\frac{1}{2N}\log \mathbf{E}_{N,\Delta}^{\otimes 2}\left[\exp\left(c\beta^2 \sum_{n=1}^{N} \mathbf{1}_{n\in\tau\cap\tau'}\right);\, N \in \tau\cap\tau'\right]$$

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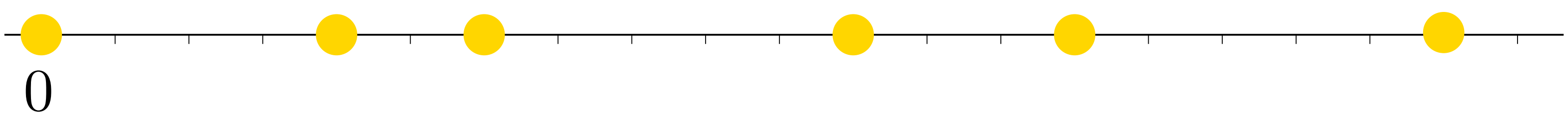
# Intersection Renewal

So one has to find the free energy of the pinning model based on the renewal $\tau \cap \tau'$, where
$\mathbf{P}_{\mathsf{F}}(\tau = n) = \exp(\Delta) K(n) \exp(-\mathsf{F} n)$ (with $\mathsf{F} = \mathsf{F}(0, \Delta)$).



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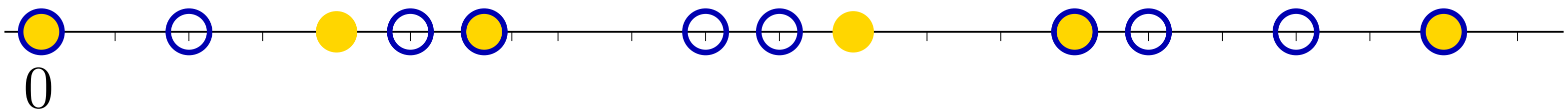

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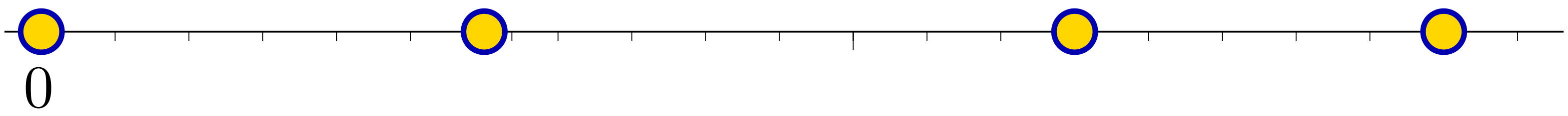
So one has to find the free energy of the pinning model based on the renewal $\tau \cap \tau'$, where
$\mathbf{P}_{\mathsf{F}}(\tau = n) = \exp(\Delta) K(n) \exp(-\mathsf{F} n)$ (with $\mathsf{F} = \mathsf{F}(0, \Delta)$).



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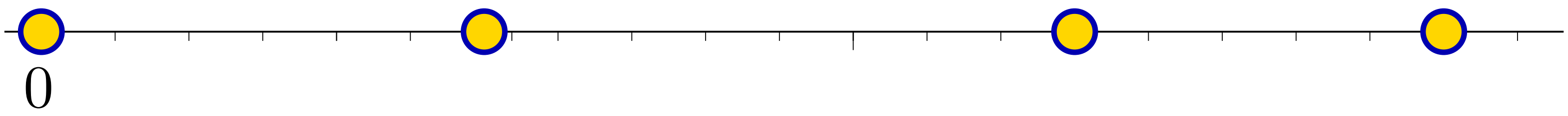


At first the question seems to be

$$\mathbb{K}_{\mathsf{F}}(n) := \mathbf{P}^{\otimes 2}\big((\tau \cap \tau')_1 = n\big) \; = ?$$

and it is, but in fact the explicit quantity is

$$\mathbb{U}_{\mathsf{F}}(n) := \mathbf{P}^{\otimes 2}\left(n \in \tau \cap \tau'\right) = \left(\mathbf{P}\left(n \in \tau\right)\right)^2$$

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The problem one has to solve is: find $B$ such that

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Key estimate:

$$\lim_{\substack{\mathsf{F}\searrow 0\\ B=o(\mathsf{F})}} \sum_{n=0}^{\infty} \exp(-Bn)\left(u_{\mathsf{F}}^2(n) - u_{\mathsf{F}}^2(\infty)\right) = \sum_{n=0}^{\infty} u_0^2(n) < \infty$$

where $u_{\mathsf{F}}(n) := \mathbf{P}_{\mathsf{F}}\left(n \in \tau\right)$.