A replica-coupling approach to disordered pinning models

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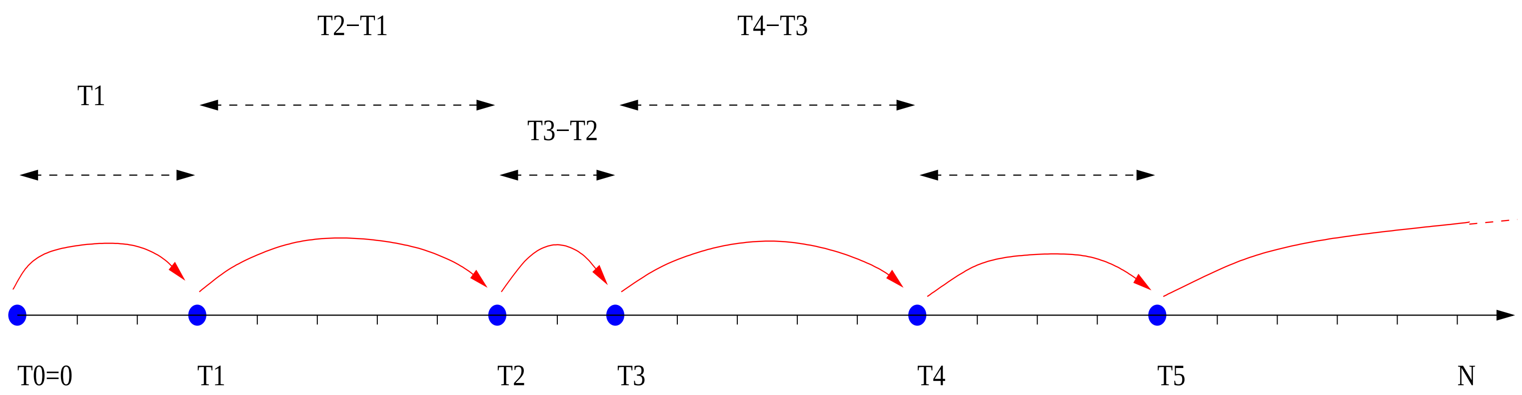
Outline of the talk

- Renewal sequences with frozen randomness
- Localization/delocalization transition
- Effect of randomness. The Harris criterion
- Irrelevant disorder: a replica coupling argument
- Relevant disorder: smoothing of the transition

The mathematical model

We start from a renewal sequence on $\mathbb{N}$:



$\{T_{i+1} - T_i\}_{i \geq 0}$: IID random variables.

We assume power law tail of the jump distribution:
$\mathbf{P}(T_1 = n) \overset{n\to\infty}{\sim} n^{-1-\alpha}$ for some $0 < \alpha < \infty$.

Notice: $\mathbf{E}(T_{i+1} - T_i) = \infty$ if $\alpha < 1$.

We call $\tau := \{T_0, T_1, T_2, \ldots\}$ and we introduce the pinning measures in finite volume N:

$$\frac{\mathbf{P}_{N,\varepsilon}(\tau \cap \{1,\ldots,N\})}{\mathbf{P}(\tau \cap \{1,\ldots,N\})} = \frac{1}{Z_{N,\varepsilon}} \exp\left(\sum_{n=1}^{N} \varepsilon_n \mathbf{1}_{\{n\in\tau\}}\right)$$

where $N \in \mathbb{N}$ (system size) and $\varepsilon_n \in \mathbb{R}$ (charges).

$\varepsilon_n > 0 \Rightarrow$ occurrence of renewal at n is favored, and viceversa

Random model: $\varepsilon_n = \beta\omega_n + h$ with $h \in \mathbb{R}$ (average pinning strength), $\beta > 0$ (strength of the disorder) and ω_n IID centered Gaussian random variables.

(Bio)-physical applications of the model

• Directed polymers in $\mathbb{Z}^{d+1}$, interacting with a defect line.
Let $\{S_n\}_{n\geq 0}$ be the Simple Random Walk on $\mathbb{Z}^d$ started at $S_0 = 0$, and $\tau := \{n : S_n = 0\}$. Then, τ is a renewal with $\alpha = 1/2$ if $d = 1$ and $\alpha = d/2 - 1$ for $d \geq 2$.

