

Universal structures in mean field models for spin glasses

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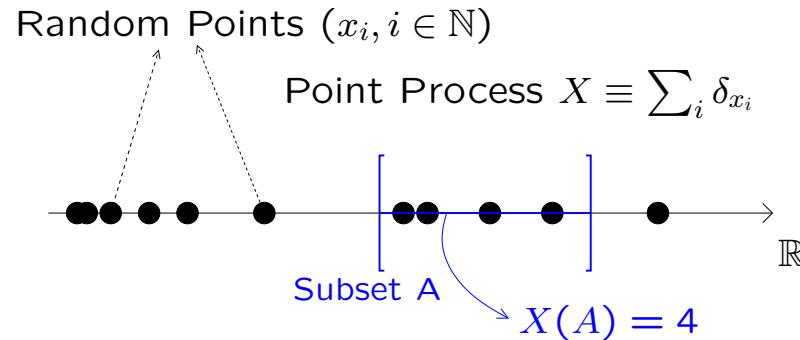
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Outline

- 1. Point Processes & Derrida-Ruelle Structures**
- 2. A nonhierarchical version of the GREM (GGREM)**
- 3. On a cavity field perturbation of the REM**

1. Point Processes & Derrida-Ruelle Structures



Poisson Point Process (PPP) of density $g(x)dx$

- $\mathbb{P}[X(A) = n] = \frac{\mu(A)^n}{n!} e^{-\mu(A)}, \quad \mu(A) = \int_A g(x)dx.$
- $A_1, \dots, A_n \subset \mathbb{R}$ disjoint $\Rightarrow X(A_1), \dots, X(A_n)$ independent

If $x_i \geq 0$ and $\sum_i x_i < \infty$ \mathbb{P} -a.s. define **Normalization**

$$\mathcal{N}\left((x_i, i \in \mathbb{N})\right) = (\bar{x}_i, i \in \mathbb{N}), \quad \text{with } \bar{x}_i = \frac{x_i}{\sum_j x_j}.$$

Additive Derrida-Ruelle Structures

Parameters $0 < \beta_1 < \dots < \beta_K < 1$. Consider $\text{PP}(\xi_i; i \in \mathbb{N}^K)$

$$\xi_i \stackrel{\text{def}}{=} \xi_{i_1}^1 + \xi_{i_1, i_2}^2 + \dots + \xi_{i_1, \dots, i_K}^K$$

- For i_1, \dots, i_{j-1} , $(\xi_{i_1, \dots, i_{j-1}, k}^j, k) \sim \text{PPP}(\beta_j e^{-\beta_j t} dt)$ on \mathbb{R}
- ξ^j independent for different j
- $(\xi_{i_1, \dots, i_{j-1}, l}^j, l)$ independent for different i_1, \dots, i_{j-1}

Multiplicative Derrida-Ruelle structures

For $\beta > \beta_K$ consider image of (ξ_i) under $x \mapsto e^{\beta x}$.

$$\eta_{\mathbf{i}} = e^{\beta(\xi_{i_1}^1 + \dots + \xi_{i_1, \dots, i_K}^K)} = \eta_{i_1}^1 \eta_{i_1, i_2}^2 \cdots \eta_{i_1, \dots, i_K}^K$$

