

About Universality for the Viana-Bray Model

Shannon Starr – University of Rochester

based on joint work with Brigitta Vermesi

Carmona and Hu's Universality Result

The Sherrington-Kirkpatrick model:

$$H_N(\sigma) = H_N(\sigma_1, \dots, \sigma_N) = \frac{1}{\sqrt{2N}} \sum_{i=1}^N \sum_{j=1}^N g_{ij} \sigma_i \sigma_j + h \sum_{i=1}^N \sigma_i$$

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ξ_{ij} 's = i.i.d. copies of ξ

$$\mathbb{E}[\xi] = 0, \quad \mathbb{E}[\xi^2] = 1 \quad \text{and} \quad \mathbb{E}[|\xi|^3] < \infty$$

Theorem [Carmona and Hu, 2004 : *Ann. I. H. Poincaré* (2006)]

Define the quenched pressure:

$$p_N(\beta, \xi) = \frac{1}{N} \mathbb{E} \log Z_N(\beta, \xi) = \frac{1}{N} \mathbb{E} \log \sum_{\sigma \in \{\pm 1\}^N} e^{-\beta H_N(\sigma, \xi)}$$

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Then

$$|p_N(\beta, \xi) - p_N(\beta, g)| \leq 9 \mathbb{E}[|\xi|^3] \frac{\beta^3}{\sqrt{N}}$$

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ξ symmetric, and $\mathbb{E}[\xi^4] < \infty$

$$\Rightarrow |p_N(\beta, \xi) - p_N(\beta, g)| = O\left(\frac{1}{\sqrt{N}}\right)$$

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This then implies a SLLN by a result of Pastur and Shcherbina

$$\text{Var}(N^{-1} \log Z_N(\beta, \xi)) \leq \frac{\beta^2}{2N} \text{Var}(\xi)$$

Appendix to “Absence of self-averaging . . .,” *J. Stat. Phys.* (1991)

Approximate Gaussian Integration by Parts

Carmona and Hu proved that

$$\mathbb{E}[\xi F(\xi)] = \mathbb{E}[\xi^2] \mathbb{E}[F'(\xi)] + \text{Remainder}$$

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Carmona and Hu's result has also been generalized to quantum spin glasses by Nick Crawford.

The Viana-Bray Model

The Viana-Bray model is a diluted spin-glass model.

$$H_N(\sigma) = \sum_{i=1}^N \sum_{j=1}^N \sum_{\mu=1}^{K_{ij}} \xi_{ij}^\mu \sigma_i \sigma_j + h \sum_{i=1}^N \sigma_i$$

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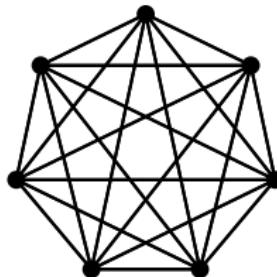
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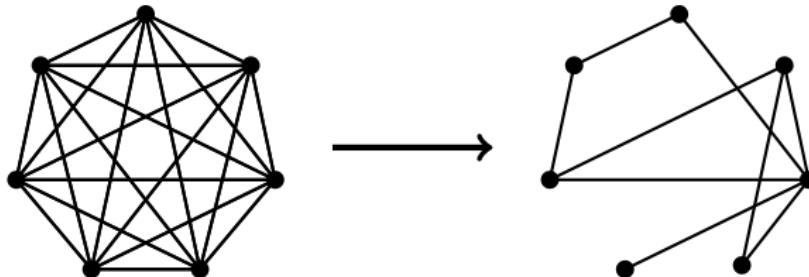
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A Generalization

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$$\mathbb{P}(\kappa = 0) = 1 - \frac{\alpha}{N} + O\left(\frac{1}{N^2}\right) \quad \text{and} \quad \mathbb{P}(\kappa = 1) = \frac{\alpha}{N} + O\left(\frac{1}{N^2}\right)$$

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This includes $\kappa \sim \text{Bernoulli}(\alpha/N)$.

Guerra's interpolation method does generalize to the Viana-Bray model.

This was used to great effect in:

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Also, the analogue of Gaussian IBP is the Poisson shift

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Could try to mimic Carmona and Hu's proof for the Vianna-Bray model.