• $(1+1)$–dimensional wetting model. In this case $\alpha = 1/2$
[Forgacs et al. '86], [Derrida, Hakim, Vannimenius '92]

• Poland-Scheraga model of DNA denaturation (separation of the two DNA strands at high temperature).
Here $\alpha \simeq 1.15$ (phenomenological value) [Kafri, Mukamel, Peliti 2000]

Free energy and depinning transition

The infinite-volume limit of the free energy exists and is non-random:

$$\lim_{N\to\infty} \frac{1}{N} \log Z_{N,\omega} = F(\beta, h) \qquad \text{almost surely}$$

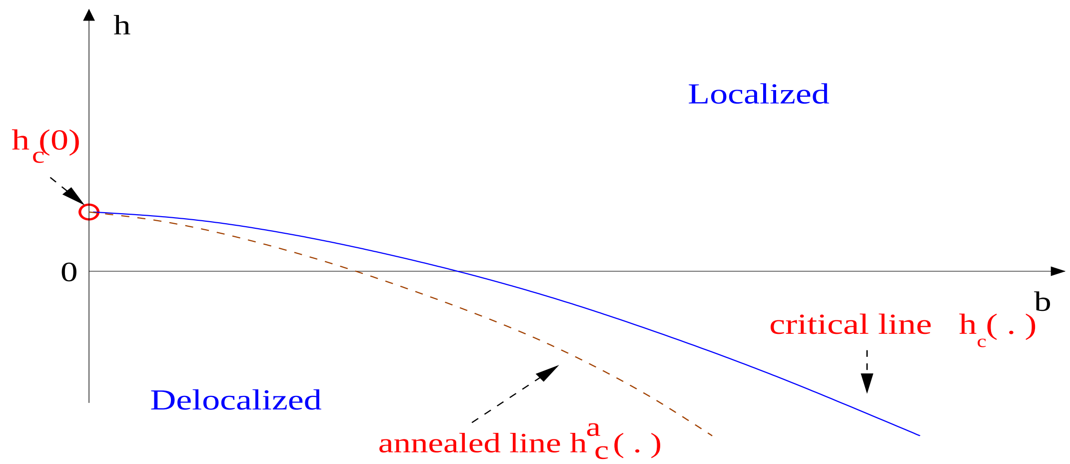
Easy fact: $F(\beta, h) \geq 0$. Indeed,

$$\frac{1}{N}\mathbb{E} \log Z_{N,\omega} \geq \frac{1}{N}\mathbb{E} \log \mathbf{E}\left\{e^{\sum_{n=1}^{N}(\beta\omega_n+h)1_{n\in\tau}};\ T_1 = N\right\}$$
$$= \frac{h}{N} + \frac{1}{N} \log \mathbf{P}(T_1 = N) \longrightarrow 0.$$

This suggests to define:

- the localized region $\mathcal{L} = \{(\beta, h) : F(\beta, h) > 0\}$
- the delocalized region $\mathcal{D} = \{(\beta, h) : F(\beta, h) = 0\}$

Qualitative picture of the phase diagram



The two regions are separated by a critical line $h_c(\beta)$:

$$\mathcal{L} = \{(\beta, h) : h > h_c(\beta)\}, \quad \mathcal{D} = \{(\beta, h) : h \leq h_c(\beta)\}$$

Easy fact: $h_c(0) = -\log \mathbf{P}(T_1 < \infty)$. Observe that $h_c(0) \geq 0$ and that $h_c(0) > 0$ iff the renewal is transient.

Annealed model

The annealed free energy is defined as

$$F^a(\beta, h) = \lim_N \frac{1}{N} \log \mathbb{E} Z_{N,\omega} = F(0, h + \beta^2/2)$$

and therefore $h_c^a(\beta) = h_c(0) - \beta^2/2$.

This is just a homogenous model with a different h

By Jensen's inequality, $F(\beta, h) \leq F^a(\beta, h)$ and $h_c(\beta) \geq h_c^a(\beta)$.

Is the annealed bound good in some situations? YES, see later.

The order parameter

The order parameter is the contact fraction:

$$L_N = \frac{1}{N}|\{\tau \cap \{1, \cdots, N\}|$$

related to the free energy by

$$\partial_h F(\beta, h) \overset{\text{almost surely}}{=} \lim_{N \to \infty} \mathbf{E}_{N,\omega}(L_N) \quad \begin{cases} > 0 & \text{in } \mathcal{L} \\ = 0 & \text{in } \mathcal{D} \end{cases}$$

$\mathbf{E}_{N,\omega}(\cdot)$: finite–volume, disorder–dependent Gibbs measure (with parameters β, h)

What happens at the critical point $h_c(\beta)$?

