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Recurrence for persistent random walks in two dimensions

Persistent (Newtonian) random walks

Persistent random walk (PRW) in $\mathbb{Z}^\nu$:

2nd order Markov chain on $\mathbb{Z}^\nu$ =
stochastic process $\{X_n\}_{n\in\mathbb{N}}$ with $X_n \in \mathbb{Z}^\nu$ s.t.

$$\begin{aligned}&\mathrm{Prob}\left(X_{n+1} \mid X_n, X_{n-1}, \ldots, X_0\right) = \\ &\mathrm{Prob}\left(X_{n+1} \mid X_n, X_{n-1}\right)\end{aligned}$$

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with $\boxed{D_n := X_n - X_{n-1}}$ = incoming direction $\simeq$ "velocity"
(whence "Newtonian" random walk)

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Will assume $\nu = 2$ and

$$\boxed{D_n \in \Delta := \{\pm e_1, \pm e_2\}}$$

$\Delta =$ fundamental directions $\Longrightarrow$ nearest-neighbor PRW

Environment: $\omega = \{\omega_x\}_{x\in\mathbb{Z}^2} \in (M_\Delta)^{\mathbb{Z}^2} =: \mathcal{E}$ (environment space)

$$\omega_x = \{\omega_x(d,d')\}_{d,d'\in\Delta} \in M_\Delta \overset{\text{def}}{\Longleftrightarrow} \sum_{d'\in\Delta} \omega_x(d,d') = 1 \ \ \forall d \in \Delta$$

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So, formally: PRW = Markov chain P_p^ω on $\mathbb{Z}^2 \times \Delta$ defined by

$$P_p^\omega(\,(X_0, D_0) = (x,d)\,) = p(x,d);$$

$$P_p^\omega(\,(X_{n+1}, D_{n+1}) = (x',d') \mid (X_n, D_n) = (x,d)\,) = \begin{cases} \omega_x(d,d'), & \text{if } x' = x + d'; \\ 0, & \text{otherwise} \end{cases}$$

p = probability on $\mathbb{Z}^2 \times \Delta$ (initial state)

E.g., $p(x,d) = \delta_{x,x_0}\,\delta_{d,d_0}$

Recurrence

Defn (recurrence). *PRW in ω with initial state p is recurrent if*

$$P_p^{\omega}(\,(X_n, D_n) = (X_0, D_0) \text{ for infinitely many } n \in \mathbb{N}\,) = 1.$$

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- (Some) inhomogeneous PRW via dual graph

Dynamical systems and cocycles

Probability-preserving dynamical system: $(\mathcal{S}, T, \mu)$ with

$T : \mathcal{S} \longrightarrow \mathcal{S}$

$\mu(T^{-1}A) = \mu(A)$, $\forall A \subset \mathcal{S}$ $\quad$ (μ T-invariant)

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Defn. $\{S_n\}_{n\in\mathbb{N}}$ *is a ν-dimensional commutative cocycle for* $(\mathcal{S}, T, \mu)$ *if* $S_0 \equiv 0$ *and*

$$S_n := \sum_{k=0}^{n-1} f \circ T^k$$

for some $f : \mathcal{S} \longrightarrow \mathbb{R}^\nu$, $f \in L^2(\mathcal{S}, \mu)$ (vector-valued Birkhoff sum).

If $f : \mathcal{S} \longrightarrow \mathbb{L}$, with $\mathbb{L}$ lattice of $\mathbb{R}^\nu$, cocycle is called discrete.

Defn. $\{S_n\}$ *is* *recurrent* *if,* $\forall \varepsilon > 0$, μ*-a.s.*

$$\|S_n\| \leq \varepsilon \quad \text{for infinitely many } n.$$

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Thm *(Schmidt '98, Conze '99)*. *If* $(\mathcal{S}, T, \mu)$ *is ergodic,* $\{S_n\}$ *is 2D and verifies the centered CLT (even with* ∞ *variance), then* $\{S_n\}$ *is recurrent.*

Application: Define $\sigma : (\mathbb{Z}^2 \times \Delta)^{\mathbb{N}} \longrightarrow (\mathbb{Z}^2 \times \Delta)^{\mathbb{N}}$ as

$$\sigma((X_0, D_0), (X_1, D_1), \dots) := ((X_1, D_1), (X_2, D_2), \dots)$$

(left shift on paths = time evolution). Then

$$\boxed{X_n - X_0 = \sum_{j=1}^{n} D_n = \sum_{k=0}^{n-1} D_1 \circ \sigma^k}$$

would be a discrete 2D cocycle for $\left((\mathbb{Z}^2 \times \Delta)^{\mathbb{N}}, \sigma, P_p^{\omega}\right)$

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$\Longrightarrow$ Must choose suitable dynamical system

Homogeneous PRWs

Propn. *Take homogeneous PRW defined by* $\omega_0 \in M_\Delta$*, irreducible aperiodic (ergodic). CLT holds. CLT is centered (thus recurrence holds)* $\iff$ $\pi =$ *stationary vector of* ω_0 *(*$\sum_d \pi(d)\omega_0(d,d') = \pi(d')$*) is balanced, i.e.*

$$\pi(e_1) = \pi(-e_1), \qquad \pi(e_2) = \pi(-e_2).$$

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Proof: Use $(\Delta^{\mathbb{N}}, \sigma, \mathcal{P}_\pi)$, where

$\sigma(D_0, D_1, \ldots) := (D_1, D_2, \ldots)$ (abuse of notation)

$\mathcal{P}_\pi =$ finite-state Markov chain on Δ with initial state π

$\Longrightarrow \mathcal{P}_\pi$ invariant (since π stationary for ω_0)

Ergodicity and CLT standard Q.E.D.

