Mean-field behaviour of long- and finite range Ising model, percolation and self-avoiding walk

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Joint work with Remco van der Hofstad (Eindhoven) and Akira Sakai (Sapporo).

The Ising model on $\mathbb{Z}^d$

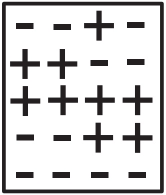
Let $\Lambda \subset\subset \mathbb{Z}^d$. Energy of a configuration $\boldsymbol{\varphi} = \{\varphi_x | x \in \Lambda\} \in \{-1,+1\}^\Lambda$ defined by

$$\mathcal{H}_\Lambda(\boldsymbol{\varphi}) := -\sum_{\{x,y\} \in \Lambda\times\Lambda} J_{\{x,y\}}\, \varphi_x\, \varphi_y .$$

Assume J symmetric and $J \geq 0$ (ferromagnetic interaction)
Example: $J_{\{x,y\}} = \mathbb{1}_{|x-y|=1}$ (nearest-neighbor model).

Probability of configuration $\boldsymbol{\varphi} \in \{-1,+1\}^\Lambda$ is $\mathrm{e}^{-\beta\mathcal{H}_\Lambda(\boldsymbol{\varphi})} / Z_\Lambda(\beta)$.
Two-point correlation function:

$$\langle \varphi_0 \varphi_x \rangle_\Lambda := \frac{\sum_{\boldsymbol{\varphi}\in\{-1,1\}^\Lambda} \varphi_0\, \varphi_x \exp(-\beta\mathcal{H}_\Lambda(\boldsymbol{\varphi}))}{\sum_{\boldsymbol{\varphi}\in\{-1,1\}^\Lambda} \exp(-\beta\mathcal{H}_\Lambda(\boldsymbol{\varphi}))} .$$



Λ

Phase transition: $\exists m, M > 0$:

$$G_\beta(x) := \lim_{\Lambda \nearrow \mathbb{Z}^d} \langle \varphi_0 \varphi_x \rangle_\Lambda \begin{cases} \leq \mathrm{e}^{-m\,|x|} & (\beta < \beta_c) \\ \geq M & (\beta > \beta_c). \end{cases}$$

Aim: Understanding of *critical* behaviour, i.e., $\beta = \beta_c$.

Critical behaviour

Physics concept: Introduce *critical exponents*, e.g.

$$G_{\beta_c}(x) \overset{|x|\nearrow\infty}{\approx} |x|^{-(d-2+\eta)}, \qquad \sum_x G_\beta(x) \overset{\beta\nearrow\beta_c}{\approx} (\beta_c - \beta)^{-\gamma}.$$

Theorem (Aizenman, Barsky, Fernández, Graham ´80s)

If $\sum_x G_{\beta_c}(x)^2 < \infty$ (bubble condition), then $\eta = 0$, $\gamma = 1$, $\delta = 3$, ...

Note: These values for critical exponents coincide with Curie-Weiss model (=Ising model on *complete* graph). Known as *mean-field values*.

Theorem (Fröhlich, Simon, Spencer '76)

For RP-version of J (reflection positivity) an *infrared bound* holds:

$$0 \le \hat{G}_\beta(k) \le \frac{\text{const}}{|k|^2} \qquad (d \ge 2, \beta < \beta_c),$$

i.e., *mean-field behaviour* above 4 dimensions.

The spin-spin coupling J

Recall:

$$G_\beta(x) := \lim_{\Lambda \nearrow \mathbb{Z}^d} \frac{\sum_{\varphi \in \{-1,1\}^\Lambda} \varphi_0 \, \varphi_x \exp(-\beta \mathcal{H}_\Lambda(\varphi))}{\sum_{\varphi \in \{-1,1\}^\Lambda} \exp(-\beta \mathcal{H}_\Lambda(\varphi))},$$

$$\mathcal{H}_\Lambda(\varphi) := -\sum_{\{x,y\} \in \Lambda \times \Lambda} J_{\{x,y\}} \, \varphi_x \, \varphi_y.$$

Question: What is the role of J?

Introduce D by $D(x) = \frac{\tanh(\beta J_{\{0,x\}})}{\sum_{y \in \mathbb{Z}^d} \tanh(\beta J_{\{0,y\}})}$.