Order of the transition in the non–random case $\beta = 0$

Transition can be either of first or of higher order:

- If $\mathbf{E}(T_{i+1} - T_i) < +\infty$: first order (e.g., $\alpha > 1$)

$$F(0, h_c(0) + \Delta) \sim \Delta \quad \text{for} \quad \Delta \searrow 0$$

The contact fraction has a jump at $h_c(0)$

- If $\mathbf{E}(T_{i+1} - T_i) = +\infty$: higher order

(Non–integrable return times, e.g. $0 \leq \alpha < 1$)

$$F(0, h_c(0) + \Delta) \sim \Delta^{1/\alpha} \quad \text{for} \quad \Delta \searrow 0$$

Random case $\beta > 0$. Harris criterion

The following is believed to be true:

- For $\alpha < 1/2$ and $\beta \ll 1$ irrelevant disorder: $h_c(\beta) = h_c^a(\beta)$ and same critical exponents : $F(\beta, h_c(\beta) + \Delta) \sim \Delta^\nu$ with $\nu = 1/\alpha$.

- For $\alpha > 1/2$ and $\beta > 0$ relevant disorder: $h_c(\beta) > h_c^a(\beta)$ and different critical exponents: $\nu > 1/\alpha$

- For $\alpha = 1/2$ marginal disorder. No complete agreement. Probably: $h_c(\beta) > h_c^a(\beta)$.

Heuristic justification of the Harris criterion

We check stability of the annealed model when disorder is added.

Recall that $F^a(\beta, h) = F(0, h - h_c^a(\beta))$ and let $h - h_c^a(\beta) := \Delta > 0$. Then,

$$F^a(\beta, h) \sim \Delta \quad \text{for} \quad \alpha > 1,$$

and

$$F^a(\beta, h) \sim \Delta^{1/\alpha} \quad \text{for} \quad \alpha < 1.$$

Next, write

$$F(\beta, h) = F^a(\beta, h) + \overbrace{[F(\beta, h) - F^a(\beta, h)]}^{R(\beta, \Delta) \leq 0}.$$

Expansion of the "error term" for $\beta \sim 0$ gives

$$R(\beta, h) = -\beta^2 \left(\partial_\Delta F(0, \Delta)\right)^2 + O(\beta^3)$$

Take Δ small. Then, $|R(\beta, h)| \sim \beta^2 \Delta^{\frac{2}{\alpha}-2}$ (or $\sim \beta^2$ if $\alpha > 1$). Therefore,

$$\beta^2 \Delta^{\frac{2}{\alpha}-2} \sim |R(\beta, h)| \gg F^a(\beta, h) \sim \Delta^{1/\alpha} \quad \text{if} \quad \alpha > 1/2$$

(relevant disorder), while

$$|R(\beta, h)| \ll F^a(\beta, h) \quad \text{if} \quad \alpha < 1/2$$

(irrelevant disorder).

For $\alpha = 1/2$, $R(\beta, h)$ is of the same order as $F^a(\beta, h)$ (marginal disorder).

Irrelevant disorder

Is the annealed free energy a good approximation of F in some cases? In a sense NO, since one can prove [F.T. '07]:
$F(\beta, h) \neq F^a(\beta, h)$ whenever $\beta > 0$ and $F^a(\beta, h) \neq 0$.
However, more importantly:

Theorem 2 [K. Alexander '06, Different method: F.T. '07]

- Let $\alpha < 1/2$, and $\beta \ll 1$. Then, $h_c(\beta) = h_c^a(\beta)$ and $\nu = 1/\alpha$.

- Let $\alpha = 1/2$ and $\beta \ll 1$. Then, $|h_c^a(\beta) - h_c(\beta)| \leq O(e^{-1/\beta^2})$

which explains why weak-disorder expansions cannot decide the question of relevance.

Relevant disorder

Theorem 1 [G. Giacomin, F.T., CMP '06 and PRL '06]
For $0 < \beta < \infty$ there exists $c(\beta) < \infty$ such that, for $0 \leq \alpha < \infty$ and $h > h_c(\beta)$,

$$0 < F(\beta, h) < (1+\alpha)\, c(\beta)(h - h_c(\beta))^2.$$

I.e., the transition is always continuous (at least of second order)

Remark: $c(\beta)$ can be large for β small: $c(\beta) \sim \beta^{-2}$.

Remark: this implies that $\mathbb{E}\,\mathbf{E}_{N,\omega}(L_N) \to 0$ at $h_c(\beta)$.
More precise estimates [F.T., JSP '07]:
If $|h - h_c(\beta)| \leq N^{-1/3}$, then $\mathbb{E}\,\mathbf{E}_{N,\omega}(L_N) = O(N^{-1/3} \log N)$