Random environments

Tóth random environments $(\mathcal{E}, \Pi)$

- Ergodic for the action of $(\tau_y \omega)_x := \omega_{x+y}$ (e.g., $\{\omega_x\}$ i.i.d.)
- Elliptic: $\exists \varepsilon > 0$ s.t. $\forall x, d, d'$, $\omega_x(d, d') \geq \varepsilon$ (can do better)
- Isotropic: Π-a.s., $\omega_x^T \in M_\Delta$ (ω_x doubly stochastic)
  $\Rightarrow$ PRW in ω "invertible" (backward dyn. given by $\omega^T = \{\omega_x^T\}$)
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Proof: Uses $\big((\Delta \times \mathcal{E})^{\mathbb{N}}, \sigma, \mathbb{P}\big)$ (point of view of the particle), then adaptation of *Kipnis-Varadhan '86* for CLT.

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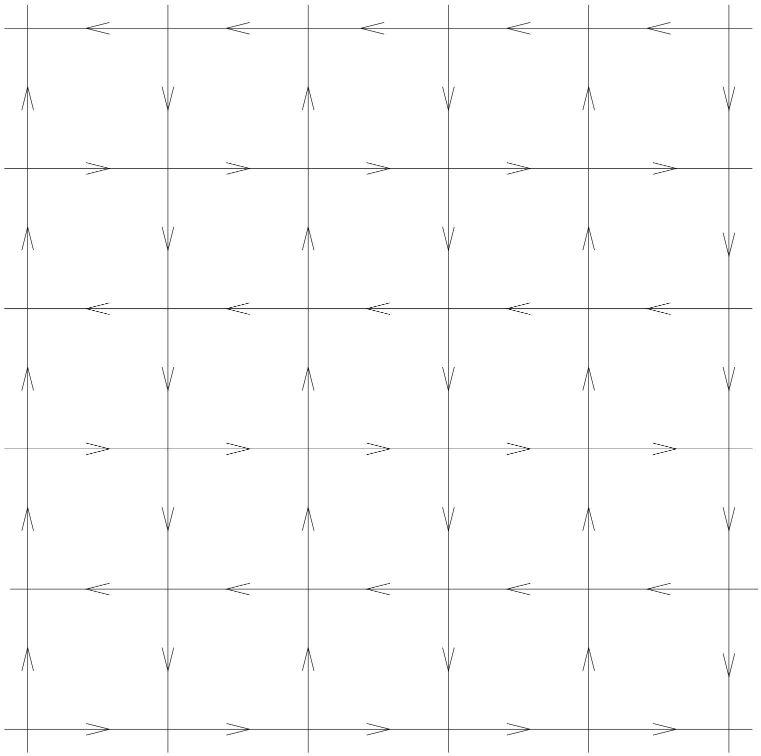
Propn. *Tóth PRWREs are a.s. recurrent.*

Digression: The Manhattan lattice



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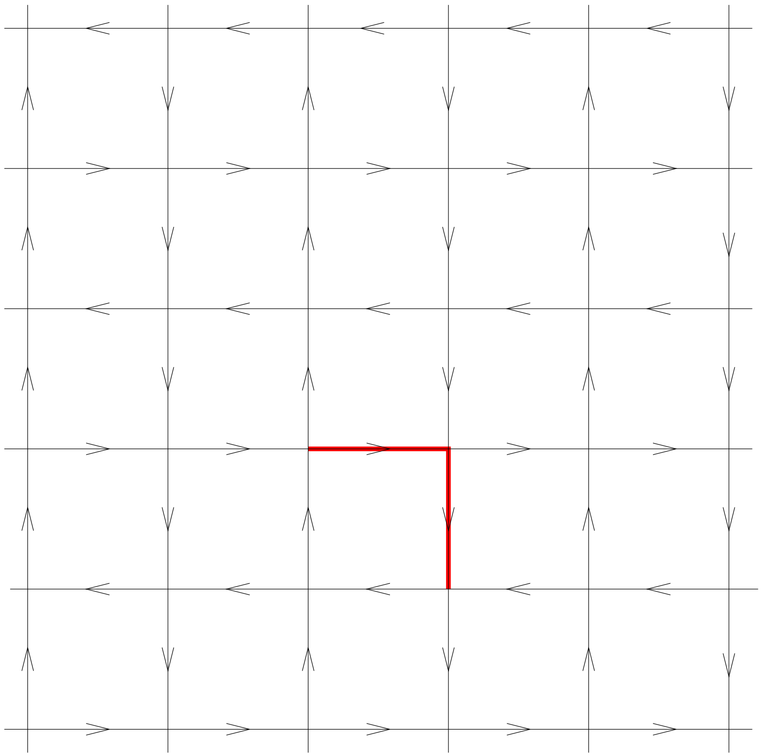