Consider three cases:

- nearest-neighbor coupling: $D(x) = \frac{1}{2d} \mathbb{1}_{\{|x|=1\}}$
- finite-variance coupling: e.g., $D(x) \approx e^{-|x/L|}$ as $|x| \to \infty$, L suff. large
- long-range coupling: e.g., $D(x) \approx |x/L|^{-(d+\alpha)}$ as $|x| \to \infty$, $\alpha > 0$, L suff. large

Result

Fourier transform:

$1 - \hat{D}(k) = 1 - \sum_{x \in \mathbb{Z}^d} e^{ik \cdot x} \; D(x) = \sum_{x \in \mathbb{Z}^d} [1 - \cos(k \cdot x)] \, D(x) \asymp |k|^{2 \wedge \alpha}.$

Theorem (HHS 2007)

Consider the Ising model in dimension

$$d \begin{cases} > 4 & \text{(finite-variance case)}, \\ > 2(2 \wedge \alpha) & \text{(power-law case)}, \end{cases}$$

and L sufficiently large. Then

$$\hat{G}_{z_c}(k) = \frac{1 + O(L^{-d})}{1 - \hat{D}(k)} \asymp \left(1 + O(L^{-d})\right) |k|^{-(2 \wedge \alpha)}. \qquad (1)$$

- Finite range models known by Sakai (CMP '07).
- Proof uses *lace expansion*.
- Result holds also for nearest-neighbor model with $O(L^{-d})$ replaced by $O(1/d)$ and d suff. large.

Random Current representation

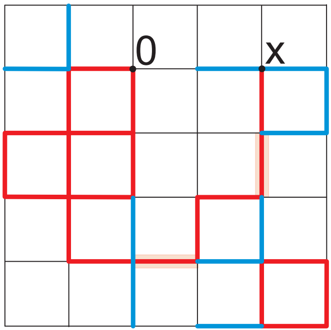
Starting point: RC representation by Griffiths, Hurst, Sherman
Denote $\mathbb{B}_\Lambda = \big\{\{x,y\} \mid x,y \in \Lambda, x \neq y\big\}$ set of *bonds* in Λ.

$$\begin{aligned}
\langle \varphi_0 \varphi_x \rangle_\Lambda &= \frac{2^{-|\Lambda|}}{Z_\Lambda} \sum_{\varphi \in \{\pm 1\}^\Lambda} \varphi_0\, \varphi_x \prod_{\{u,v\} \in \mathbb{B}_\Lambda} \exp(-\beta J_{\{u,v\}}\, \varphi_u\, \varphi_v) \\
&= \frac{2^{-|\Lambda|}}{Z_\Lambda} \sum_{\varphi \in \{\pm 1\}^\Lambda} \varphi_0\, \varphi_x \sum_{\mathbf{n} \in \mathbb{N}^{\mathbb{B}_\Lambda}} \underbrace{\Big(\prod_{b \in \mathbb{B}_\Lambda} \frac{(\beta J_b)^{n_b}}{n_b!} \Big)}_{=: w_\lambda(\mathbf{n})} \prod_{v \in \Lambda} \varphi_v^{\sum_{\bar v \in \Lambda} n_{\{v,\bar v\}}} \\
&= \sum_{\mathbf{n} \in \mathbb{N}^{\mathbb{B}_\Lambda}} \frac{w_\lambda(\mathbf{n})}{Z_\Lambda} \prod_{v \in \Lambda} \left(\frac{1}{2} \sum_{\varphi_v = \pm 1} \varphi_v^{\mathbb{1}\{v \in \{0\} \triangle \{x\}\} + \sum_{\bar v \in \Lambda} n_{\{v,\bar v\}}} \right) \\
&= \sum_{\partial \mathbf{n} = \{0\} \triangle \{x\}} \frac{w_\lambda(\mathbf{n})}{Z_\Lambda} \qquad \left(\partial \mathbf{n} := \Big\{ v \in \Lambda \colon \sum_{\bar v} n_{\{v,\bar v\}} \text{ is odd} \Big\} \right)
\end{aligned}$$

Lace expansion for the Ising model (Sakai '07)

This gives rise to a representation involving bonds:

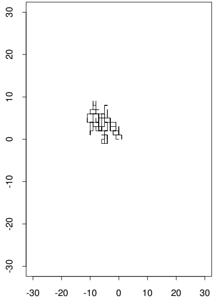
$$\langle \varphi_0 \varphi_x \rangle_\Lambda = \sum_{\substack{\mathbf{n} \in \mathbb{N}^{\mathbb{B}_\Lambda} \\ \partial \mathbf{n} = \{0\} \triangle \{x\}}} \frac{w_\lambda(\mathbf{n})}{Z_\Lambda}$$

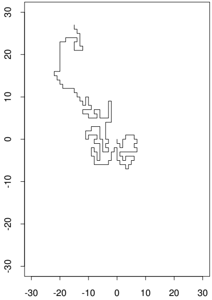


Perform *lace expansion* using ideas from lace expansion for **percolation**.
Use of BK-inequality replaced by use of RHS source switching lemma.

