A tractable multivariate default model based on a stochastic time-change

Jan-Frederik Mai[1] and Matthias Scherer[2]

Abstract

A stochastic time-change is applied to introduce dependence to a portfolio of credit-risky assets whose default times are modeled as random variables with arbitrary distribution. The dependence structure of the vector of default times is completely separated from its marginal default probabilities, making the model analytically tractable. This separation is achieved by restricting the time-change to suitable Lévy subordinators which preserve the marginal distributions. Jumps of the Lévy subordinator are interpreted as times of excess default clustering. Relevant for practical implementations is that the parameters of the time-change allow for an intuitive economical explanation and can be calibrated independently of the marginal default probabilities. On a theoretical level, a so-called time normalization allows to compute the resulting copula of the default times. Moreover, the exact portfolio-loss distribution and an approximation for large portfolios under a homogeneous portfolio assumption are derived. Given these results, the pricing of complex portfolio derivatives is possible in closed-form. Three different implementations of the model are proposed using a compound Poisson subordinator, a Gamma subordinator, and an Inverse Gaussian subordinator. In each case using two parameters to adjust the dependence structure, the model is capable of capturing the full range of dependence patterns from independence to complete comonotonicity. A simultaneous calibration to portfolio-CDS spreads and CDO tranche spreads is carried out to demonstrate the model's applicability.

Keywords: Lévy subordinator; Cuadras-Augé copula; CDO pricing; portfolio-loss process; multivariate default model

1 Introduction

Several multivariate default models rely on the technique of conditional independence. The default of each firm in the considered credit portfolio depends on mostly two sources of randomness; one firm-individual risk factor, the other being a market factor representing the uncertainty affecting all firms simultaneously. Conditional on the market factor, all default events are independent, which makes these models analytically tractable. This idea was first developed by [25] and put into a copula framework by [18]. In these models, idiosyncratic and market factors are modeled using normal distributions. As this assumption constrains the range of possible dependence structures, [14] generalized the

[1]HVB-Institute for Mathematical Finance, Technische Universität München, Boltzmannstr. 3, 85748 Garching, e-mail: mai@ma.tum.de

[2]HVB-Institute for Mathematical Finance, Technische Universität München, Boltzmannstr. 3, 85748 Garching, e-mail: scherer@ma.tum.de

model using t-distributions, [17] using NIG distributions, and [1] using a general Lévy framework. On a high level, our model is based on the same approach; yet, the common factor is replaced by a stochastic process which introduces a dynamic aspect.

The major difficulty in setting up a dynamic portfolio model is the evolution of the dependence structure. It has not been accomplished yet to impose a given dependence structure on arbitrary stochastic processes, an issue which was recently explored by [4] on a theoretical level. A popular approach to introduce dependence to stochastic processes is to subordinate them with a common stochastic time-change. This bears the additional advantage of naturally implying a dynamic model. Recent examples for advanced models in the context of credit-risk are [13], [16], and [12]. Our approach pursues a similar idea as these models. However, the crucial innovation lies on the explicit separation of dependence structure and marginal default probabilities. This property turns out to be fruitful for the derivation of several theoretical results and is also convenient for calibration purposes.

Another popular approach to capture the risk of correlated defaults is the class of doubly-stochastic default models, see e.g. [7], [9], and [26]. These models introduce dependence to individual default events, defined as the first jump of some Poisson process, by assuming correlated default intensities. Often, common risk factors with economical interpretation are used to drive the individual intensities. Consequently, conditional on these risk factors all default events are independent. Evidence is found in [7] that such models fail to explain excess clustering. Moreover, these models typically do not support joint defaults with positive probability. This is due to the fact that the hazard process is defined as the integrated intensity and is thus continuous. In contrast, our approach models the hazard process directly as a Lévy subordinator, which allows for jumps. This induces positive probabilities of joint defaults.

The starting point for the construction of our default model is the idea of a common stochastic time-change underlying the individual default times. A high level of tractability is achieved if the time-change is suitably chosen. More precisely, introducing a time normalization we keep the marginal distributions of the involved processes invariant under the time-change. This leads to surprisingly many qualitative results on the model structure. In particular, the survival copula of all default times is computed and it is shown that the bivariate marginals are of Cuadras-Augé type. Additional analytical results include pairwise default correlations, the exact portfolio-loss distribution under a homogeneous portfolio assumption, an approximation for large portfolios, and the conditional portfolio-loss distribution given earlier default events. Finally, we implement the model using different choices for the time-change and perform a calibration to CDO tranches and portfolio-CDS spreads of the European iTraxx.

The paper is organized as follows. Section 2 provides mathematical notions and properties of Lévy subordinators. Section 3 explains the construction of the model. In Section 4 we derive and explore the implied dependence structure of the model. Section 5 prepares analytical formulas for pricing applications under the assumption of a homogeneous portfolio. Section 6 applies the model to the pricing of a CDO contract and presents a calibration of the model. Finally, Section 7 concludes.

2 Mathematical preliminaries

Let us first introduce some notations used throughout this paper. For two real numbers x, y we denote by $x \vee y$ the larger and by $x \wedge y$ the smaller of both. The binomial law with n trials and success probability $p \in [0, 1]$ is abbreviated by $Bin(n, p)$, the exponential law with parameter $\lambda > 0$ by $Exp(\lambda)$. Other distributions are usually denoted by means of their cumulative distribution function, mostly by the letters F and G. By writing $X \sim F$ we mean that the random variable X has distribution given by F. Moreover, for a stochastic process $\Lambda = \{\Lambda_t\}_{t \geq 0}$ we denote by $[\Lambda] = \sigma(\Lambda_t \, : \, t \geq 0)$ the σ-algebra generated by the complete path of Λ. Finally, for a sample $E_1, \ldots, E_n$ of n i.i.d. random variables, we denote by $E_{(1)} \leq \cdots \leq E_{(n)}$ the order statistic of $E_1, \ldots, E_n$. We use the same notation for real numbers $x_1, \ldots, x_n$, i.e. $x_{(1)} \leq \cdots \leq x_{(n)}$. It is further convenient to additionally define $x_{(0)} = 0$ in this case.

Throughout this paper we work on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A one-dimensional Lévy process on this probability space is a càdlàg stochastic process $\Lambda = \{\Lambda_t\}_{t \geq 0}$, starting at $\Lambda_0 = 0$, which has independent and stationary increments and is stochastically continuous. Standard references for Lévy processes include [22], [3], [2], [23], and [5]; the latter two give applications to mathematical finance. Replacing the time in our default model requires the additional condition that almost all paths of Λ are non-decreasing; processes of this subclass are called *Lévy subordinators*.

One fundamental property of Lévy processes is the possibility of characterizing them completely in terms of few characteristics. In case of a subordinator, these are a drift $\mu \geq 0$ and a positive measure ν on $(0, \infty)$. Given these, the Lévy subordinator Λ is completely determined by its Laplace transform

$$\mathcal{L}[\Lambda_t](\lambda) = \mathbb{E}[e^{\lambda \Lambda_t}] = e^{t \, \Psi(\lambda)}, \;\; \Psi(\lambda) = \mu \, \lambda + \int_0^\infty (e^{\lambda \, t} - 1) \, \nu(dt), \;\; \lambda \leq 0, \qquad (1)$$

where Ψ is called *Laplace exponent* of Λ. Note that $\Psi : (-\infty, 0] \to (-\infty, 0]$ is strictly increasing unless $\Lambda_t \equiv 0$.

Heuristically speaking, Λ is a process that grows linearly with constant drift $\mu \geq 0$ and is affected by random upward jumps. The Lévy measure ν of the interval $[x, \infty)$ gives the expected number of jumps of Λ that are greater than or equal to x within one unit of time. In this paper we extensively exploit the convenient structure of the Laplace transform to construct the model. The following lemma is used in Example 2.3 to provide an intuitive interpretation of a so-called time normalization, which is defined in Definition 2.2.

Lemma 2.1 (Interpretation of the Laplace exponent)
Let Λ be a Lévy subordinator with characteristics (μ, ν) and corresponding Laplace exponent Ψ. Then, for each $\lambda < 0$ we have

$$\mu + \mathbb{E}\Big[\#\big\{t \in [0, \mathbb{E}[E]] \,:\, \Delta\Lambda_t \geq E\big\}\Big] = \frac{\Psi(\lambda)}{\lambda},$$

where E is an exponentially distributed random variable with parameter $-\lambda$, independent of the subordinator Λ, and $\Delta\Lambda_t = \Lambda_t - \lim_{u\uparrow t}\Lambda_u$. In particular, if λ is a fixpoint for Ψ, the right-hand side equals one.

Proof
Let $\lambda < 0$ and $E \sim Exp(-\lambda)$, independent of Λ. We divide Equation (1) by λ to get

$$\frac{\Psi(\lambda)}{\lambda} = \mu + \int_0^\infty \frac{e^{\lambda\, t} - 1}{\lambda}\, \nu(dt).$$

Since $1 - \exp(\lambda\, t) = \mathbb{P}(E \leq t)$ and $-\lambda^{-1} = \mathbb{E}[E]$, we obtain from Tonelli's theorem

$$\begin{aligned}
\int_0^\infty \frac{e^{\lambda\, t} - 1}{\lambda}\, \nu(dt) &= \mathbb{E}[E] \int_0^\infty \mathbb{E}[\mathbb{1}_{\{E \leq t\}}]\, \nu(dt) \\
&= \mathbb{E}[E]\, \mathbb{E}\left[\int_0^\infty \mathbb{1}_{\{E \leq t\}}\, \nu(dt)\right] \\
&= \mathbb{E}[E]\, \mathbb{E}[\nu([E, \infty))] \\
&= \mathbb{E}\Big[\#\big\{t \in [0, \mathbb{E}[E]] \,:\, \Delta\Lambda_t \geq E\big\}\Big],
\end{aligned}$$

establishing the claim. □

To accomplish a separation of dependence structure and marginal distributions we impose the following normalization in what follows.