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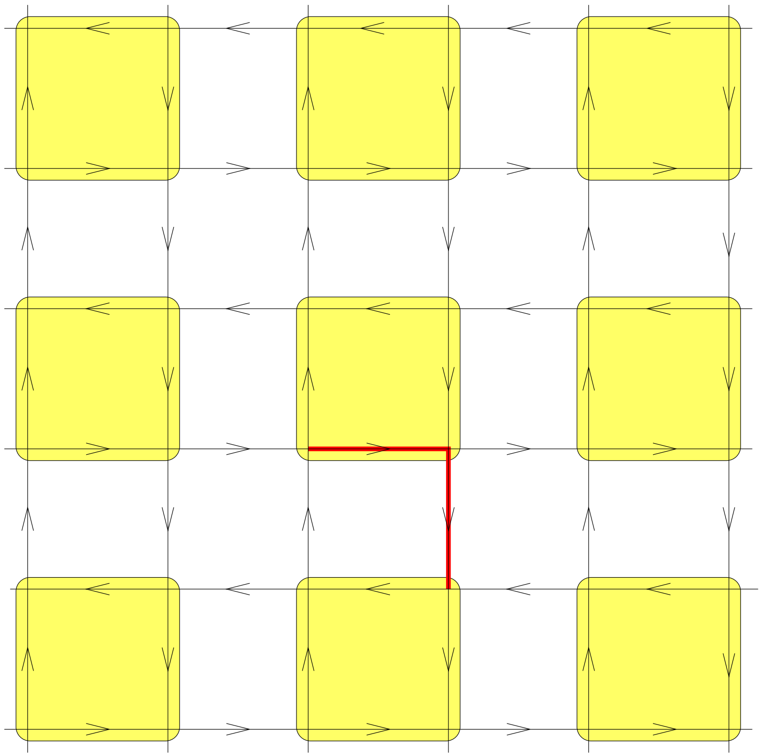

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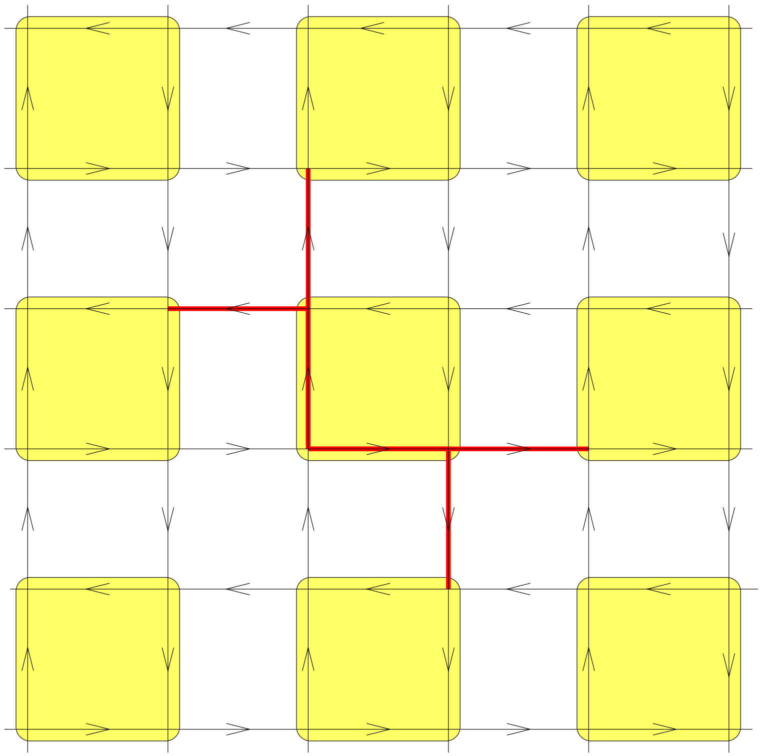
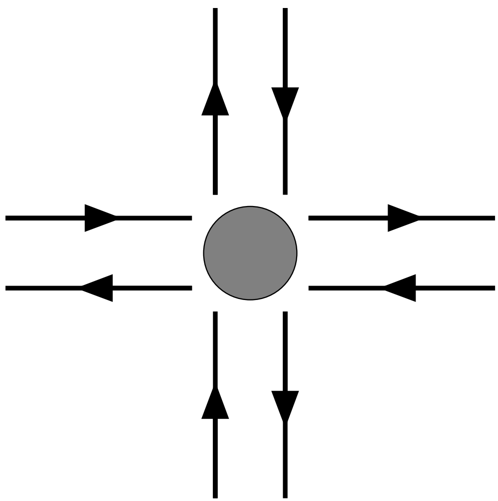