Self-avoiding walk

A *self-avoiding walk* is random walk conditioned to visit every site at most once.





Realization 200-step (nearest-neighbor) SRW and SAW for $d = 2$.

Interesting: The asymptotic difference between SRW and SAW (in terms of critical exponents) disappears in dimension $d \geq 5$.

Recall versions of D:

- nearest-neighbor coupling: $D(x) = \frac{1}{2d}\mathbb{1}_{\{|x|=1\}}$
- finite-variance coupling: e.g., $D(x) \approx \mathrm{e}^{-|x/L|}$ as $|x| \to \infty$, L large
- long-range coupling: e.g., $D(x) \approx |x/L|^{-(d+\alpha)}$ as $|x| \to \infty$, L large

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Let $\mathcal{W}_n(x)$ be set of n-step walks from the origin 0 to x:

$$\mathcal{W}_n(x) = \{(w_0, \ldots, w_n) \mid w_0 = 0,\ w_n = x,\ w_i \in \mathbb{Z}^d, 1 \le i \le n-1\}$$

Call $w \in \mathcal{W}_n(x)$ **self-avoiding** if $w_i \neq w_j$ for $i \neq j$ with $i, j \in \{0, \ldots, n\}$. Define *SAW two-point function* by $c_0(x) = \delta_{0,x}$ and, for $n \ge 1$,

$$c_n(x) := \sum_{w \in \mathcal{W}_n(x)} \prod_{i=1}^{n} D(w_i - w_{i-1})\, \mathbb{1}_{\{w \text{ is self-avoiding}\}}.$$

Self-avoiding walk Green's function:

$$G_z(x) := \sum_{n=0}^{\infty} c_n(x) z^n$$

Question: How does $G_z(x)$ behave?

Self-avoiding walk - result

Theorem (HHS)

Let d sufficiently large in the nearest-neighbor case, or $d > 4$ and L sufficiently large in the finite-variance spread-out case, or $d > 2(2 \wedge \alpha)$ and L sufficiently large in the spread-out power-law case. Then

$$\hat{G}_z(k) = \frac{1 + O(L^{-d})}{\chi(z)^{-1} + z[1 - \hat{D}(k)]} \tag{2}$$

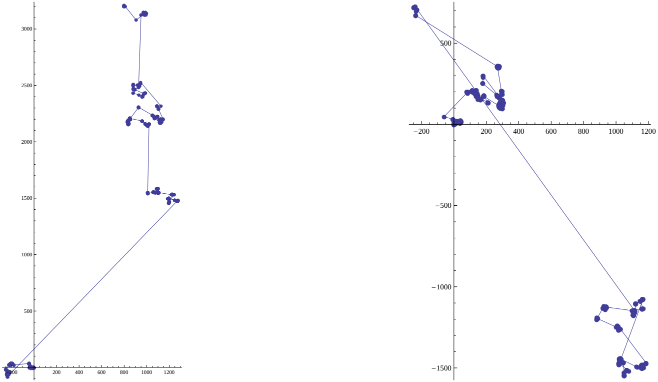
uniformly for $z \in [0, z_c)$ and $k \in [-\pi, \pi)^d$.

Here, $\chi(z) = \sum_x G_z(x) = \hat{G}_z(0)$ is the *susceptibility*.
The critical value z_c is the radius of convergence for the power series defining G_z; it is characterized by $z_c = \sup\{z \mid \chi(z) < \infty\}$.
In the nearest-neighbor case, $O(L^{-d})$ replaced with $O(1/d)$.
As for the Ising model,

$$\hat{G}_{z_c}(k) = \left(1 + O(L^{-d})\right)[1 - \hat{D}(k)]^{-1} \asymp \left(1 + O(L^{-d})\right)|k|^{-(2\wedge\alpha)}$$

and this implies *mean-field behaviour*.

Long-range walks



Realizations 500-step long-range random walk with $\alpha = 0.7$ for $d = 2$.