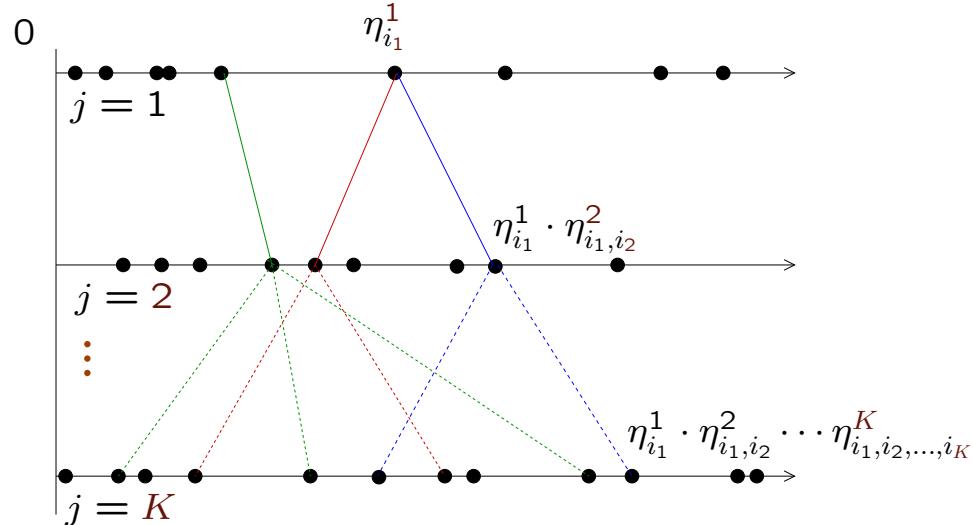
- $(\eta_{i_1, \dots, i_{k-1}, j}^k, j \in \mathbb{N})$ is PPP($m_k t^{-m_k - 1} dt$) with $m_k = \beta_k / \beta$
- $\max_{\mathbf{i}} \eta_{\mathbf{i}} < \infty, \quad \sum_{\mathbf{i}} \eta_{\mathbf{i}} < \infty, \quad \mathbb{P}\text{-a.s.}$
- $\sum_{\mathbf{i}} \bar{\eta}_{\mathbf{i}} = 1 \rightsquigarrow (\bar{\eta}_{\mathbf{i}})$ Random Probability on \mathbb{N}^K
- $\sum_{\mathbf{i}} \delta_{\bar{\eta}_{\mathbf{i}}} \stackrel{\text{(law)}}{=} \mathcal{N}(\text{PPP}(m_k t^{-m_k - 1} dt)) \Rightarrow$ forgets structure

Retaining the K-levels tree structure

Order (η_i) downwards, random bijection

$$\begin{aligned}\pi : \quad \mathbb{N} &\rightarrow \quad \mathbb{N}^K \\ j &\mapsto \quad \pi(j), \quad \eta_{\pi(j)} \quad j^{\text{th}}\text{-largest element in } \{\eta_i\}\end{aligned}$$

Overlaps $q(i, i') = \max_r \left\{ \pi(i)_1, \dots, \pi(i)_r = \pi(i')_1, \dots, \pi(i')_r \right\}$



Bolthausen-Sznitman coalescent

Introduce Random equivalence relations on \mathbb{N}

$$i \sim_k i' \iff q(i, i') \geq k.$$

Keep track of random equivalence classes for $k = 0, 1, \dots, K$ we
get a **clustering** of random partitions

$$\left(\mathcal{Z}_K = \text{'singletons'}, \dots, \mathcal{Z}_0 = \text{'one single set'} \right)$$

Parisi Picture for SK-Model.

Gibbs measure $\mathcal{G}_{\beta,N}(\sigma) = e^{\beta H_N(\sigma)} / Z_N(\beta)$ on $\Sigma_N = \{\pm 1\}^N$,

$$H_N(\sigma) = \frac{1}{\sqrt{N}} \sum_{1 \leq i < j \leq N} g_{ij} \sigma_i \sigma_j$$

- Emergence of **Pure states**: disjoint $A_\alpha \subset \Sigma_N$, $\bigcup_\alpha A_\alpha = \Sigma_N$

$$\lim_{N \rightarrow \infty} \sum_\alpha \delta_{\mathcal{G}_{\beta,N}(A_\alpha)} \stackrel{(d)}{=} \mathcal{N}(\text{PPP}(t^{-m_K-1} dt))$$

- $d_N(\sigma, \tau) = 1 - \left(\frac{1}{N} \sum_i \sigma_i \tau_i \right)^2$ becomes **ultrametric**

$$\lim_N \mathbb{P} \otimes \mathcal{G}_{\beta,N,\cdot}^{\otimes 2} \left[d_N(\sigma, \tau) \leq \max_\rho \{ d_N(\sigma, \rho), d_N(\rho, \tau) \} \right] = 1$$

- Tree structure $(\mathcal{Z}_K, \dots, \mathcal{Z}_0)$ on pure states (possibly $K = \infty$?)