Symmetric RW is recurrent

The dual graph

Goal: *Map 2nd order RW on $\mathbb{Z}^2$ into 1st order RW on some graph Γ*

For $x \in \mathbb{Z}^2$, consider incoming/outgoing displacements:



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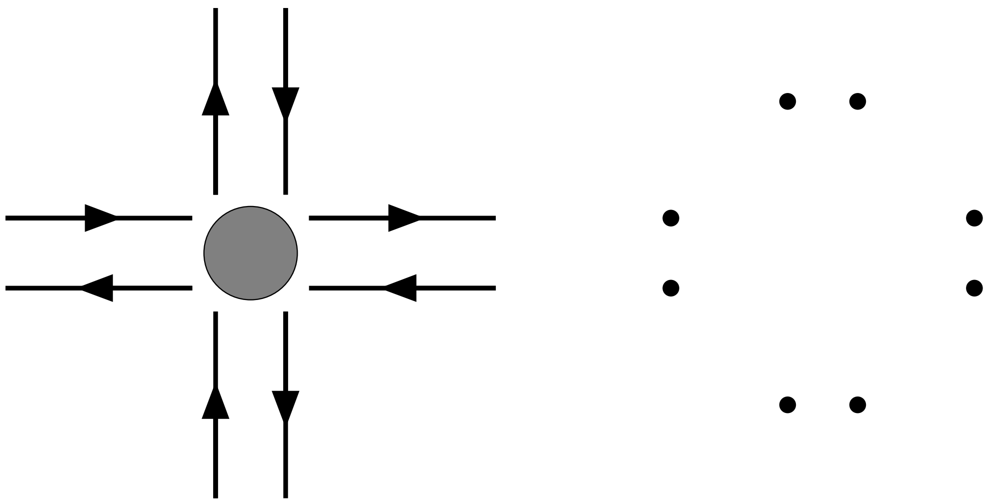

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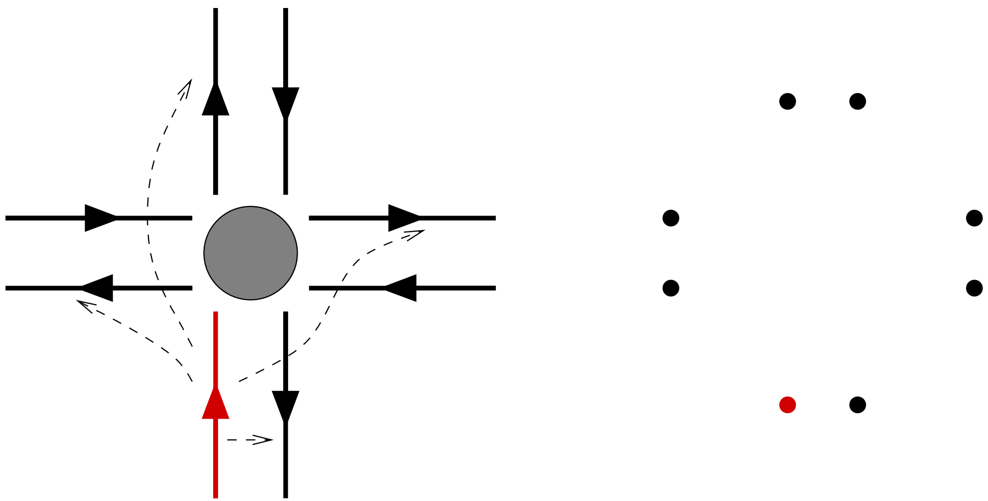