Definition 2.2 (Time normalization (TN))
Let F be a cumulative distribution function with $F(0) = 0$. Moreover, let $\Lambda = \{\Lambda_t\}_{t\geq 0}$ be a stochastic process which is almost surely non-decreasing and satisfies $\Lambda_0 = 0$. We say that Λ satisfies (TN) for the distribution *F if $\mathbb{E}[F(\Lambda_t)] = F(t)$ holds $\forall t \geq 0$.*

The following example is crucial for the construction of the multivariate default model and shows how F and Λ can be specified such that Definition 2.2 applies. The convenient structure of the Laplace exponent of a Lévy subordinator is exploited here.

Example 2.3
Let $F(t) = \big(1 - \exp(\lambda\, t)\big)\, \mathbb{1}_{\{t>0\}}$ be the cumulative distribution function of an exponential random variable with parameter $-\lambda > 0$. Let Λ be a Lévy subordinator with characteristics (μ, ν) and corresponding Laplace exponent Ψ. Then, it holds that

$$\begin{aligned}
\Lambda \text{ satisfies (TN) for } F \;&\Leftrightarrow\; \Psi(\lambda) = \lambda \\
&\Leftrightarrow\; \mu = 1 - \int_0^\infty \frac{e^{\lambda\, t} - 1}{\lambda}\, \nu(dt).
\end{aligned}$$

At this point, let us briefly explain the intuition behind this example. Let E be exponentially distributed with parameter $-\lambda$ and independent of the Lévy subordinator Λ. Further suppose that Λ satisfies (TN) for $Exp(-\lambda)$. It is demonstrated above that this is equivalent to λ being a fixpoint of the Laplace exponent Ψ. By Lemma 2.1 this is equivalent to

$$\mathbb{E}\Big[\#\big\{t \in [0, \mathbb{E}[E]] \,:\, \Delta\Lambda_t \geq E\big\}\Big] = 1 - \mu. \tag{2}$$

By the lack of memory property of the exponential distribution, the stochastic process $\mathbb{1}_{\{t \leq E\}}$ is a continuous-time Markov chain with intensity matrix

$$Q = \begin{pmatrix} \lambda & -\lambda \\ 0 & 0 \end{pmatrix}.$$

Subordinating this Markov chain with Λ, it follows from the Lévy properties of Λ that $\mathbb{1}_{\{\Lambda_t \leq E\}}$ is again a continuous-time Markov chain, compare [24]. In general, the intensity matrix of the subordinated process is not invariant under the time-change. However, due to condition (TN), both Markov processes have the same distribution, i.e. the same intensity matrix Q. This Markov chain is degenerate, since it starts in state 1, remains there for the time E, and is then absorbed by state 0. The intuitive interpretation of Equation (2) is that during the expected lifetime of the Markov chain, the average number of jumps of the Lévy subordinator exceeding this lifetime must be equal to $1 - \mu \in [0, 1]$. Otherwise, the intensity matrix of the subordinated process is not equal to the original matrix Q.

3 The multivariate default model

We consider n defaultable firms and denote their default times by $(\tau^1, \ldots, \tau^n)$. The i-th default time τ^i is constructed in such a way that it has the pre-specified continuous and strictly increasing cumulative distribution function G_i with $G_i(0) = 0$, $G_i(t) < 1$ for all $t \geq 0$, $i = 1, \ldots, n$. Allowing for (almost) arbitrary marginal distributions gives convenient freedom for a latter implementation of the model. Then, the cumulative hazard function $h_i(t) = -\log\big(1 - G_i(t)\big)$ is well-defined for $i = 1, \ldots, n$, and $h_i(t)$ tends to infinity as t does.

Let $E_1, \ldots, E_n$ be i.i.d. random variables with $Exp(1)$-distribution on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let Λ be an independent Lévy subordinator satisfying (TN) for the unit exponential law, i.e. with Laplace exponent Ψ satisfying $\Psi(-1) = -1$ according to Example 2.3. We define the stochastic processes $A^i = \{A^i_t\}_{t \geq 0}$ for $i = 1, \ldots, n$ as $A^i_t := \mathbb{1}_{\{E_i \leq \Lambda_{h_i(t)}\}}$. Furthermore, for each $i = 1, \ldots, n$ define

$$\tau^i := \inf\{t \geq 0 \,:\, A^i_t = 1\} = \inf\{t \geq 0 \,:\, E_i \leq \Lambda_{h_i(t)}\}.$$

Finally, our model consists of the $\{0,1\}^n$-valued stochastic process $(A^1, \ldots, A^n)$ or equivalently of the random vector $(\tau^1, \ldots, \tau^n)$. Note that the processes A^i are dependent, as they are similarly affected by the common factor Λ. Let us remark that the default times in our framework may alternatively be considered as first-jump times of Poisson processes subject to a common stochastic clock.

Theorem 3.1 (Distributional properties)

(a) For fixed $t \geq 0$ and $i = 1, \ldots, n$, each A_t^i follows a Bernoulli distribution with probability of success $G_i(t)$. Under $\mathbb{P}(.|[\Lambda])$, A_t^i has success probability $1 - \exp(-\Lambda_{h_i(t)})$.

(b) For each $i = 1, \ldots, n$ the random time τ^i has law G_i. This shows that the univariate marginals are invariant under the time-change.

(c) Let $\alpha_\nu := 2 + \Psi(-2)$. For $i \neq j$, the correlation coefficient $Corr\big[A_t^i, A_t^j\big]$, i.e. the default correlation of firms i and j up to time t is equal to

$$\frac{G_i(t) + G_j(t) - 1 + \Big(1 - \big(G_i(t) \wedge G_j(t)\big)\Big)^{1-\alpha_\nu} \Big(1 - \big(G_i(t) \vee G_j(t)\big)\Big)}{\sqrt{G_i(t) - G_i(t)^2}\,\sqrt{G_j(t) - G_j(t)^2}} \\ - \frac{G_i(t)\, G_j(t)}{\sqrt{G_i(t) - G_i(t)^2}\,\sqrt{G_j(t) - G_j(t)^2}}.$$

(d) Let $i \neq j$. We assume the existence of the limits $G_k^{'}(0) := \lim_{t\downarrow 0} G_k(t)/t = h_k^{'}(0) > 0$ for $k \in \{i, j\}$. Moreover, we assume the existence of an $\epsilon > 0$ such that $h_i(t) \leq h_j(t)$ for all $t \in [0, \epsilon]$. Then, it holds that

$$\lim_{t\downarrow 0} Corr\big[A_t^i, A_t^j\big] = \sqrt{\frac{G_i^{'}(0)}{G_j^{'}(0)}}\,\alpha_\nu.$$

This shows that the model is able to allow for a positive limit of default correlations; a property which is important for the pricing of correlation driven products with short maturity.

Proof

(a) Clearly, A_t^i follows a Bernoulli distribution. We compute the success probability

$$\begin{aligned}\mathbb{P}(A_t^i = 1) &= \mathbb{P}(E_i \leq \Lambda_{h_i(t)}) = \mathbb{E}\big[\mathbb{P}(E_i \leq \Lambda_{h_i(t)} | \Lambda_{h_i(t)})\big] \\ &= \mathbb{E}\big[1 - e^{-\Lambda_{h_i(t)}}\big] = 1 - e^{h_i(t)\,\Psi(-1)} \\ &= 1 - e^{-h_i(t)} = G_i(t),\end{aligned}$$

where in the fifth equality we use $\Psi(-1) = -1$. From the above calculation we can immediately derive the second claim.

(b) This follows from part (a), since $\mathbb{P}(\tau^i \leq t) = \mathbb{P}(A_t^i = 1) = G_i(t)$.

(c) By part (a), $Var(A_t^k) = G_k(t) - G_k(t)^2$ for $k \in \{i,j\}$. Moreover, by conditional independence, we get

$$\begin{aligned}\mathbb{E}\big[A_t^i A_t^j\big] &= \mathbb{E}\big[\mathbb{P}(E_i \le \Lambda_{h_i(t)} \,|\, [\Lambda])\, \mathbb{P}(E_j \le \Lambda_{h_j(t)} \,|\, [\Lambda])\big] \\ &= \mathbb{E}\big[(1 - e^{-\Lambda_{h_i(t)}})\,(1 - e^{-\Lambda_{h_j(t)}})\big] \\ &= 1 - e^{-h_i(t)} - e^{-h_j(t)} + \mathbb{E}\big[e^{-\Lambda_{h_i(t)} - \Lambda_{h_j(t)}}\big] \\ &= G_i(t) + G_j(t) - 1 \\ &\quad + \exp\big((h_i(t) \wedge h_j(t))\,(1 + \Psi(-2)) - (h_i(t) \vee h_j(t))\big).\end{aligned}$$

In the last equality we use the Lévy properties of Λ. From these calculations the claim is obvious since $\exp\big(-h_k(t)\big) = 1 - G_k(t)$ for $k \in \{i,j\}$.