Universality conjecture.

The above picture is conjectured to be universal; i.e. given any mean field model of spin glasses on some configuration space, then, in the thermodynamical limit:

- the configuration space is **hierarchically** organised
- the configuration space splits into **pure states** with statistics given by **normalized PPP** (Derrida-Ruelle structures)
- **Tree structure** on the Pure states according to Bolthausen-Sznitman coalescent

Instructive examples where this can be proved?

2. A nonhierarchical version of the GREM

- Parameters: $I = \{1, \dots, n\}$, $(\gamma_i, i \in I)$, $\sum_i \gamma_i = 1$, and $(a_J, J \subset I)$ positive, $\sum_J a_J = 1$
- Configurations $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$ with $\sigma_i = 1, \dots, 2^{\gamma_i N}$
- For $\sigma_J = (\sigma_i, i \in J)$ and $X_{\sigma_J}^J \stackrel{\text{iid}}{\sim} \mathcal{N}(0, a_J N)$ Hamiltonian

$$X_\sigma = \sum_{J \subset I} X_{\sigma_J}^J$$

- $\mathcal{G}_{\beta, N}(\sigma) = e^{\beta X_\sigma} / \sum_\tau e^{\beta X_\tau}$, $f_N(\beta) = \frac{1}{N} \log \sum_\tau e^{\beta X_\tau}$.

To **chain** $T = \{A_1 \subset A_2 \dots \subset A_K\}$ associate **GREM** $\tilde{X}_\alpha(T)$.

E.g. **Triangle-GREM** : $X_\sigma = X_{\sigma_1, \sigma_2}^{\{1,2\}} + X_{\sigma_1, \sigma_3}^{\{1,3\}} + X_{\sigma_2, \sigma_3}^{\{2,3\}}$,

$$\#\sigma_i = 2^{\gamma_i N}, \quad \text{var}\left(X_{\sigma_i, \sigma_j}^{\{i,j\}}\right) = N a_{ij}.$$

Chain $T = \left\{ \{1,2\}, \{1,2,3\} \right\} \Rightarrow \tilde{X}_\alpha(T) = \tilde{X}_{\alpha_1}^{\{1\}}(T) + \tilde{X}_{\alpha_1, \alpha_2}^{\{1,2\}}(T)$

$$\begin{aligned} \#\alpha_1 &= 2^{(\gamma_1 + \gamma_2)N} & \#\alpha_2 &= 2^{(\gamma_3)N} \\ \text{var}\left(\tilde{X}_{\alpha_1}^{\{1\}}(T)\right) &= N a_{12} & \text{var}\left(\tilde{X}_{\alpha_1, \alpha_2}^{\{1,2\}}(T)\right) &= N (a_{13} + a_{23}) \end{aligned}$$

Thm (Bolthausen-K.). **GREM universality**

- ▶ *GGREM-Free Energy exists, self-averaging.*
- ▶ \exists chain $T = \{A_1, A_2, \dots, A_K\}$

$$f(\beta) = f(\beta, T)$$

- ▶ T is minimal,

$$f(\beta) = \min_S f(\beta, S)$$

But Ultrametricity \iff Hamiltonian irreducible

Thm (Bolthausen-K.). **Ultrametricity for GGREM.**

- *Marked Point Process on $\mathbb{R} \times \mathbb{R} \times 2^I$*

$$\Xi_N \stackrel{\text{def}}{=} \sum \delta_{X_\sigma - a_N, X_\tau - a_N; \hat{q}(\sigma, \tau)}$$