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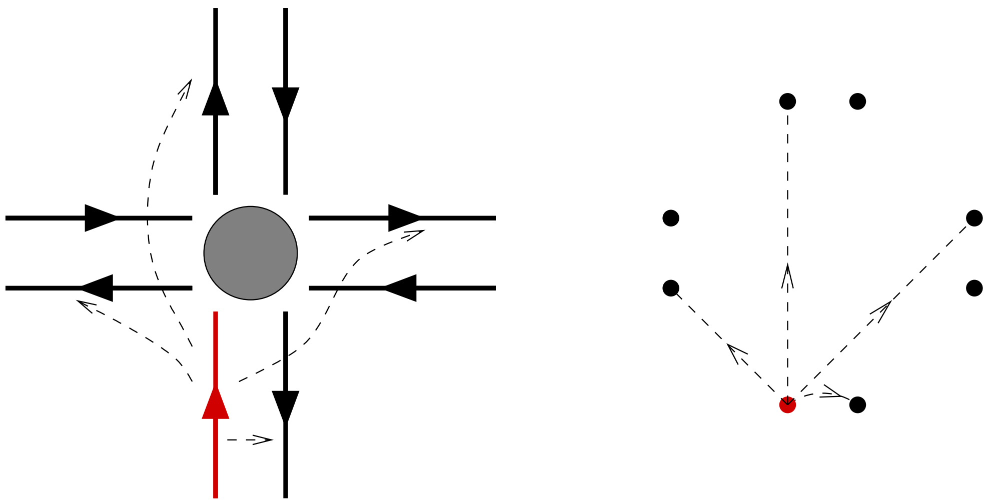
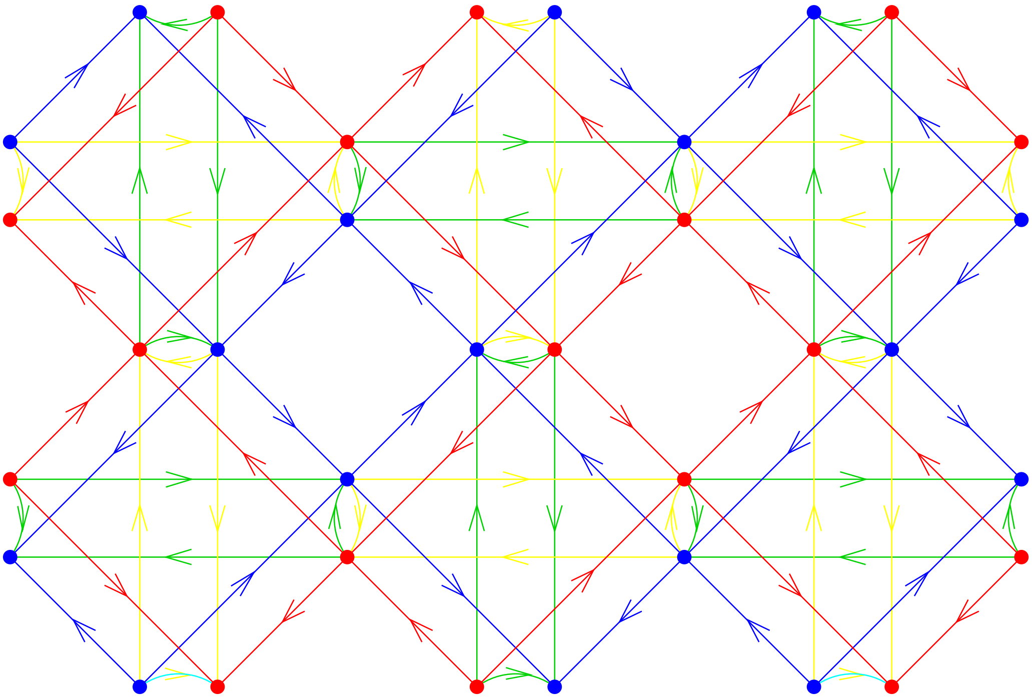


Γ looks like



Two Manhattan lattices (blue and red) with opposite orientations connected by other links (green and yellow)

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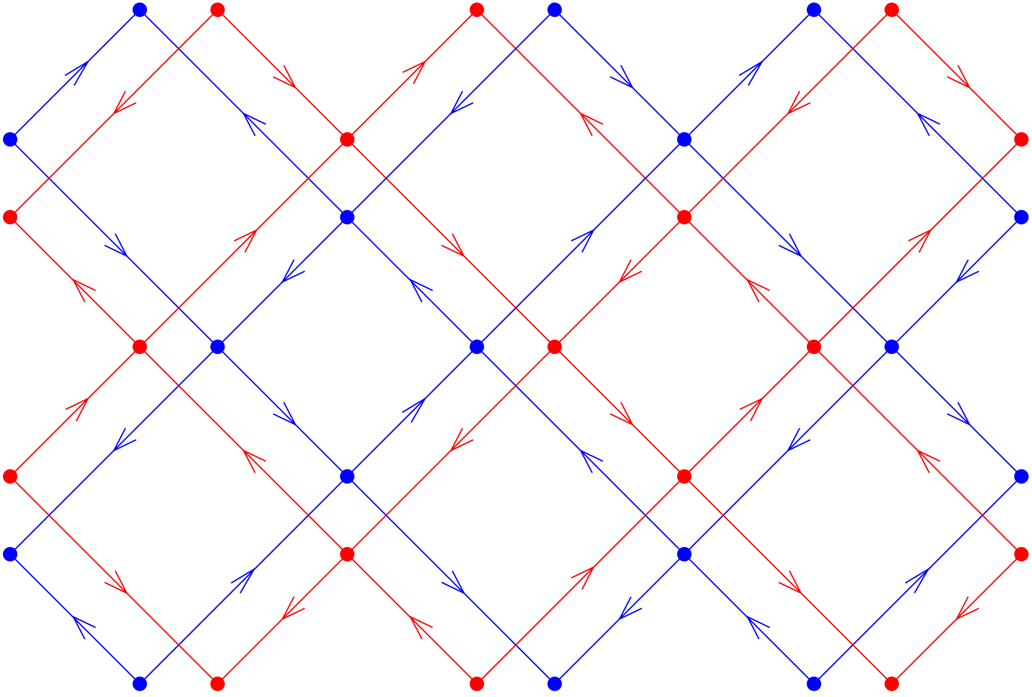