Another approach: properties of the pressure function

Given any Hamiltonian, and any ξ_{ij} note that

$$\frac{1}{N} \mathbb{E} \log \sum_{\sigma \in \{\pm 1\}^N} e^{\beta H_N(\sigma) + \beta \xi_{ij} \sigma_i \sigma_j}$$

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$$\hookrightarrow = \frac{1}{N} \mathbb{E} \log \sum_{\sigma \in \{\pm 1\}^N} [\cosh(\beta \xi_{ij}) + \sigma_i \sigma_j \sinh(\beta \xi_{ij})] \frac{e^{\beta H_N(\sigma)}}{Z_N(\beta)}$$

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Finally using Taylor series for $\log(1 + x)$,

$$\begin{aligned} & \frac{1}{N} \mathbb{E} \log \sum_{\sigma \in \{\pm 1\}^N} e^{\beta H_N(\sigma) + \beta \xi_{ij} \sigma_i \sigma_j} - p_N(\beta) \\ &= \frac{1}{N} \mathbb{E} \log \cosh(\beta \xi_{ij}) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \mathbb{E}[\langle \sigma_i \sigma_j \rangle^n] \frac{1}{N} \mathbb{E}[\tanh^n(\beta \xi_{ij})] \end{aligned}$$

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This motivates defining

$$a_0(\xi) = \mathbb{E}[\log \cosh(\xi)]$$

$$a_n(\xi) = \mathbb{E}[\tanh^n(\xi)] \quad \text{for } n = 1, 2, \dots$$

as some sort of generalized moments.

Let ξ be a random variable, with β included,

$$H_N(\sigma, \xi) = \sum_{i=1}^N \sum_{j=1}^N \xi_{ij} \sigma_i \sigma_j + h \sum_{i=1}^N \sigma_i$$

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Theorem: Given two random variables ξ and η ,

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Proof: “Discrete interpolation” (?). Replace one ξ_{ij} by one η_{ij} at a time.
The connectivity shift type calculation from the last slide applies, and
 $|\langle \sigma_i; \sigma_j \rangle| \leq 1$.

There are N^2 pairs (i, j) which gives a multiplier N^2 ,
but there is a pre-factor $\frac{1}{N}$ built-in to $p_N(\xi)$. \square

Application to the SK model: Suppose $\mathbb{E}[\xi] = 0$ and $\mathbb{E}[\xi^2] = 1$. Then

$$\lim_{N \rightarrow \infty} \left| p_N \left(\sqrt{\beta/2N} \xi \right) - p_N \left(\sqrt{\beta/2N} g \right) \right| = 0$$

But the limit is purely existential: no rate of decay unless $\mathbb{E}[|\xi|^3] < \infty$.

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$$H_N(\sigma, \xi, \kappa) = \sum_{i=1}^N \sum_{j=1}^N \sum_{\mu=1}^{\kappa_{ij}} \xi_{ij}^\mu \sigma_i \sigma_j + h \sum_{i=1}^N \sigma_i$$

and similar definition of $p_N(\xi, \kappa)$.

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Suppose $K_n \sim \text{Poisson}(\alpha/N)$, for $n = 1, 2, \dots$,

suppose $\kappa_1, \kappa_2, \dots$ defines a sequence of random variables such that

$$\lim_{N \rightarrow \infty} N \mathbb{P}(\kappa_N = 1) = \alpha \quad \text{and} \quad \lim_{N \rightarrow \infty} N \mathbb{E}[\kappa_N] = \alpha$$

and suppose g denotes a $\text{Normal}(0, 1)$.

Application to the SK model: Suppose $\mathbb{E}[\xi] = 0$ and $\mathbb{E}[\xi^2] = 1$. Then

$$\lim_{N \rightarrow \infty} \left| p_N \left(\sqrt{\beta/2N} \xi \right) - p_N \left(\sqrt{\beta/2N} g \right) \right| = 0$$

But the limit is purely existential: no rate of decay unless $\mathbb{E}[|\xi|^3] < \infty$.

Application to the Viana-Bray model: Define

$$H_N(\sigma, \xi, \kappa) = \sum_{i=1}^N \sum_{j=1}^N \sum_{\mu=1}^{\kappa_{ij}} \xi_{ij}^\mu \sigma_i \sigma_j + h \sum_{i=1}^N \sigma_i$$

and similar definition of $p_N(\xi, \kappa)$.

Suppose $K_n \sim \text{Poisson}(\alpha/N)$, for $n = 1, 2, \dots$,

suppose $\kappa_1, \kappa_2, \dots$ defines a sequence of random variables such that

$$\lim_{N \rightarrow \infty} N \mathbb{P}(\kappa_N = 1) = \alpha \quad \text{and} \quad \lim_{N \rightarrow \infty} N \mathbb{E}[\kappa_N] = \alpha$$

and suppose g denotes a $\text{Normal}(0, 1)$.

$$\lim_{N \rightarrow \infty} |p_N(\kappa_N, \sqrt{\beta} g) - p_N(K_N, \sqrt{\beta} g)| = 0.$$