(d) Firstly, by the assumption on the existence and positivity of the limits $G_k'(0)$ for $k \in \{i,j\}$, we obtain by L'Hospital's rule

$$\lim_{t \downarrow 0} \sqrt{\frac{G_i(t) - G_i(t)^2}{G_j(t) - G_j(t)^2}} = \sqrt{\frac{G_i'(0)}{G_j'(0)}}, \quad \lim_{t \downarrow 0} \sqrt{\frac{G_j(t) - G_j(t)^2}{G_i(t) - G_i(t)^2}} = \sqrt{\frac{G_j'(0)}{G_i'(0)}}.$$

Using this and the second assumption $h_i \le h_j$ near zero, we can again apply L'Hospital's rule to the result in part (c) and obtain the claim. □

Remark 3.2 (Link to Bernoulli Mixture Models)
Note that the result on the correlation in part (c) of Theorem 3.1 can also be stated in terms of the mixing variable Λ

$$Corr\big[A_t^i, A_t^j\big] = \frac{Cov\big[1 - e^{-\Lambda_{h_i(t)}}, 1 - e^{-\Lambda_{h_j(t)}}\big]}{\sqrt{Var\big[A_t^i\big]}\,\sqrt{Var\big[A_t^j\big]}}.$$

In the exchangeable case $h_1 = \cdots = h_n =: h$, the covariance of the default indicators is given by the variance of the mixing variable $1 - \exp(-\Lambda_{h(t)})$. This result was already stated in [10] in the context of so-called Bernoulli Mixture Models.

Example 3.3 (Possible marginal distributions)
Note that our construction induces dependence to models with arbitrary marginal default probabilities. If for example $\lambda^i = \{\lambda_t^i\}_{t \ge 0}$, $i = 1, \ldots, n$, are deterministic non-negative intensities, we can define the marginal distribution functions as $G_i(t) = 1 - \exp(-\int_0^t \lambda_s^i\, ds)$. In this case, the cumulative hazard functions take the form $h_i(t) = \int_0^t \lambda_s^i\, ds$.

The parameter $\alpha_\nu = \Psi(-2) + 2$ turns out to be an important measure for the induced dependence, see Theorem 4.1. Therefore, we explore the effect of the choice of Lévy measure on α_ν in what follows. Lemma 3.4 shows that high dependence requires the time-change process to be of pure jump type. The larger the drift $\mu \ge 0$, the smaller is the dependence coefficient α_ν. However, let us stress that Lemma 3.4 is not restrictive concerning the dependence, since μ is implicitly defined via the time normalization, compare Example 2.3. Therefore, if large dependence is required, μ automatically becomes small.

Lemma 3.4 (Upper bound for α_ν)
We have $\alpha_\nu \leq 1-\mu$. Moreover, α_ν is represented by

$$\alpha_\nu = \int_0^\infty (1-e^{-t})^2\, \nu(dt).$$

Proof
According to condition (TN), Example 2.3 shows that the drift μ is completely determined by the Lévy measure of Λ and is given by $\mu = 1 - \int_0^\infty \big(1-\exp(-t)\big)\, \nu(dt)$. Hence, we obtain

$$\begin{aligned}\alpha_\nu &= 2 + \Psi(-2) \\ &= 2 - 2\,\mu + \int_0^\infty (e^{-2\,t}-1)\, \nu(dt) \\ &= 2 - 2 + \int_0^\infty 2\,(1-e^{-t}) + (e^{-2\,t}-1)\, \nu(dt) \\ &= \int_0^\infty (1-e^{-t})^2\, \nu(dt).\end{aligned}$$

From this, we obtain

$$\alpha_\nu = \int_0^\infty (1-e^{-t})^2\, \nu(dt) \leq \int_0^\infty (1-e^{-t})\, \nu(dt) = 1-\mu. \qquad \square$$

Example 3.5 (Poisson Subordinator)
Suppose the assumptions of Theorem 3.1 (d) to hold. For the special choice $\nu(dt) = \big(1-\exp(-J)\big)^{-1}\, \mathbb{1}_{\{J \in dt\}}$, where $J>0$, we obtain by a simple calculation

$$\lim_{J\downarrow 0}\lim_{t\downarrow 0} Corr[A_t^i, A_t^j] = \lim_{J\downarrow 0}\sqrt{\frac{G_i'(0)}{G_j'(0)}}\,(1-e^{-J}) = 0. \tag{3}$$

Choosing the Lévy measure as above corresponds to a pure jump subordinator of Poisson type $\Lambda_t = J\,N_t$, where N is a Poisson process with intensity given by $\big(1-\exp(-J)\big)^{-1}$. For a given jump size J, this is precisely the intensity such that the time-change Λ satisfies (TN) for the unit exponential distribution. We have that $\big(1-\exp(-J)\big)^{-1}$ increases to infinity as J decreases to zero. Hence, the subordinator Λ changes its structure from finite- to infinite activity. Then, Equation (3) implies that in short time periods, the default correlation vanishes when changing from a model with few big jumps to one with infinitely many small jumps. The following lemma states the same qualitative result more formally.

Lemma 3.6 (α_ν for small jumps)
Suppose the Lévy subordinator Λ satisfies at least one of the following two properties:

1. *The Lévy measure ν is absolutely continuous with respect to the Lebesque measure.*
2. *Λ is of compound Poisson type.*

Then, the function

$$f : [0,\infty] \to [0,1], \;\; x \mapsto \int_0^x (1-e^{-t})^2 \,\nu(dt),$$

is continuous at zero from the right, with $\lim_{x\downarrow 0} f(x) = 0$.

Proof
The assertion follows from the fundamental theorem of calculus if ν is absolutely continuous with respect to the Lebesque measure. Now suppose Λ is a compound Poisson process, i.e. $\nu(dx) = \beta \int_{\{dx\}} dD(t)$, where D is the jumpsize-distribution function and $\beta > 0$ is the jump intensity. Then, let $\{x_n\}_{n\in\mathbb{N}}$ be any sequence in $[0,\infty)$ tending to zero. We find the estimate

$$0 \le f(x_n) = \int_0^{x_n} (1-e^{-t})^2 \,\nu(dt) \le \max_{t\in[0,x_n]} \big\{(1-e^{-t})^2\big\}\, \beta \big(D(x_n) - D(0)\big).$$

Letting $n \to \infty$ yields the result. □

Note that most popular Lévy subordinators satisfy either one of the conditions of Lemma 3.6. The qualitative statement we can infer from Lemma 3.6 is that for small $\epsilon > 0$, subordinators with Lévy measure ν satisfying $\nu\big([\epsilon,\infty)\big) = 0$ imply only a small dependence coefficient $\alpha_\nu = f(\epsilon)$. Hence, significant dependence is only induced by (large) jumps of Λ.

Figure 1 shows a simulation of the model with $n = 125$ firms over a period of 100 years. For this example, we set $h(t) = h_i(t) = 0.0038\,t$, corresponding to an exponential distribution with mean 263.16 for all default times, and $\Lambda_t = \mu\, t + \sum_{i=1}^{N_t} J_i$, where the intensity of the Poisson process N is set to 2.14 and the mean of the exponentially distributed random variables J_i is 0.13. Hence, $\mu = 0.748$ in order for Λ to satisfy (TN) for the unit exponential law. The upper graph shows the distribution of the i.i.d. unit exponential random variables $\{E_i\}$ that occur before $\Lambda_{h(100)}$, whereas the lower graph shows the dependent default times $\{\tau^i\}$ that occur before time $t = 100$. The solid line shows the path of $\Lambda_t^{-1} := \inf\{s \ge 0 \,:\, \Lambda_{h(s)} \ge t\}$ in the upper graph and the path of $\Lambda_{h(t)}$ in the lower graph. Jumps of $\Lambda_{h(t)}$ correspond to stagnancy of Λ_t^{-1}. Hence, all E_i's in the upper plot that fall into a phase of stagnancy of Λ_t^{-1} are piled up in the lower plot. Let us point out again that this yields the desired property of default clustering as illustrated in the lower graph of Figure 1.

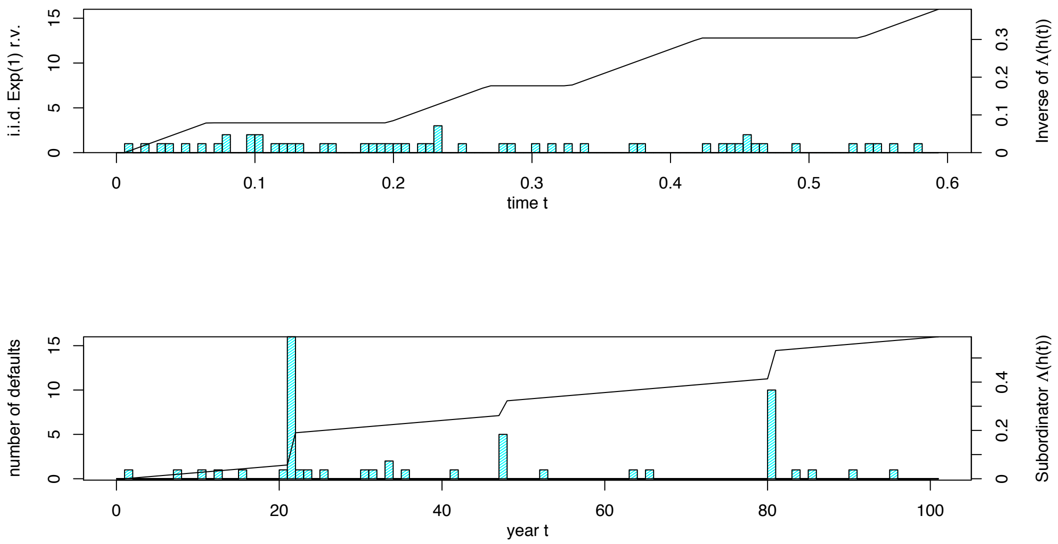

Figure 1 One realization of dependent defaults in our model.

4 Dependence structure

In this section we derive the implied copula of the time-changed default times in our model. This link between stochastic processes, default times, and copulas is remarkable from a theoretical point of view and useful for the interpretation of the calibrated parameters from a practical perspective. In a first step, we investigate the dependence structure between any two firms in our portfolio. Then, we generalize Theorem 4.1 to n firms in Theorem 4.3.

Theorem 4.1 (Implied copula for two firms)
We have

$$C_{\alpha_\nu}\big(\mathbb{P}(\tau^1 > x), \mathbb{P}(\tau^2 > y)\big) = \mathbb{P}\big(\tau^1 > x, \tau^2 > y\big),$$

where $C_\alpha : [0,1]^2 \to [0,1],\ C_\alpha(x,y) = (x \vee y)^{1-\alpha}(x \wedge y)$, *is the Cuadras-Augé copula with parameter* $\alpha \in [0,1]$.

Proof
On the one hand, we have

$$\begin{aligned}
\mathbb{P}\big(\tau^1 > x, \tau^2 > y\big) &= \mathbb{P}\big(A^1_x = 0, A^2_y = 0\big) \\
&= \mathbb{P}\big(E_1 > \Lambda_{h_1(x)}, E_2 > \Lambda_{h_2(y)}\big) \\
&= \mathbb{E}\Big[\mathbb{P}\big(E_1 > \Lambda_{h_1(x)}, E_2 > \Lambda_{h_2(y)} \big| [\Lambda]\big)\Big] \\
&= \mathbb{E}\Big[\mathbb{P}\big(E_1 > \Lambda_{h_1(x)} \big| [\Lambda]\big)\, \mathbb{P}\big(E_2 > \Lambda_{h_2(y)} \big| [\Lambda]\big)\Big] \\
&= \mathbb{E}\Big[e^{-\Lambda_{h_1(x)} - \Lambda_{h_2(y)}}\Big] \\
&= \exp\big(\left(h_1(x) \wedge h_2(y)\right)\left(1 + \Psi(-2)\right) - \left(h_1(x) \vee h_2(y)\right)\big).
\end{aligned}$$

On the other hand, we see

$$\begin{aligned}
&C_{\alpha_\nu}\big(1 - G_1(x), 1 - G_2(y)\big) \\
&= \Big(\big(1 - G_1(x)\big) \vee \big(1 - G_2(y)\big)\Big)^{1-\alpha_\nu} \Big(\big(1 - G_1(x)\big) \wedge \big(1 - G_2(y)\big)\Big) \\
&= \Big(e^{-h_1(x)} \vee e^{-h_2(y)}\Big)^{1-(2+\Psi(-2))} \Big(e^{-h_1(x)} \wedge e^{-h_2(y)}\Big) \\
&= \exp\big(\left(h_1(x) \wedge h_2(y)\right)\left(1 + \Psi(-2)\right) - \left(h_1(x) \vee h_2(y)\right)\big).
\end{aligned}$$

Hence, the claim is proved. □

Let us remark at this point that the family $C_\alpha : [0,1]^2 \to [0,1]$, for $\alpha \in [0,1]$, given by

$$C_\alpha(x, y) := (x \vee y)^{1-\alpha}\,(x \wedge y),$$

of *Cuadras-Augé* copulas was originally presented by [6]. Moreover, we note that choosing a Lévy subordinator Λ corresponds to determining the dependence parameter α_ν.

As long as we support jumps of Λ and have identical marginal default probabilities, we have that $\mathbb{P}(\tau^1 = \tau^2) > 0$. In probabilistic terms, this corresponds to a singular component of C_{α_ν}. Translated into our multivariate default model, this corresponds to multiple defaults at the same time. Interestingly, one can even compute the probability of joint default. Namely, it follows from [6] that

$$\mathbb{P}\big(G_1(\tau^1) = G_2(\tau^2)\big) = \frac{\alpha_\nu}{2 - \alpha_\nu}.$$

Note that $G_1 = G_2$ allows for the desired property of joint defaults, as illustrated in Figure 1, which distinguishes the present framework from many other default models.

Furthermore, note that the complete range of dependence patterns from independence to complete comonotonicity is covered. If $\alpha_\nu = 0$, e.g. if $\Lambda_t = t$, the firms are independent.

In this case the copula boils down to $C_0(x,y) = x\,y$. On the other hand, $\alpha_\nu = 1$ corresponds to the upper Fréchet bound $C_1(x,y) = \min\{x,y\}$, which implies that both default times are completely dependent. Note that we can easily construct a sequence of Lévy subordinators $\{\Lambda_k\}_{k\in\mathbb{N}}$ with corresponding Lévy measures $\{\nu_k\}_{k\in\mathbb{N}}$ satisfying (TN) for $Exp(1)$ and yielding $\alpha_{\nu_k} \to 1$ as $k \to \infty$, see e.g. Example 6.2.

The parameter α_ν allows for another intuitive interpretation, namely it equals the parameter of *upper-tail dependence* of the copula C_{α_ν}, compare [15], page 33. This is equivalent to the parameter of *lower-tail dependence* of the respective survival copula $\hat{C}_{\alpha_\nu}$ of C_{α_ν}, which is precisely the copula of (τ^1, τ^2). Therefore, we have several representations for α_ν, namely

$$\begin{aligned}
\alpha_\nu = \Psi(-2) + 2 &= \int_0^\infty (e^{-t} - 1)^2\, \nu(dt) && \text{(analytical formula)}\\
&= \lim_{u\downarrow 0} \mathbb{P}\left(\tau^1 \leq G_1^{-1}(u) \,\middle|\, \tau^2 \leq G_2^{-1}(u)\right) && \text{(lower-tail dependence)}\\
&= \lim_{t\downarrow 0} \operatorname{Corr}\big[A_t^1, A_t^2\big]. && \text{(correlation at zero)}
\end{aligned}$$

Note that the last equality requires $G_1 = G_2$ and the existence of $G_1'(0) > 0$, according to Theorem 3.1 (d).

Remark 4.2
Using the notation $\hat{C}_{\alpha_\nu}(x,y) = x + y - 1 + C_{\alpha_\nu}(1-x, 1-y)$ for the actual copula of (τ^1, τ^2), we can rewrite the result of Theorem 3.1 (c) as

$$Corr\big[A_t^i, A_t^j\big] = \frac{\hat{C}_{\alpha_\nu}(G_i(t), G_j(t)) - G_i(t)\, G_j(t)}{\sqrt{G_i(t) - G_i(t)^2}\,\sqrt{G_j(t) - G_j(t)^2}}.$$

With more effort it is further possible to compute the implied copula of the joint distribution of n default times. This is presented in the following theorem.

Theorem 4.3 (Implied copula for n firms)
Define the function $C : [0,1]^n \to [0,1]$ by

$$(x_1, \ldots, x_n) \mapsto \prod_{i=1}^{n} (x_{(n-i+1)})^{-\Psi(-(n+1-i)) + \Psi(-(n-i))}.$$

Then, we have

$$C\big(\mathbb{P}(\tau^1 > x_1), \mathbb{P}(\tau^2 > x_2), \ldots, \mathbb{P}(\tau^n > x_n)\big) = \mathbb{P}\big(\tau^1 > x_1, \tau^2 > x_2, \ldots, \tau^n > x_n\big).$$

Proof
This proof is an extension of the proof for $n = 2$ in Theorem 4.1. First, by the Lévy

properties of Λ, we get for arbitrary $x_1, x_2, \dots, x_n \in [0,\infty)$

$$\mathbb{E}\left[e^{-\sum_{i=1}^n \Lambda_{x_i}}\right] = \prod_{i=1}^n \mathbb{E}\left[e^{-(n+1-i)\,\Lambda_{(x_{(i)}-x_{(i-1)})}}\right]$$
$$= \prod_{i=1}^n \exp\Big(\big(x_{(i)} - x_{(i-1)}\big)\,\Psi\big(-(n+1-i)\big)\Big).$$

From this, using conditional independence, it is straightforward to calculate

$$\begin{aligned} W(x_1,\dots,x_n) :&= \mathbb{P}\big(\tau^1 > x_1, \tau^2 > x_2, \dots, \tau^n > x_n\big) \\ &= \mathbb{P}\big(E_1 > \Lambda_{h_1(x_1)}, E_2 > \Lambda_{h_2(x_2)}, \dots, E_n > \Lambda_{h_n(x_n)}\big) \\ &= \mathbb{E}\left[\prod_{i=1}^n e^{-\Lambda_{h_i(x_i)}}\right] \\ &= \mathbb{E}\left[e^{-\sum_{i=1}^n \Lambda_{h_i(x_i)}}\right] \\ &= \prod_{i=1}^n \exp\Big(\big(h(x)_{(i)} - h(x)_{(i-1)}\big)\,\Psi\big(-(n+1-i)\big)\Big). \end{aligned}$$

In the last equation, the i-th largest element of the set $\{h_1(x_1),\dots,h_n(x_n)\}$ is denoted by $h(x)_{(i)}$ with the convention $h(x)_{(0)} = 0$. The claim now follows immediately from Sklar's theorem, which gives

$$C(x_1,\dots,x_n) = W\left(G_1^{-1}(1-x_1), G_2^{-1}(1-x_2), \dots, G_n^{-1}(1-x_n)\right),$$

since $x \mapsto h_i\big(G_i^{-1}(1-x)\big) = -\log x$ is strictly decreasing for $i = 1,\dots,n$. Note that $\Psi(0) = 0$. □

Remark 4.4
If we specify $\Lambda_t = J\,N_t$ as in Example 3.5 with $J > 0$ and intensity of N given by $\big(1-\exp(-J)\big)^{-1}$, it follows that the induced copula is given by

$$C(x_1,\dots,x_n) = \prod_{i=1}^n \big(x_{(n-i+1)}\big)^{(1-\alpha_\nu)^{n-i}},$$

where $\alpha_\nu = 1 - \exp(-J)$. This copula is the exchangeable special case of the multivariate distribution presented in [6]. However, this does not hold true for general subordinators.

5 Homogeneous portfolio assumption

The central quantity for most pricing formulas of portfolio derivatives is the distribution of the overall portfolio-loss process. To begin with, we define the zero-recovery relative-loss process $L^n = \{L^n_t\}_{t\geq 0}$ as

$$L^n_t := \frac{1}{n}\sum_{i=1}^n A^i_t.$$

L_t^n gives the fraction of defaulted names in the portfolio up to time t. Combined with the recovery rate, this quantity is later used to compute the portfolio-loss process. In this section we further assume $G_1 = \cdots = G_n =: G$, i.e. all default times are equal in distribution. We denote the corresponding cumulative hazard function by $h(t) = -\log(1-G(t))$. The following theorem shows that the presented model with homogeneous firms is an Exchangeable Bernoulli Mixture Model in the sense of [10] for every fixed $t \geq 0$. This provides an approximation of the portfolio-loss distribution for large portfolios and gives the exact distribution of L_t^n.

Theorem 5.1 (Portfolio-loss distribution)

(a) Under $\mathbb{P}(.|[\Lambda])$, $n\,L_t^n \sim Bin\big(n, 1-e^{-\Lambda_{h(t)}}\big)$.

(b) For fixed $t \geq 0$, L_t^n tends to the mixing variable $1-\exp(-\Lambda_{h(t)})$ $\mathbb{P}$-a.s. and in L^2, as n tends to infinity.

(c) For $m \in \{0, 1, \ldots, n\}$,

$$\mathbb{P}\big(n\,L_t^n = m\big) = \binom{n}{m} \sum_{l=0}^{m} (-1)^l \binom{m}{l} (1-G(t))^{-\Psi\big(-(n+l-m)\big)}.$$

Proof

(a) Conditional on $[\Lambda]$, $\{A_t^i\}_{i=1,\ldots,n}$ are independent and follow a Bernoulli distribution with success probability $1-\exp(-\Lambda_{h(t)})$, compare Theorem 3.1.

(b) According to [10] we have a Bernoulli Mixture Model for fixed $t \geq 0$ with mixing variable $1-\exp(-\Lambda_{h(t)})$. The almost sure convergence then follows from [20], page 357 ff. To show L^2-convergence, we compute $\mathbb{E}\big[L_t^n\big] = G(t)$ as well as

$$\mathbb{E}\big[L_t^n\,(1-e^{-\Lambda_{h(t)}})\big] = \mathbb{E}\big[(1-e^{-\Lambda_{h(t)}})^2\big],$$

$$\mathbb{E}\big[(L_t^n)^2\big] = \frac{G(t)}{n} + \frac{n-1}{n}\,\mathbb{E}\big[(1-e^{-\Lambda_{h(t)}})^2\big].$$

From this, it follows that

$$\begin{aligned}\mathbb{E}\Big[\big(L_t^n - (1-e^{-\Lambda_{h(t)}})\big)^2\Big] &= \mathbb{E}\big[(L_t^n)^2\big] - 2\,\mathbb{E}\big[L_t^n\,(1-e^{-\Lambda_{h(t)}})\big] + \mathbb{E}\big[(1-e^{-\Lambda_{h(t)}})^2\big] \\ &= \frac{G(t)}{n} + \frac{n-1}{n}\,\mathbb{E}\big[(1-e^{-\Lambda_{h(t)}})^2\big] - \mathbb{E}\big[(1-e^{-\Lambda_{h(t)}})^2\big] \\ &\overset{n\to\infty}{\longrightarrow} 0.\end{aligned}$$

(c) It can be computed that

$$\begin{aligned}\mathbb{P}\Big(\sum_{k=1}^{n} A_t^i = m\Big) &= \mathbb{E}\Big[\mathbb{P}\Big(\sum_{k=1}^{n} \mathbb{1}_{\{E_i \leq \Lambda_{h(t)}\}} = m \,\Big|\, \Lambda_{h(t)}\Big)\Big] \\ &= \mathbb{E}\left[\binom{n}{m}(1-e^{-\Lambda_{h(t)}})^m\,(e^{-\Lambda_{h(t)}})^{n-m}\right] \\ &= \binom{n}{m}\,\mathbb{E}\Big[(1-e^{-\Lambda_{h(t)}})^m\,(e^{-\Lambda_{h(t)}})^{n-m}\Big].\end{aligned}$$

The term inside the expectation may be expanded using the binomial formula

$$(1 - e^{-\Lambda_{h(t)}})^m \, (e^{-\Lambda_{h(t)}})^{n-m} = \sum_{l=0}^{m} \binom{m}{l} (-1)^l \, e^{-(l+n-m)\,\Lambda_{h(t)}},$$

the claim then follows easily. □

Being able to update the value of credit derivatives, when they are already on the run and defaults in the portfolio are observed, is important for practical applications. Let us assume that at some point in time $u > 0$ already k or more firms in the portfolio have defaulted. Given this information one can then update the value of the considered credit derivative. For this purpose, the distribution of L_t^n conditional on the number of defaults up to time $u < t$ is required. This conditional distribution is computed below and exemplarily illustrated in Figure 2.

Theorem 5.2 (Updated portfolio-loss distribution)
Let $u \leq t$. For $n \geq m > k \geq 1$ we have

$$\begin{aligned}
\mathbb{P}\big(n\, L_t^n \geq m,\, n\, L_u^n \geq k\big) &= \frac{n!}{(k-1)!\,(m-1-k)!\,(n-m)!} \\
&\times \sum_{i=0}^{m-1-k} \binom{m-1-k}{i} \sum_{l=0}^{k-1} \binom{k-1}{l} \frac{(-1)^{i+l}}{(1+i+n-m)} \\
&\times \Bigg[\frac{1-(1-G(u))^{-\Psi(-(l+n+1-k))}}{1+n+l-k} - \frac{(1-G(t))^{-\Psi(-(1+i+n-m))}}{l+m-k-i} \\
&+ \frac{(1-G(t))^{-\Psi(-(1+i+n-m))}\,(1-G(u))^{-\big(\Psi(-(1+n+l-k))-\Psi(-(1+i+n-m))\big)}}{l+m-k-i} \Bigg].
\end{aligned}$$

In particular, for $n \geq m > k \geq 0$ we obtain the updated portfolio-loss distribution, conditional on the number of defaults that have been observed up to time u, via

$$\mathbb{P}\big(n\, L_t^n \geq m \,\big|\, n\, L_u^n = k\big) = \frac{\mathbb{P}\big(n\, L_t^n \geq m,\, n\, L_u^n \geq k\big) - \mathbb{P}\big(n\, L_t^n \geq m,\, n\, L_u^n \geq k+1\big)}{\mathbb{P}\big(n\, L_u^n = k\big)},$$

where we compute the terms on the right-hand side as shown above and in Theorem 5.1. Moreover, for each $l = 1, \ldots, n$ note that $\{n\, L_t^n \geq l,\, n\, L_u^n \geq l\} = \{n\, L_u^n \geq l\}$ almost surely.

Proof
Using the joint density of the pair of order statistics $(E_{(k)}, E_{(m)})$*, compare [8] page* 12*, we compute*

$$\begin{aligned}
&\mathbb{P}\big(n\, L_t^n \geq m, n\, L_u^n \geq k\big) \\
&= \mathbb{P}\big(E_{(k)} \leq \Lambda_{h(u)},\, E_{(m)} \leq \Lambda_{h(t)}\big) \\
&= \frac{n!}{(k-1)!\,(m-1-k)!\,(n-m)!}\,\mathbb{E}\Big[\int_0^{\Lambda_{h(u)}} \int_{x_k}^{\Lambda_{h(t)}} e^{-x_k - x_m} \\
&\quad \times (1-e^{-x_k})^{k-1}\,(e^{-x_k} - e^{-x_m})^{m-1-k}\, e^{-(n-m)\,x_m}\, dx_m\, dx_k\Big] \\
&= \frac{n!}{(k-1)!\,(m-1-k)!\,(n-m)!} \sum_{i=0}^{m-1-k} \binom{m-1-k}{i} \\
&\quad \times \sum_{l=0}^{k-1} \binom{k-1}{l} \frac{(-1)^{i+l}}{(1+i+n-m)}\,\mathbb{E}\Big[\frac{1}{1+n+l-k} \\
&\quad \times \Big(1 - e^{-\Lambda_{h(u)}\,(l+n+1-k)}\Big) - \frac{1}{l+m-k-i}\Big(e^{-\Lambda_{h(t)}\,(1+i+n-m)}\Big) \\
&\quad + \frac{1}{l+m-k-i}\Big(e^{-\Lambda_{h(t)}\,(1+i+n-m) - \Lambda_{h(u)}\,(l+m-k-i)}\Big)\Big].
\end{aligned}$$

Given the last equality, the claim follows easily using the Lévy properties of Λ.

Since for $A, B, C \in \mathcal{F}$ *with* $A \subset B$ *we have that* $C \cap (B \setminus A) = (C \cap B) \setminus (C \cap A)$*, the second claim follows by setting* $C = \{n\, L_t^n \geq m\}$*,* $A = \{n\, L_u^n \geq k+1\}$*, and* $B = \{n\, L_u^n \geq k\}$*. We obtain*

$$\begin{aligned}
\mathbb{P}\big(n\, L_t^n \geq m \,\big|\, n\, L_u^n = k\big) &= \mathbb{P}(C \,|\, B \setminus A) = \frac{\mathbb{P}(C \cap (B \setminus A))}{\mathbb{P}(B \setminus A)} = \frac{\mathbb{P}(C \cap B) - \mathbb{P}(C \cap A)}{\mathbb{P}(B \setminus A)} \\
&= \frac{\mathbb{P}\big(n\, L_t^n \geq m,\, n\, L_u^n \geq k\big) - \mathbb{P}\big(n\, L_t^n \geq m,\, n\, L_u^n \geq k+1\big)}{\mathbb{P}\big(n\, L_u^n = k\big)}. \quad \square
\end{aligned}$$

Remark 5.3 (Markov-Property of the Model)
Note that the formula in Theorem 5.2 can be simplified massively using the following consideration. Let $N^1, \dots, N^n$ *be independent Poisson processes with unit intensity, independent of* Λ*. Then, clearly the process* $N_{\Lambda_{h(t)}} := \big(N^1_{\Lambda_{h(t)}}, \dots, N^n_{\Lambda_{h(t)}}\big)$ *is an additive process, i.e. it has independent increments. Note that* $N_{\Lambda_{h(t)}}$ *is a Lévy process if* $h(t) = \lambda\, t$ *for some* $\lambda > 0$*, that is if the marginal default probability is given by the exponential distribution. If we denote by* $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$ *the natural filtration of* $N_{\Lambda_{h(t)}}$*, then* $\mathcal{F}_t$ *contains the information about how many defaults have occured up to time t*[3]*. Thus, we can conclude that for* $0 \leq k \leq m \leq n$ *and* $u \leq t$ *it holds that*

$$\mathbb{P}\Big(n\, L_t^n = m \,\Big|\, n\, L_u^n = k\Big) = \mathbb{P}\Big((n-k)\, \tilde{L}_{t-u}^{n-k} = m-k\Big),$$

[3]Note that $\tau^i \leq t$ if and only if $N^i_{\Lambda_{h(t)}} \geq 1$.

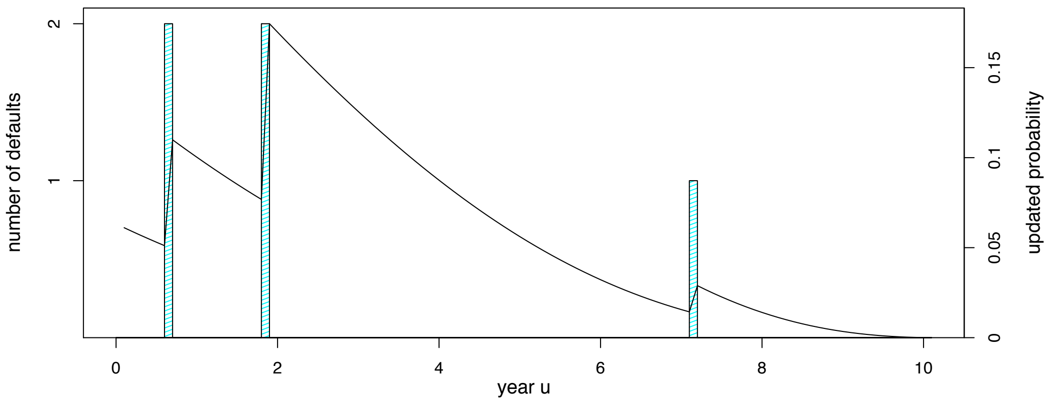

Figure 2 This figure illustrates a path of $\{\mathbb{P}(n\,L_T^n \geq k\,|\,L_u^n)\}_{u\in[0,T]}$, where $n = 20$, $k = 10$, and $T = 10$. The bars in the plot represent default events. We observe that these events cause upward jumps in the probability of at least k defaults at time T.

where $\tilde{L}$ is an independent copy of L, the single difference being that the cumulative hazard function h which occurs in the definition of L is replaced by the function $\tilde{h}(t) = h(t+u) - h(u)$. Thus, the issue of updating values of credit derivatives when they are already on the run may be put down to the formula of Theorem 5.1.

6 Applications

Given a portfolio of credit-risky assets, the presented approach allows to introduce dependence to the individual default times. These default times, combined with the recovery rate, define the portfolio-loss process which is the relevant quantity for most risk-management purposes, e.g. if quantiles of the portfolio-loss need to be computed, and for the pricing of derivatives on credit portfolios.

The analytical formula of the distribution of the portfolio-loss process, obtained under a homogeneous portfolio assumption, makes the model attractive for practical applications. However, let us stress that even if one is not willing to assume a homogeneous portfolio, the model might be solved via a fast Monte Carlo simulation, which only requires the simulation of one Lévy subordinator and n trigger variables in each run. In the following section we exemplarily treat the pricing of CDO tranches to demonstrate the applicability as well as the model's fitting capability. Choices for Λ include a compound Poisson, a Gamma (Γ), and an Inverse Gaussian (IG) time-change. However, other choices of Λ may similarly be implemented and might further improve the calibration results.

6.1 The tranches of a Collateralized Debt Obligation (CDO)

A portfolio-CDS is a swap contract in which the insurance buyer exchanges periodic premium payments against default compensations of the insurance seller resulting from losses in some credit portfolio. CDOs are constructed by partitioning this credit portfolio in tranches with different seniority. A tranche of a CDO represents a certain loss piece of the overall portfolio which is defined via its lower and upper attachment points. The insurance seller receives periodic premium payments depending on the remaining nominal and the spread of this tranche. In return, the insurance buyer is compensated for losses affecting this tranche. Pricing of a tranche corresponds to assessing the spread such that the expected discounted payment streams of this tranche agree.

We fix a payment schedule $\mathcal{T} = \{t_0 = 0 < t_1 < \cdots < t_M = T\}$ and assume a constant recovery rate $R = 40\%$ for all companies. Based on the overall portfolio-loss $(1 - R)L_t^n$ up to time t, the loss affecting tranche j of a CDO up to time t is given by

$$L_{t,j} = \min\big(\max\big(0, (1-R)\, L_t^n - l_j\big), u_j - l_j\big),$$

where l_j and u_j denote the lower and upper attachment points of tranche j. The remaining nominal of the portfolio at time t is given by $Nom_t = 1 - L_t^n$, of tranche j by $Nom_{t,j} = u_j - l_j - L_{t,j}$. Given the payment schedule $\mathcal{T}$, the annualized portfolio-CDS spread s^{CDS} (quoted in bp), and the discount factors d_{t_k} corresponding to the time points t_k, the expected discounted premium and default legs of the portfolio-CDS are given by

$$EDPL = \mathbb{E}\left[\sum_{k=1}^{M} d_{t_k}\, s^{CDS}\, \Delta t_k \left(Nom_{t_k} + \frac{Nom_{t_{k-1}} - Nom_{t_k}}{2}\right)\right],$$

$$EDDL = \mathbb{E}\left[\sum_{k=1}^{M} d_{t_k}\, (1-R)\, (L_{t_k}^n - L_{t_{k-1}}^n)\right],$$

where $\Delta t_k = t_k - t_{k-1}$. For tranche j, the corresponding legs are given by

$$EDPL_j = \mathbb{E}\left[\sum_{k=1}^{M} d_{t_k}\, s^j\, \Delta t_k \left(Nom_{t_k,j} + \frac{Nom_{t_{k-1},j} - Nom_{t_k,j}}{2}\right)\right],$$

$$EDDL_j = \mathbb{E}\left[\sum_{k=1}^{M} d_{t_k}\, (L_{t_k,j}^n - L_{t_{k-1},j}^n)\right].$$

Accrued interest is considered by assuming companies to default at the midpoint of two payment dates. The fair spreads s^{CDS} and s^j of the portfolio CDS and the tranches of the CDO, respectively, are computed by equating the respective expected discounted premium and default leg and solving for the spread. It became market standard to assume a running spread of 500 bp plus an *upfront payment* (*up*), quoted as a percentage of the nominal, for the equity tranche.

6.2 Compound Poisson subordinators

Specifying the subordinating Lévy process Λ to be of compound Poisson type, the Lévy measure ν has the form $\nu(dy) = \beta \int_{\{dy\}} dD(t)$, where D is a cumulative distribution function on the positive axis. This choice of Lévy measure corresponds to the subordinator $\Lambda_t = \mu\, t + \sum_{k=1}^{N_t} J_k$, where $\{J_k\}_{k\in\mathbb{N}}$ are i.i.d. random variables with distribution given by D and N is a Poisson process with intensity β which is independent of $\{J_k\}_{k\in\mathbb{N}}$. That means the subordinator Λ has upward jumps of magnitude J_k and the expected number of jumps within a unit time interval is β. In order for Λ to satisfy (TN) for the unit exponential law, the parameters β and D have to satisfy the relation

$$\beta \leq \frac{1}{1-\mathbb{E}[e^{-J_1}]}. \tag{4}$$

After β and D are specified according to Condition (4), the drift μ of the Lévy subordinator Λ is defined as $\mu = 1 - \beta\,(1 - \mathbb{E}[\exp(-J_1)])$. Using the large portfolio approximation from Theorem 5.1 (b), for the pricing of CDO contracts one needs to compute the expected loss of tranche $[l, u]$, i.e.

$$\mathbb{E}\Big[g\big((1-R)\,(1-e^{-\Lambda_{h(t)}})\big)\Big], \quad g(x) = \min\big(\max\big(x-l, 0\big), u-l\big), \quad 0 \leq l < u \leq 1. \tag{5}$$

In the current setup of a compound Poisson subordinator, this expectation is explicitly given by

$$\sum_{k=0}^{\infty} \mathbb{P}(N_{h(t)} = k)\, \mathbb{E}\Big[g\big((1-R)\,(1-e^{-\Lambda_{h(t)}})\big)\,\Big|\, N_{h(t)} = k\Big].$$

For the inner expectation in the case $k \geq 1$ we obtain the expression

$$\int_0^{u-l} \Big(1 - D^{*k}\Big(-\log\big(1 - \frac{x+l}{1-R}\big) - \mu\, h(t)\Big)\Big)\, dx,$$

where D^{*k} denotes the k-th convolution of the distribution D. This means that a distribution function D implies a tractable model if the k-th convolution is known explicitly.

Before we further specify the jump-distribution D, we proof a lemma which shows that the time normalization (4) is not restrictive, i.e. we can achieve strong dependence in this model framework. In particular, the dependence coefficient α_ν may take any value in $[0, 1)$.

Lemma 6.1 (Admissible Parameters)
(β, D) are called admissible if Condition (4) holds.

1. *All $\beta \in (0, 1]$ are admissible, independent of the choice of D. In particular, we may choose D such that $\mathbb{E}[J_1]$ is arbitrarily large and are still able to find an intensity β such that (β, D) is admissible.*

2. *Let* $\{D_n\}_{n\in\mathbb{N}}$ *be a sequence of jump distribution functions such that* $\mathbb{E}[J_1]$ *tends to zero as* n *tends to infinity. Then, we have that* $(1-\mathbb{E}[\exp(-J_1)])^{-1}$ *tends to infinity as well. That means the admissible range for* β *tends to* $(0,\infty)$.

Proof
Since the exponential function is convex, it follows from Jensen's inequality that

$$\frac{1}{1-\mathbb{E}[e^{-J_1}]} \geq \frac{1}{1-e^{-\mathbb{E}[J_1]}} \geq 1.$$

From this, both results follow immediately. □

Example 6.2
Let us briefly demonstrate that the time normalization allows α_ν *to take any value in* $[0,1)$. *For example, let* $\Lambda_t = \mu\, t + J\, N_t$, *where* $J \geq 0$ *and* $N = \{N_t\}_{t\geq 0}$ *is a Poisson process with intensity* $\beta > 0$, *which corresponds to* $D(t) = \mathbb{1}_{\{J\leq t\}}$. *For a given jump size* J, *Condition (4) restricts the intensity by* $\beta \leq 1/(1-\exp(-J))$. *Note that if* $J=0$, β *is arbitrary. Then, the drift* μ *is determined by* $\mu = 1-\beta\,(1-\exp(-J))$ *and the dependence coefficient* α_ν *satisfies*

$$0 \leq \alpha_\nu = \beta\left(1-e^{-J}\right)^2 \leq 1-e^{-J} \leq 1.$$

Both bounds in the above inequality are sharp in the following sense: Given any value α *in* $[0,1)$, *there exists an admissible pair of parameters* (J,β) *such that the corresponding coefficient* α_ν *equals* α. *This shows that the time normalization (TN) is only a technical issue and not a restriction on the achievable dependence structures.*

When fitting our model to market quotes we found it convenient to specify D as the exponential distribution with parameter $\eta > 0$. Thus, the dependence within the presented framework is determined by the two parameters (η,β). Note that μ is indirectly specified via $\Psi(-1) = -1$. Let us further remark that other distributions on the positive axis may alternatively be used to model the jumps which might further improve the calibration results.

6.3 Infinite acitvity subordinators

Besides compound Poisson subordinators, we also apply infinite activity subordinators to our framework. Processes of this class jump infinitely often within a unit interval of time. Well-known examples of this class are the *Gamma* (Γ) *subordinator* and the *Inverse Gaussian (IG) subordinator*. Their popularity is due to the fact that their Lévy measure as well as the density of their underlying infinitely divisible distribution are known explicitly. Let $\Lambda^\Gamma = \{\Lambda^\Gamma_t\}_{t\geq 0}$ and $\Lambda^{IG} = \{\Lambda^{IG}_t\}_{t\geq 0}$ be a Gamma subordinator, respectively an IG subordinator, with parameters η, $\beta > 0$. According to [23], page 52 ff, Λ_t^Γ follows a $\Gamma(\beta\, t, \eta)$-distribution with density

$$f_\Gamma(x) = \frac{\eta^{\beta\, t}}{\Gamma(\beta\, t)} x^{\beta\, t-1}\, e^{-x\,\eta}\, \mathbb{1}_{\{x\geq 0\}},$$

and Λ_t^{IG} follows a $IG(\beta\, t, \eta)$-distribution with density

$$f_{IG}(x) = \frac{\beta\, t}{\sqrt{2\,\pi}} x^{-\frac{3}{2}}\, e^{\eta\,\beta\, t}\, e^{-\frac{1}{2}(\beta^2\, t^2/x+\eta^2\, x)}\, \mathbb{1}_{\{x\geq 0\}}.$$

The corresponding Lévy measures are given as

$$\nu_\Gamma(dx) = \beta\, e^{-\eta\, x}\, \frac{1}{x}\, \mathbb{1}_{\{x>0\}}\, dx, \quad \nu_{IG}(dx) = \frac{1}{\sqrt{2\,\pi}}\, \beta\, x^{-\frac{3}{2}}\, e^{-\frac{1}{2}\,\eta^2\, x}\, \mathbb{1}_{\{x>0\}}\, dx.$$

For the calibration, we specify the subordinator Λ as $\Lambda_t = \mu\, t + \Lambda_t^i$, for $i \in \{\Gamma,\, IG\}$. Note that μ is indirectly specified by the Lévy measure of the subordinator via $\Psi(-1) = -1$. Hence, the dependence structure is determined by the pair of parameters (η, β). Moreover, note that the required expectations (5) can be computed efficiently using numerical integration, since the densities are known in closed form.

6.4 The calibration to iTraxx quotes

The payment streams for the calibration are in concordance with the iTraxx conventions, i.e. quarter-yearly premium payments and attachment points $[0\%, 3\%]$, $[3\%, 6\%]$, $[6\%, 9\%]$, $[9\%, 12\%]$, and $[12\%, 22\%]$ for the tranches. Further required input are discount factors, these were obtained from risk-free par yields. Market quotes, to which we calibrate our model, comprise the portfolio-CDS spreads with maturities three and five years, the upfront payment for the first tranche, and spreads for the remaining tranches; all tranche spreads for contracts maturing in five years. We use one week of daily data from the seventh series of iTraxx Europe ranging from June 20, 2007, to June 26, 2007, and run a calibration for each of these days.

Due to the seperation of dependence structure and marginal default probabilities, we can proceed in two steps. At first, we calibrate the marginal distribution G, for which we assume a piecewise linear intensity. To be precise, we let

$$1 - G(t) = e^{-\int_0^t \lambda(s)\, ds}, \quad \lambda(s) = \lambda_3\, \min\{s, 3\} + \lambda_5\, (s-3)\, \mathbb{1}_{\{s>3\}},$$

where λ_3 and λ_5 are positive intensity parameters which we calibrate to the portfolio-CDS spreads for a three- and a five-year contract, respectively. This calibration is done via two succeeding bisection procedures. Secondly, we fix λ_3 and λ_5 and calibrate the parameters of the subordinator (η, β), specifying the dependence, to observed market spreads of the tranches of the CDO. For this, we define a grid for η. Given η, the equality

$\Psi(-1) = -1$ defines an interval for β on which the required time normalization holds. On this interval, we choose β such that the observed upfront payment is matched perfectly by our model; if possible. This is done by a bisection procedure again. This procedure is guaranteed to find at most one solution, as the upfront payment is monotone in β. After completion of these steps, among the obtained parameter pairs that perfectly explain the upfront payment we choose (η, β) to be the minimizer of the sum of absolute deviations of market to model spreads over all tranches $j \geq 2$. Due to the closed-form expression of the portfolio-loss distribution, the whole calibration requires only few seconds on a Mac iBook G4 with 1.2 GHz for all choices of Λ. The results of the calibration are shown in Table 1.

Table 1 Fitted CDO prices (5 years).