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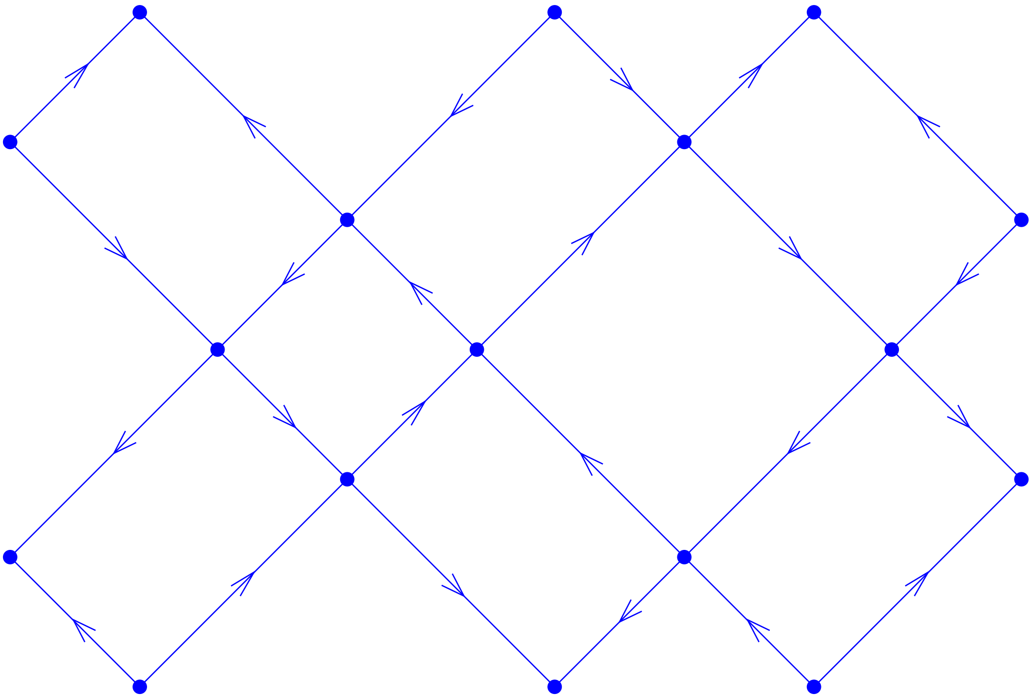

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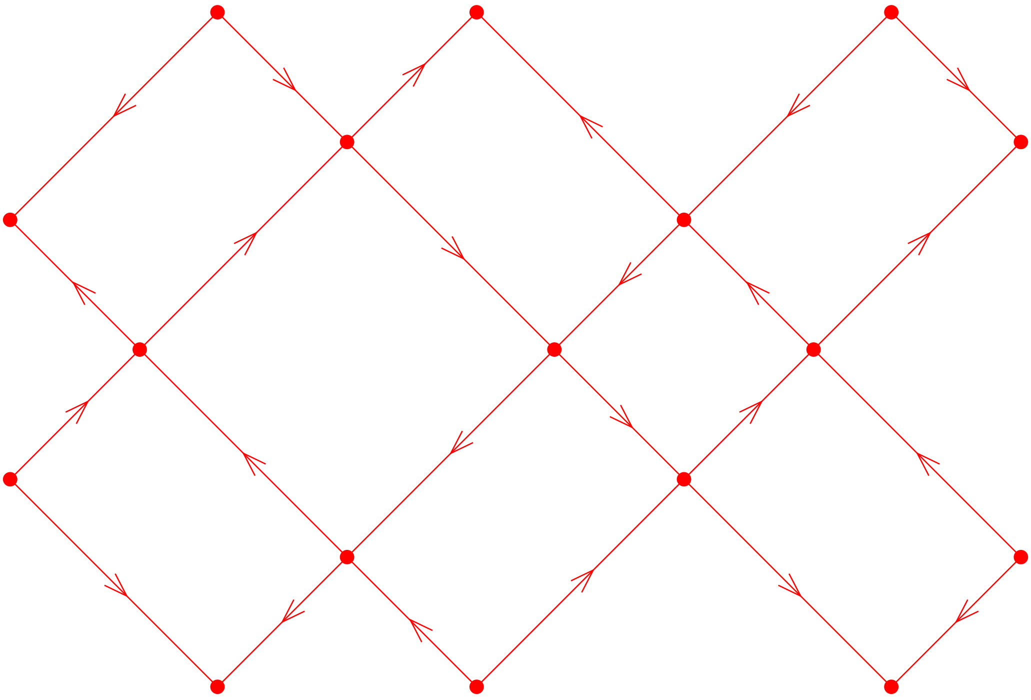

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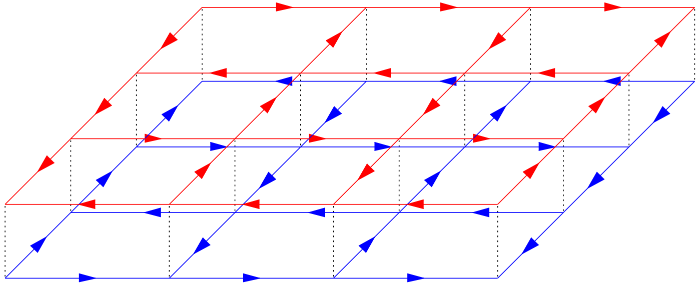
3D rendering:

Blue: lower level
Red: upper level



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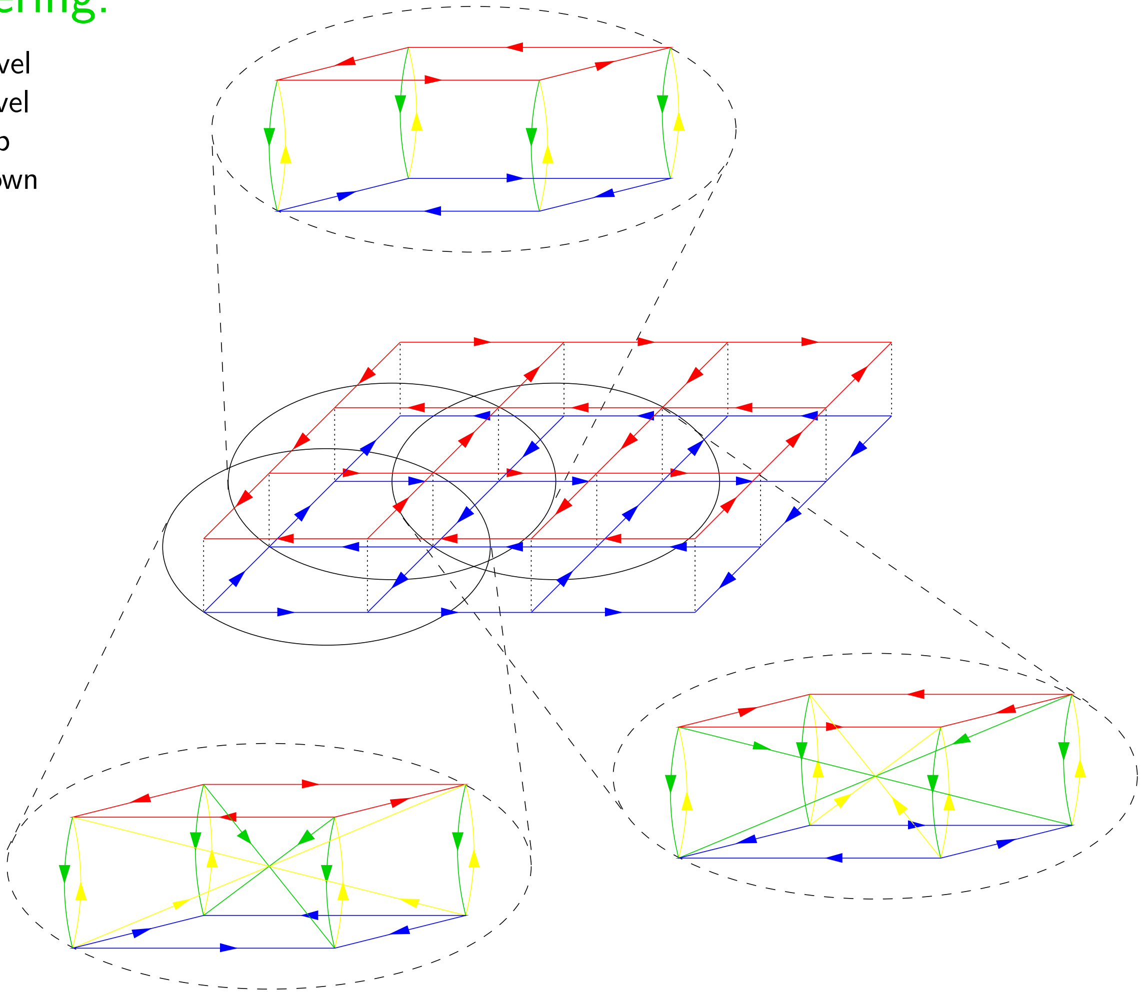
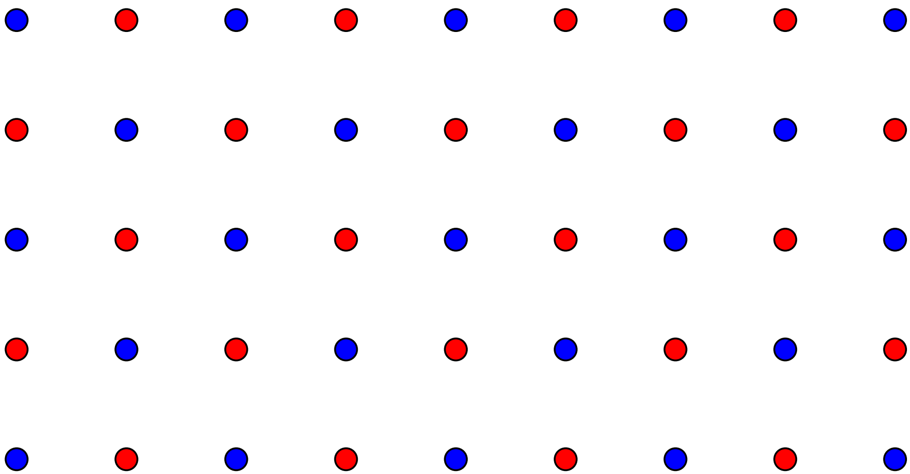