overlap $\hat{q}(\sigma, \tau) = J \iff \sigma_i = \tau_i, i \in J.$

- $T = \{A_j\}$ minimal chain. Marked PP on $\mathbb{R} \times \mathbb{R} \times 2^I$,

$$\Xi \stackrel{\text{def}}{=} \sum \delta_{\xi_i, \xi_{i'}; \check{q}(i, i')}$$

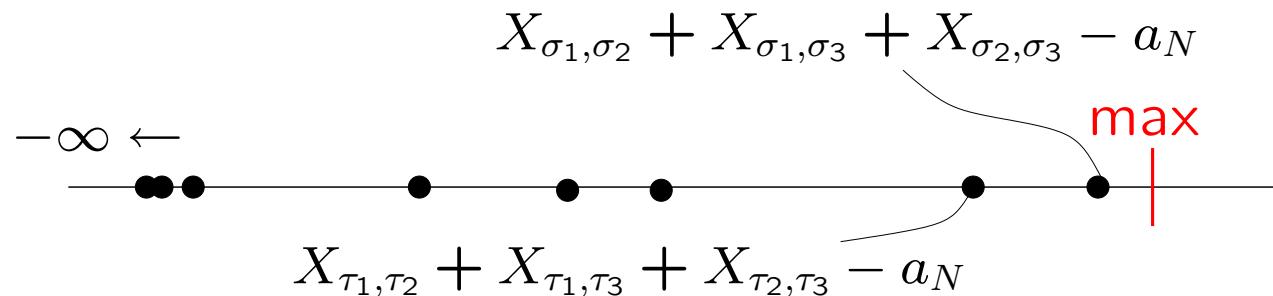
$$\xi_i = \xi_{i_1}^1 + \cdots + \xi_{i_1, \dots, i_K}^K \quad \text{and} \quad (\xi^j) \text{ PPP}(\kappa_j \beta_j e^{-\beta_j t} dt)$$

overlap $\check{q}(i, i') = A_j$ if $\max_r \{(i_1, \dots, i_r) = (i'_1, \dots, i'_r)\} = j.$

- If X_σ is irreducible, $\Xi_N \rightarrow \Xi$ weakly.

Mechanism behind ultrametricity (energy levels)

Triangle-GREM, chain $T = \{\{1, 2\}, \{1, 2, 3\}\}$



Energy vs. Entropy: for $N \gg 1$, with $\mathbb{P} \approx 1$:

- If $\sigma_2 = \tau_2$ then $\sigma_1 = \tau_1$.
- If $\sigma_3 = \tau_3$ then $\sigma_1 = \tau_1, \sigma_2 = \tau_2$.

⇒ Overlap only in the Chain!

Thm (Bolthausen-K.). **Full Parisi Picture for GGREM.**

- ▶ Let $\beta > \beta_K$, $m_K = \beta_K/\beta$, and $\eta_i = \exp [\beta \xi_i]$.

$$\begin{aligned} \sum \delta_{\mathcal{G}_{\beta,N}(\sigma), \mathcal{G}_{\beta,N}(\tau); \hat{\mathbf{q}}(\sigma, \tau)} \\ \longrightarrow \sum \delta_{\overline{\eta_i}, \overline{\eta_{i'}}; \check{\mathbf{q}}(i, i')} \\ \stackrel{(d)}{=} \mathcal{N} \left(PPP^{(2)}(t^{-m_K-1}) \right) \otimes \text{"BS-coalescent".} \end{aligned}$$

In particular, **ultrametricity**:

$$\lim_{N \rightarrow \infty} \mathbb{E} \left[\left\langle \hat{q}(\sigma, \sigma') \cap \hat{q}(\sigma', \sigma'') \subset \hat{q}(\sigma, \sigma'') \right\rangle_{\beta, N, \cdot} \right] = 1.$$

- ▶ Similar results for $\beta_j < \beta < \beta_{j+1}$ for $j = 1, \dots, K-1$.

4. On a cavity field perturbation of the REM

$\alpha \in \Sigma_N \stackrel{\text{def}}{=} \{1, \dots, 2^N\}$. Hamiltonian on $\Sigma_N \times \{\pm 1\}^N$

$$H_N(\alpha, \sigma) = \sum_{i=1}^N X_{\alpha,i} + \sum_{i=1}^N g_{\alpha,i} \sigma_i \quad \equiv \quad \text{"REM + Cavity"},$$

$X_{\alpha,i}, g_{\alpha,i}$ iid normal.

Summing σ 's out, **Free Energy** reads

$$\begin{aligned} f_N(\beta) &= \frac{1}{N} \log \sum_{\alpha, \sigma} \exp [H_N(\alpha, \sigma)] = \\ &= \frac{1}{N} \log \sum_{\alpha} \exp \left[\sum_i \beta X_{\alpha,i} + \log \cosh(\beta g_{\alpha,i}) \right] + \log 2. \end{aligned}$$

\Rightarrow This suggests study of

$$f_N(\phi) = \frac{1}{N} \log \sum_{\alpha=1}^{2^N} \exp \left[\sum_{i=1}^N \phi(X_{\alpha,i}) \right]$$

with $\phi : \mathbb{R} \rightarrow \mathbb{R}$ and $X_{\alpha,i}$ iid with distribution μ .

By 2nd-moment on empirical measure $L_{N,\alpha} = \frac{1}{N} \sum_i \delta_{X_{\alpha,i}}$

$$\lim_N f_N(\phi) = \sup_{\nu} \left\{ \int \phi d\nu - H(\nu \mid \mu) : H(\nu \mid \mu) \leq \log 2 \right\},$$

with $H(\nu \mid \mu)$ the standard relative entropy.

How to solve the variational problem?

Optimal measure turns out to be Gibbs measure, ν_m for $m \in [0, 1]$

$$\frac{d\nu_m}{d\mu} = \frac{\exp[m\phi]}{\int \exp[m\phi] d\mu}.$$

HighTemp $H(\nu_1 | \mu) < \log 2 \Rightarrow \nu_1$ optimal (i.e. $m := 1$).

LowTemp Otherwise m_\star solution to $H(\nu_m | \mu) = \log 2$.

Thm. Suppose ϕ, μ s.t. LowTemp.

$$\sum_{\alpha} \delta_{\sum_i \phi(X_{\alpha,i}) - a_N} \rightarrow PPP\left(\frac{1}{m_\star} e^{-m_\star t} dt\right)$$

for $a_N = N f_N(\phi) + O(\log N)$.

Rem. No conditions on ϕ, μ . **Universality of PPP.**

Thm (Bolthausen-K.). **Free Energy of REM+Cavity**

$$\lim_{N \rightarrow \infty} f_N(\beta) = \begin{cases} \beta^2 & \beta \leq \beta_\star, \\ \frac{\beta^2}{2} m_\star + \frac{E[\cosh(\beta g)^{m_\star} \log \cosh(\beta g)]}{E[\cosh(\beta g)^{m_\star}]} - \log 2 & \beta \geq \beta_\star, \end{cases}$$

► β_\star unique solution to $E[\cosh(\beta g) \log \cosh(\beta g)] = e^{\beta^2/2} \log 2$

► $m_\star = m_\star(\beta) \in (0, 1)$ unique solution of

$$\frac{\beta^2}{2} m^2 - \log E[\cosh(\beta g)^m] + m \frac{E[\cosh(\beta g)^m \log \cosh(\beta g)]}{E[\cosh(\beta g)^m]} = \log 2.$$

$\langle \cdot \rangle_{\beta, N} \stackrel{\text{def}}{=} \sum_{\alpha, \sigma} (\cdot) \mathcal{G}_{\beta, N}(\alpha, \sigma)$ average w.r.t. Gibbs measure.

Thm. Let

$$A_{\alpha} \stackrel{\text{def}}{=} \{(\alpha, \sigma) : \sigma \in \{\pm 1\}^N\}$$

$$\widehat{\mathcal{G}}_{\beta, N}(\alpha) \stackrel{\text{def}}{=} \mathcal{G}_{\beta, N}(A_{\alpha}), \quad \langle \cdot \rangle_{\beta, N}^{\alpha} \stackrel{\text{def}}{=} \langle \cdot \mathbf{1}_{\alpha} \rangle_{\beta, N} / \widehat{\mathcal{G}}_{\beta, N}(\alpha)$$

$$\langle \cdot \rangle_{\beta, N} = \sum_{\alpha} \langle \cdot \rangle_{\beta, N}^{\alpha} \widehat{\mathcal{G}}_{\beta, N}(\alpha).$$

If $\beta > \beta_*$ the $\{A_{\alpha}\}$ are **pure states**

- ▶ $\bigcup_{\alpha} A_{\alpha} = \Sigma_N \times \{\pm 1\}^N$
- ▶ $\sum_{\alpha} \delta_{\widehat{\mathcal{G}}_{\beta, N}(\alpha)} \rightarrow \mathcal{N}\left(PPP(t^{-m_*-1} dt)\right)$

$\langle \cdot \rangle_{\beta, N, \omega}^{\otimes 2} \stackrel{\text{def}}{=} \text{average over } (\Sigma_N \times \{\pm 1\}^N) \times (\Sigma_N \times \{\pm 1\}^N) \text{ w.r.t.}$

$$\mathcal{G}_{\beta, N, \omega}(\alpha, \sigma) \times \mathcal{G}_{\beta, N, \omega}(\alpha', \sigma')$$

Thm. Ultrametricity. Let $\beta > \beta_\star$.

- ▶ $\lim_{N \rightarrow \infty} \mathbb{E} \left\langle 1_{\alpha \neq \alpha'} q_N(\sigma, \sigma')^2 \right\rangle_{\beta, N, \cdot}^{\otimes 2} = 0.$
 - ▶ $\lim_{N \rightarrow \infty} \mathbb{E} \left\langle 1_{\alpha = \alpha'} (q_N(\sigma, \sigma') - \textcolor{red}{q}_\star)^2 \right\rangle_{\beta, N, \cdot}^{\otimes 2} = 0,$
- $$\textcolor{red}{q}_\star \stackrel{\text{def}}{=} \frac{\mathbb{E} [\tanh^2(\beta g) \exp(m_\star \log \cosh(\beta g))]}{\mathbb{E} [\exp(m_\star \log \cosh(\beta g))]},$$

$$\langle \cdot \rangle_{\beta_1, \beta_2, N, \omega}^{\otimes 2} \stackrel{\text{def}}{=} \text{average w.r.t. } \mathcal{G}_{\beta_1, N, \omega} \otimes \mathcal{G}_{\beta_2, N, \omega}$$

Thm. Chaos in temperature. Let $\beta_1, \beta_2 > \beta_\star$. If $\beta_1 \neq \beta_2$:

- ▶ $\lim_{N \rightarrow \infty} \mathbb{E} \langle 1_{\alpha=\alpha'} \rangle_{\beta_1, \beta_2, N, \cdot}^{\otimes 2} = 0,$

Thus relevant energies

$$H_N(\alpha, \sigma) = X_\alpha + \sum_{i=1}^N g_{\alpha, i} \sigma_i$$

are temperature-dependent

$$\alpha = \alpha(\beta)$$

Summarizing:

*Parisi Theory predicts **universality**: of ultrametricity, of PPP and Derrida-Ruelle structures. We have presented simple models where these issues can be addressed:*

- **GGREM**: ultrametricity arises naturally from Energy vs. Entropy competition
- **REM+Cavity**: universality of PPP and Derrida-Ruelle structures (any ϕ, μ), one-step RSB...

What about more "realistic" models?

GREM+Cavity: work in progress...