Model(η, β)	(λ_3, λ_5)	CDS 3y, 5y	up	s^2	s^3	s^4	s^5	E_a	α_ν
Market 6-20-07		$(11.50, 21.60)$	7.13	47.00	12.30	5.60	2.10		
EXP$(10.28, 2.48)$	$(1.31, 1.62)$	$(11.50, 21.60)$	7.13	47.00	26.85	14.86	4.31	25.94	3.6
$\Gamma(5.48, 1.57)$	$(1.31, 1.62)$	$(11.50, 21.60)$	7.13	47.00	24.02	13.49	4.68	22.10	3.8
IG$(2.59, 1.00)$	$(1.31, 1.62)$	$(11.50, 21.60)$	7.13	47.00	22.13	12.37	4.66	19.07	3.9
Market 6-21-07		$(12.59, 22.78)$	8.48	50.47	13.28	6.11	2.43		
EXP$(9.72, 2.42)$	$(1.43, 1.68)$	$(12.59, 22.78)$	8.48	50.48	29.73	17.00	5.23	30.13	3.9
$\Gamma(5.05, 1.48)$	$(1.43, 1.68)$	$(12.59, 22.78)$	8.48	50.48	26.56	15.33	5.61	25.68	4.1
IG$(2.44, 0.94)$	$(1.43, 1.68)$	$(12.59, 22.78)$	8.48	50.28	24.40	14.03	5.58	22.37	4.3
Market 6-22-07		$(13.00, 23.36)$	9.65	55.08	14.49	6.68	2.69		
EXP$(11.93, 2.93)$	$(1.48, 1.71)$	$(13.00, 23.36)$	9.65	55.10	28.87	14.60	3.61	23.25	3.3
$\Gamma(6.57, 1.97)$	$(1.48, 1.71)$	$(13.00, 23.36)$	9.65	55.09	26.12	13.62	4.16	20.06	3.5
IG$(2.88, 1.19)$	$(1.48, 1.71)$	$(13.00, 23.36)$	9.65	55.07	24.30	12.78	4.34	17.57	3.7
Market 6-25-07		$(13.56, 24.33)$	10.88	59.00	15.16	6.82	2.90		
EXP$(12.12, 3.03)$	$(1.54, 1.78)$	$(13.56, 24.33)$	10.88	58.97	30.63	15.34	3.74	24.83	3.3
$\Gamma(6.66, 2.03)$	$(1.54, 1.78)$	$(13.56, 24.33)$	10.88	59.00	27.81	14.39	4.35	21.68	3.5
IG$(2.92, 1.23)$	$(1.54, 1.78)$	$(13.56, 24.33)$	10.88	58.96	25.73	13.40	4.47	18.71	3.6
Market 6-26-07		$(13.70, 24.12)$	11.87	63.70	16.26	7.35	3.18		
EXP$(22.25, 5.87)$	$(1.56, 1.75)$	$(13.70, 24.12)$	11.87	63.70	19.84	5.79	0.62	7.70	2.1
$\Gamma(14.65, 5.48)$	$(1.56, 1.75)$	$(13.70, 24.12)$	11.87	63.70	18.65	5.95	0.82	6.14	2.2
IG$(4.64, 2.72)$	$(1.56, 1.75)$	$(13.70, 24.12)$	11.87	63.70	17.96	6.05	0.99	5.18	2.4

Table 1 contains the market quotes, the calibrated marginal probability parameters (λ_3, λ_5) (in promille), the dependence parameters (η, β) for the corresponding model, the upfront payment (in percent), and the corresponding tranche spreads (in bp). Moreover, the absolute deviation of model spreads to market spreads E_a (in bp) and the implied dependence parameter α_ν (in percent) are given. We observe that minimizing the sum of absolute errors results in the fact that the second tranche is also matched each time. The reason for this phenomenon might be that the spread of this tranche is much higher than those of tranches 3, 4, and 5. Furthermore, the large clipping from tranche spread 2 to tranche spread 3, as observed in the market, causes most fitting problems. This decrease from tranche 2 to tranche 3 seems to be hard to capture for other well-known models as well, see e.g. the calibration results in [17], [1], and [11]. Overall, we conclude that the calibration results are satisfying for the considered period. Still, we point out that other choices for the marginal distributions or the time-change may further improve the calibration results, which will be subject of further investigations.

7 Conclusion

We constructed a model for a portfolio of credit-risky assets using a conditional independence approach with a stochastic time-change as common factor. Our main objective was to keep the parameters of the dependence structure separated from the parameters of the marginal default probabilities. This was achieved by choosing a suitable Lévy subordinator as stochastic clock, satisfying a so-called time normalization. This time normalization was shown to be intuitive and not too restricting to allow for a variety of different dependence patterns. In particular, we showed that the complete range from independence up to the upper Fréchet bound is admissible for the model.

Due to the separation of univariate marginals and dependence structure, the implied copula of the default times could be computed explicitly, constituting an appealing link between copulas and Lévy subordinators. For a pair of firms the induced copula was found to be of Cuadras-Augé type, which enabled us to further investigate the effect of the model parameters on the firms' dependence. In particular, the parameter of upper-tail dependence of the induced bivariate Cuadras-Augé copula may be considered as an intrinsic dependence measure in the presented framework. All these results were obtained independently of the choice of marginal distribution of the individual default times, which could be chosen quite arbitrarily. Interesting for credit-risk applications is the singular component of the copula. In other words, the model naturally supports joint defaults and a contagion effect.

Under the homogeneous-portfolio assumption, the model was introduced as a dynamic extension of Exchangeable Bernoulli Mixture Models to the case when the mixing variable is time-dependent. In this framework we were able to derive exact formulas for the portfolio-loss distribution. Moreover, we derived the loss distribution conditional on the information about earlier defaults which makes it possible to update the value of credit derivatives when they are already on the run and defaults are observed. In particular, it was illustrated that the model exhibits a contagion effect.

Finally, an approximation for large portfolios was presented and the model's efficiency was demonstrated by calibrating it to observed CDO spreads. To this end, the Lévy subordinator was specified in three different ways, as compound Poisson, Gamma, and Inverse Gaussian process. Due to the tractability of the model, the calibration to market quotes is done within few seconds in all cases. The fit to market data exhibited a satisfying performance, and was best in the Inverse Gaussian case.

References

[1] H. Albrecher, S.A. Ladoucette, and W. Schoutens,
A Generic One-Factor Lévy Model for Pricing Synthetic CDOs, in Advances in Mathematical Finance (Applied and Numerical Harmonic Analysis), *Birkhäuser Boston* (2007) pp. 259–278.

[2] D. Applebaum, Lévy processes and Stochastic Calculus, *Cambridge University Press* (2004).

[3] J. Bertoin, Lévy processes, *Cambridge University Press* (1996).

[4] T.R. Bielecki, A. Vidozzi, and L. Vidozzi, Study of Dependence for Some Stochastic Processes, *working paper, available at http://www.iit.edu/ bielecki/Credit-Risk/BJVV.pdf* (2007).

[5] R. Cont and P. Tankov, Financial Modelling With Jump Processes, *Chapman and Hall/CRC Financial Mathematics Series* (2004).

[6] C.M. Cuadras and J. Augé, A continuous general multivariate distribution and its properties, *Communications in Statistics - Theory and Methods* **10:4** (1981) pp. 339–353.

[7] S.R. Das, D. Duffie, N. Kapadia, and L. Saita, Common Failings: How Corporate Defaults are Correlated, *Journal of Finance* **62:1** (2007) pp. 93 – 117.

[8] H.A. David and H.N. Nagaraja, Order Statistics, *Wiley Series in Probability and Statistics* (2003).

[9] D. Duffie, L. Saita, and K. Wang, Multi-Period Corporate Default Prediction with Stochastic Covariates, *The Journal of Financial Economics* Vol. **83**, (2007) pp. 635–665.

[10] R. Frey and A.J. McNeil, Modelling Dependent Defaults, *working paper, available at http://e-collection.ethbib.ethz.ch/show?type=bericht&nr=273* (2001).

[11] M. Hofert and M. Scherer, CDO pricing with nested Archimedean copulas, *working paper, available at http://www.mathematik.uni-ulm.de/numerik/preprints/2008/CDOpricingAC.pdf* (2008).

[12] T.R. Hurd, Credit Risk modelling using time-changed Brownian motion, *working paper, available at http://www.math.mcmaster.ca/tom/HurdTCBM.pdf* (2007).

[13] T.R. Hurd and A. Kuznetsov, Affine Markov chain model of multifirm credit migration, *The Journal of Credit Risk* **3:1** (2007) pp. 3–29.

[14] J. Hull and A. White, Valuation of a CDO and an n-th to Default CDS without Monte Carlo Simulation, *Journal of Derivatives* **12:2** (2004) pp. 8–23.

[15] H. Joe, Multivariate Models and Dependence Concepts, *Chapman and Hall/CRC* (1997).

[16] M.S. Joshi and A.M. Stacey, Intensity Gamma: A new approach to Pricing Portfolio Credit Derivatives, *Risk Magazine* (July 2006) pp. 78–83.

[17] A. Kalemanova, B. Schmid, and R. Werner, The normal inverse Gaussian distribution for synthetic CDO pricing, *The Journal of Derivatives* **14:3** (2007) pp. 80–93.

[18] D.X. Li, On Default Correlation: A Copula Function Approach, *The Journal of Fixed Income* **9:4** (2000) pp. 43–54.

[19] A.W. Marshall and I. Olkin, A multivariate exponential distribution, *Journal of the American Statistical Association* **62** (1967) pp. 30–44.

[20] A.J. McNeil, R. Frey, and P. Embrechts, Quantitative Risk Management, *Princeton University Press* (2005).

[21] R.B. Nelsen, An Introduction to Copulas, *Springer-Verlag* (1998).

[22] K.-I. Sato, Lévy Processes and infinitely divisible distributions, *Cambridge University Press* (1999).

[23] W. Schoutens, Lévy Processes in Finance: Pricing Financial Derivatives, *Wiley Series in Probability and Statistics* (2003).

[24] A.J. Stam, Derived Stochastic Processes, *Compositio Mathematica* **17** (1965-1966) pp. 102–140.

[25] O. Vasicek, Probability of Loss on a Loan Portfolio, *Working Paper, KMV Corporation* (1987).

[26] F. Yu, Correlated Defaults in Intensity-based Models, *Mathematical Finance* **17:2** (2007) pp 155–173.