Application: Further examples of recurrence

Set $\mathbb{Z}^2_{\text{even}} := \left\{(x^1, x^2) \in \mathbb{Z}^2 \;\middle|\; x^1 + x^2 \in 2\mathbb{Z}\right\}$, $\mathbb{Z}^2_{\text{odd}} := \mathbb{Z}^2 \setminus \mathbb{Z}^2_{\text{even}}$



$\mathbb{Z}^2_{\text{even}} =$ "checkerboard" subgroup of $\mathbb{Z}^2$

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(1) Inhomogeneous forward probability

For $\zeta_x \in [0, 1]$ (non-random), define $\omega = \{\omega_x\}$ via the following:

For $x \in \mathbb{Z}^2_{\text{even}}$, $d = e_1$,

$$\omega_x(d, \mathbf{F}) = \zeta_x, \ \omega_x(d, \mathbf{L}) = \omega_x(d, \mathbf{R}) = (1 - \zeta_x)/2, \ \omega_x(d, \mathbf{B}) = 0$$

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($\mathbf{F} = \mathbf{F}(d) = d$ (Forward), $\mathbf{B} = \mathbf{B}(d) = -d$ (Backward), $\mathbf{L}$ = Left, $\mathbf{R}$ = Right)

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$\Longrightarrow$ Symmetric Left-Right PRW with extra Forward displacements

Remark. *Forward displacements need not be statistically balanced, namely the local drift* $\delta_\omega(x,d) := \sum_{d' \in \Delta} \omega_x(d,d')d'$ *may not average out to zero:*

$$\lim_{\Lambda \nearrow \mathbb{Z}^2} \frac{1}{4|\Lambda|} \sum_{\substack{x \in \Lambda \\ d \in \Delta}} \delta_\omega(x,d) \text{ can be } \neq 0$$

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Propn 1. *If, for at least one* $x \in \mathbb{Z}^2_{\mathrm{even}}$, $\zeta_x > 0$ *and, for at least one* $y \in \mathbb{Z}^2_{\mathrm{odd}}$, $\zeta_y > 0$, *the PRW in* ω *is recurrent for all initial conditions.*

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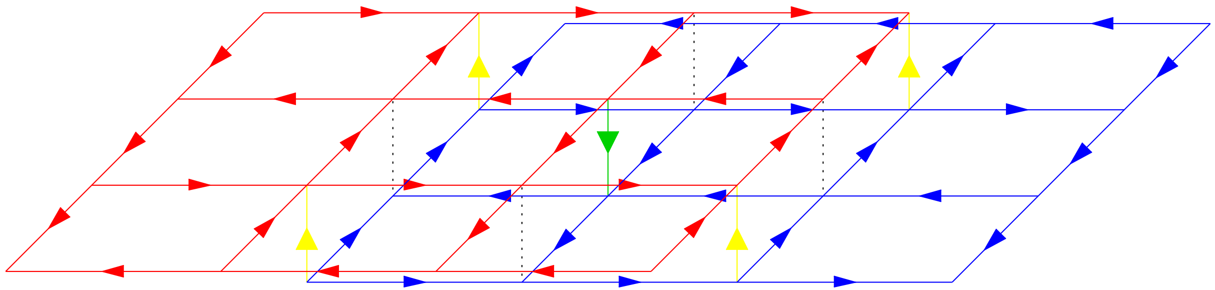
Proof:



(2) Inhomogeneous backward probability

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For $x \in \mathbb{Z}^2_{\text{even}}, \; d = e_1,$

$$\omega_x(d,\mathbf{F}) = 0, \; \omega_x(d,\mathbf{L}) = \omega_x(d,\mathbf{R}) = (1-\zeta_x)/2, \; \omega_x(d,\mathbf{B}) = \zeta_x$$

For $x \in \mathbb{Z}^2_{\text{even}}, \; d \neq e_1,$

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$\Longrightarrow$ Symmetric Left-Right PRW with extra Backward displacements

Once again, Backward displacements need not be statistically balanced

Propn 2. *Suppose that, for at least one* $x \in \mathbb{Z}^2_{\mathrm{even}}$*,* $\zeta_x > 0$ *and, for at least one* $y \in \mathbb{Z}^2_{\mathrm{odd}}$*,* $\zeta_y > 0$*. Suppose also that there is no* $x \in \mathbb{Z}^2_{\mathrm{even}}$ *such that* $\zeta_x = \zeta_{x-e_1} = 1$*. Then the PRW in* ω *is recurrent for all initial conditions.*

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Proof: