# A new approach to the giant component problem

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[1] This is based on joint work with Svante Janson

# Random graphs

- Take a graph $G = (V, E)$, finite or infinite.
- Choose a parameter $0 < p < 1$; keep each edge in $E$ with probability $p$ and remove it with probability $1 - p$, independently of all other edges.
- What is the component structure of the induced graph, in particular the size and structure of the largest component $C_1$, as a function of the edge probability $p$?
- Also, how big is the second largest component, $C_2$? Under what conditions is there a *giant* component, dominating all the others in size?
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# Earlier work

- These questions have received a great deal of attention in the context of the well-known Erdös-Rényi random graph $G(n, p)$, and this model is now more or less fully understood.
- Here there are $n$ vertices, and every pair of distinct vertices are connected by an edge with probability $p = p(n)$.
- A phase transition occurs when $p = \frac{1}{n}$.
- More precisely, consider $p(n) = \frac{1+\epsilon}{n}$, with $\epsilon \to 0$.
- When $\epsilon n^{1/3} \to -\infty$, then the maximum component size is 'about' $\epsilon^{-2} \log n$.
- When $|\epsilon n^{1/3}| \leq C$ then the maximum component size is 'about' $n^{2/3}$ (and there are several components of this size).
- Finally, when $\epsilon n^{1/3} \to \infty$, then with high probability there is a unique 'giant' component, of size approximately $2\epsilon n(1 + o(1))$, and the second largest component is of the order about $\epsilon^{-2} \log n$.

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- In the random graph $G(n, p)$, suppose that $p = \lambda/n$.
- Then the degree sequence is approximately Poisson, in the sense that the number of vertices of degree $k$ is asymptotically 'close' to $ne^{-\lambda}\lambda^k/k!$.
- Let $\lambda = 1$, and let $D$ denote a Poisson random variable with mean 1. Note, for later use, that we have $\mathbb{E}[D(D-2)] = 0$.

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# Our main results

- Here we study the largest component of a random (multi)graph on $n$ vertices with a given degree sequence, letting $n \longrightarrow \infty$.
- Under some regularity conditions on the degree sequence we give conditions on its asymptotic shape that imply that **whp** all the components are small; and other conditions that imply that **whp** there is a giant component, and the sizes of its vertex and edge sets satisfy a law of large numbers.
- Under suitable conditions, these are the only two possibilities.

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- In particular, we recover the results by Molloy and Reed on the size of the largest component in a random graph with a given degree sequence.
- We further obtain a new sharp result for the giant component just above the threshold, generalising the case of $G(n, p)$ with $np = 1 + \omega(n)n^{-1/3}$, where $\omega(n) \to \infty$ arbitrarily slowly.

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# Comparison to earlier studies

- Molloy and Reed (1995) found a threshold for the appearance of a giant component in a random graph on $n$ vertices with a given degree sequence.
- Molloy and Reed (1998) found the size of the giant above the critical threshold (away from the critical window).
- They analyse an edge deletion algorithm that finds the components and approximate the random process by a differential equation.
- Their proof is rather long and complicated, and uses a bound of the order $n^{1/4}$ on the maximum vertex degree.
- Recently, Kang and Seierstad (2007) considered the near-critical behaviour, again assuming the maximum vertex degree does not exceed $n^{1/4-\epsilon}$.
- Using singularity analysis of generating functions, they determine the size of the giant close to the critical window

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# Our methods and assumptions

- Our method is based on the properties of empirical distributions of independent random variables, and leads to simple proofs.
- Like Molloy and Reed, we work directly with the configuration model, exposing the edges one by one as they are needed.
- Unlike Molloy and Reed, we do not use differential equations. (In fact, we use a variant of the method used in JL (2007) to study the $k$-core of a random graph.)

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- In the supercritical and subcritical regime, we require only conditions on the second moment of the asymptotic degree distribution.
- In the critical regime, we need a 4th moment condition, but go all the way to the critical window, without any separation.
- We work with random graphs with given vertex degrees. Results for other random graphs, such as $G(n, p)$ and $G(n, m)$ follow immediately by conditioning on the vertex degrees.

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# No separation from criticality

- It is striking that we are able to go all the way to criticality.
- Indeed, in many other models, logarithmic or even larger separation is hard to get rid of.
- Examples of such difficulties include
  - van der Hofstad and L. (2008+) in the case of percolation on the Cartesian product of two complete graphs on $n$ vertices, where logarithmic separation occurs.
  - Borgs, Chayes, van der Hofstad, Slade and Spencer (2006) in the case of percolation on the $n$-cube, where the separation is polynomial in the number of vertices;
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# Notation and model

- For a graph $G$, let $v(G)$ and $e(G)$ denote the numbers of vertices and edges in $G$; also $v_k(G)$ is the number of vertices of degree $k$, for $k \geq 0$.
- Let $n \in \mathbb{N}$ and let $(d_i)_1^n$ be a sequence of non-negative integers, such that $\sum_{i=1}^n d_i$ is even.
- We let $G(n, (d_i)_1^n)$ be a random graph with degree sequence $(d_i)_1^n$, uniformly chosen among all possibilities (tacitly assuming that there is any such graph at all).
- We work with *multigraphs*, i.e. there may be multiple edges and loops.

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- We work with *multigraphs*, i.e. there may be multiple edges and loops.

# Our model

- Precisely, we let $G^*(n,(d_i)_1^n)$ be the *random multigraph with given degree sequence* $(d_i)_1^n$, defined by the configuration model.
- That is, take a set of $d_i$ *half-edges* for each vertex $i$, and combine the half-edges into pairs by a uniformly random matching of the set of all half-edges.
- $G^*(n,(d_i)_1^n)$ is not exactly uniformly distributed: there is a weight with a factor $1/j!$ for every edge of multiplicity $j$, and a factor $1/2$ for every loop.
- But conditioned on the multigraph being a (simple) graph, we obtain $G(n,(d_i)_1^n)$, the uniformly distributed random graph with the given degree sequence.

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- Precisely, we let $G^*(n, (d_i)_1^n)$ be the *random multigraph with given degree sequence* $(d_i)_1^n$, defined by the configuration model.
- That is, take a set of $d_i$ *half-edges* for each vertex $i$, and combine the half-edges into pairs by a uniformly random matching of the set of all half-edges.
- $G^*(n, (d_i)_1^n)$ is not exactly uniformly distributed: there is a weight with a factor $1/j!$ for every edge of multiplicity $j$, and a factor $1/2$ for every loop.
- But conditioned on the multigraph being a (simple) graph, we obtain $G(n, (d_i)_1^n)$, the uniformly distributed random graph with the given degree sequence.

# More notation

- We write $m = m(n) := \frac{1}{2}\sum_{i=1}^{n} d_i$ and $n_k = n_k(n) := \#\{i : d_i = k\}$, for $k \geq 0$.
- Thus $m$ is the number of edges and $n_k$ is the number of vertices of degree $k$ in $G(n, (d_i)_1^n)$ (or $G^*(n, (d_i)_1^n)$).
- We consider asymptotics as $n \to \infty$, and all unspecified limits below are as $n \to \infty$.
- We say that an event holds **whp** (*with high probability*), if it holds with probability tending to 1 as $n \to \infty$.
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# Conditions on degree sequence

## Condition

*For each* $n$, $(d_i)_1^n = (d_i^{(n)})_1^n$ *is a sequence of non-negative integers such that* $\sum_{i=1}^n d_i$ *is even. Furthermore,* $(p_k)_{k=0}^\infty$ *is a probability distribution independent of n such that*

1. $n_k/n = \#\{i : d_i = k\}/n \to p_k$ *as* $n \to \infty$, *for every* $k \geq 0$;
2. $\lambda := \sum_k kp_k \in (0, \infty)$;
3. $\sum_i d_i^2 = O(n)$;
4. $p_1 > 0$.

# Condition - interpretation

- Let $D_n$ be the degree of a random (uniformly chosen) vertex in $G(n,(d_i)_1^n)$ or $G^*(n,(d_i)_1^n)$; thus
$$\mathbb{P}(D_n = k) = n_k/n. \tag{1}$$
Note that $\mathbb{E}\, D_n = n^{-1}\sum_{i=1}^n d_i = 2m/n$.
- Let $D$ be a random variable with the distribution $\mathbb{P}(D = k) = p_k$. Then the above implies
$$D_n \xrightarrow{\mathrm{d}} D, \tag{2}$$
so $D$ describes the asymptotic distribution of the degree of a random vertex in $G(n,(d_i)_1^n)$.
- Also by the above, $\lambda = \mathbb{E}\, D \in (0,\infty)$, $\mathbb{P}(D = 1) > 0$, and
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- In particular, the Condition implies that the $D_n$ are uniformly integrable, and thus implies $\mathbb{E}\, D_n \to \mathbb{E}\, D$, i.e.

$$\frac{2m}{n} = n^{-1} \sum_{i=1}^{n} d_i \to \lambda. \tag{4}$$

- Let

$$g(x) := \sum_{k=0}^{\infty} p_k x^k = \mathbb{E}\, x^D, \tag{5}$$

the probability generating function of the probability distribution $(p_k)_{k=0}^{\infty}$.

- Let

$$h(x) := xg'(x) = \sum_{k=1}^{\infty} k p_k x^k, \tag{6}$$

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$$H'(1) = 2\lambda - \sum_k k^2 p_k = \mathbb{E}(2D - D^2) = -\mathbb{E}\, D(D-2). \quad (8)$$

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# First theorem: supercritical and subcritical case

### Theorem

*Assume the Condition, and let $\mathcal{C}_1$ and $\mathcal{C}_2$ be the largest and second largest components of $G(n,(d_i)_1^n)$.*

1. *If* $\mathbb{E}\,D(D-2)=\sum_k k(k-2)p_k>0$, *then there is a unique* $\xi\in(0,1)$ *such that* $H(\xi)=0$, *or equivalently* $g'(\xi)=\lambda\xi$, *and* $v(\mathcal{C}_1)/n\xrightarrow{\mathrm{p}}1-g(\xi)>0$, $v_k(\mathcal{C}_1)/n\xrightarrow{\mathrm{p}}p_k(1-\xi^k)$, *for every* $k\geq 0$, *and* $e(\mathcal{C}_1)/n\xrightarrow{\mathrm{p}}\frac{1}{2}\lambda(1-\xi^2)$, *while* $v(\mathcal{C}_2)/n\xrightarrow{\mathrm{p}}0$ *and* $e(\mathcal{C}_2)/n\xrightarrow{\mathrm{p}}0$.
2. *If* $\mathbb{E}\,D(D-2)=\sum_k k(k-2)p_k\leq 0$, *then* $v(\mathcal{C}_1)/n\xrightarrow{\mathrm{p}}0$ *and* $e(\mathcal{C}_1)/n\xrightarrow{\mathrm{p}}0$.

*The same holds for* $G^*(n,(d_i)_1^n)$.

# Second theorem: critical case

## Theorem

*Assume the Condition and that* $\mathbb{E}\, D(D-2) \ = \sum_k k(k-2)p_k = 0$. *Assume also that* $\alpha_n := \mathbb{E}\, D_n(D_n - 2) = \sum_{i=1}^n d_i(d_i - 2)/n > 0$ *and that* $n^{1/3}\alpha_n \to \infty$, *and that* $\sum_{i=1}^n d_i^{4+\eta} = O(n)$ *for some* $\eta > 0$. *Let* $\beta := \mathbb{E}\, D(D-1)(D-2)$. *Then,* $\beta > 0$ *and*

$$v(\mathcal{C}_1) = \frac{2\lambda}{\beta} n\alpha_n + o_{\mathrm{p}}(n\alpha_n),$$

$$v_k(\mathcal{C}_1) = \frac{2}{\beta} kp_k n\alpha_n + o_{\mathrm{p}}(n\alpha_n), \ \textit{for every } k \geq 0,$$

$$e(\mathcal{C}_1) = \frac{2\lambda}{\beta} n\alpha_n + o_{\mathrm{p}}(n\alpha_n),$$

*while* $v(\mathcal{C}_2) = o_{\mathrm{p}}(n\alpha_n)$ *and* $e(\mathcal{C}_2)/n = o_{\mathrm{p}}(n\alpha_n)$. *The same results hold for* $G^*(n, (d_i)_1^n)$.

# Remarks

## Remark

The moment condition in the critical case means that $\mathbb{E}\, D_n^{4+\eta} < \infty$; it thus implies that $D_n^2$ and $D_n^3$ are uniformly integrable. Combined with our earlier assumptions, it implies that $\mathbb{E}\, D_n^2 \to \mathbb{E}\, D^2$ and $\mathbb{E}\, D_n^3 \to \mathbb{E}\, D^3$. In particular, we have

$$\alpha_n := \mathbb{E}\, D_n(D_n - 2) \to \mathbb{E}\, D(D-2) = 0 \tag{9}$$

and

$$\beta_n := \mathbb{E}\, D_n(D_n - 1)(D_n - 2) \to \mathbb{E}\, D(D-1)(D-2) = \beta. \tag{10}$$

We do not think that this is best possible; we conjecture that it is enough to assume that $D_n^3$ are uniformly integrable, i.e. $\mathbb{E}\, D_n^3 \to \mathbb{E}\, D^3 < \infty$.

# Remarks

Our assumption that $\sum_i d_i^2 = O(n)$ and $n^{-1} \sum d_i \to \lambda$ implies that

$$\liminf \mathbb{P}\big(G^*(n,(d_i)_1^n) \text{ is a simple graph}\big) > 0, \qquad (11)$$

see for instance Bollobás (2001), McKay (1985) or McKay and Wormald (1991) under some extra condition on $\max d_i$ and Janson (2007+) for the general case.

Since we obtain $G(n,(d_i)_1^n)$ by conditioning $G^*(n,(d_i)_1^n)$ on being a simple graph, and all results in our Theorems are (or can be) stated in terms of convergence in probability, the results for $G(n,(d_i)_1^n)$ follow from the results for $G^*(n,(d_i)_1^n)$ by this conditioning.

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The assumption $\sum_{i=1}^{n} d_i^2 = O(n)$ is used in our proof mainly for the reduction to $G^*(n,(d_i)_1^n)$.

In fact, the proof of our 'non-critical' theorem for $G^*(n,(d_i)_1^n)$ holds with simple modifications also if it is replaced by the weaker condition that $D_n$ are uniformly integrable, or equivalently, $\mathbb{E}\, D_n \to \mathbb{E}\, D$.

It might also be possible to extend the theorem for $G(n,(d_i)_1^n)$ too, under some weaker assumption than our condition, by combining estimates of $\mathbb{P}(G^*(n,(d_i)_1^n) \text{ is simple})$ from McKay and Wormald (1991) with more precise estimates of the error probabilities in our proof, but we have not pursued this.

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# Remarks

- Note that our condition excludes the case $p_1 = 0$.
- In this case, $\mathbb{E}\, D(D-2) = \sum_{k=3}^{\infty} k(k-2)p_k \geq 0$, with strict inequality as soon as $p_k > 0$ for some $k \geq 3$. Different kinds of behaviour can occur.
- First, if $p_1 = 0$ and $\mathbb{E}\, D(D-2) > 0$, i.e. if $p_1 = 0$ and $\sum_{k \geq 3} p_k > 0$, then all but $o_{\mathrm{p}}(n)$ vertices and edges belong to a single giant component. Hence, the conclusions of the first theorem hold with $\xi = 0$. (In this case, $H(x) > 0$ for every $x \in (0,1)$.)
- The case $p_1 = 0$ and $\mathbb{E}\, D(D-2) = 0$, i.e. $p_k = 0$ for all $k \neq 0, 2$, is much more exceptional. (In this case, $H(x) = 0$ for all $x$.)

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## $p_k = 0,\ k \neq 0, 2$: examples

- Since isolated vertices do not matter, let us assume $p_0 = 0$ too and consider thus the case $p_2 = 1$.
- If all $d_i = 2$ (random 2-regular graph), the components are cycles. In the multigraph the distribution of cycle lengths is given by the Ewens's sampling formula $\mathrm{ESF}(1/2)$, and thus $v(\mathcal{C}_1)/n$ converges to a non-degenerate distribution on $[0, 1]$. The same is true for $v(\mathcal{C}_2)/n$ (and for $v(\mathcal{C}_3)/n, \dots$), so there are several large components.
- Second case with $p_2 = 1$: add a small number of vertices of degree 1: i.e. let $n_1 \to \infty$, $n_1/n \to 0$, and $n_2 = n - n_1$.) Then $v(\mathcal{C}_1) = o_{\mathrm{P}}(n)$.
- A third case with $p_2 = 1$: add a small number of vertices of degree 4 (i.e., $n_4 \to \infty$, $n_4/n \to 0$, and $n_2 = n - n_4$). Then $v(\mathcal{C}_1) = n - o_{\mathrm{P}}(n)$, so there is a giant component containing almost everything. (The case $\varepsilon = 0$ again.)

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# Potential extensions

- Quantitative results, e.g a central limit theorem for the size of the giant component, as for the $k$-core in JL (2007+).
- Large deviation estimates.
- Inside the transition window, where $\alpha_n = O(n^{1/3})$, an appropriate scaling seems to lead to convergence to Gaussian processes, like in Aldous (1997); similar results on the distribution of the sizes of the largest components could be obtained.
- We have not given more precise bounds on $\mathcal{C}_2$. Direct analysis of the Markov process $(A(t), V_0(t), V_1(t), \ldots)$ can show that the largest component has size $O(\log n)$ in the subcritical phase, and that so does the supercritical second largest component, but we have not pursued this.
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- Quantitative results, e.g a central limit theorem for the size of the giant component, as for the $k$-core in JL (2007+).
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# $G(n,p)$, $G(n,m)$ and other random graphs

- Can apply our results to some other random graphs by conditioning on vertex degrees, whenever the random graph conditioned on the degree sequence is uniformly distributed.
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- For $G(n, p)$ with $np \to \lambda$ or $G(n, m)$ with $2m/n \to \lambda$, where $0 < \lambda < \infty$, the assumptions hold with $D \sim \text{Po}(\lambda)$.
- Thus $g(x) = e^{\lambda(x-1)}$, $h(x) = \lambda x e^{\lambda(x-1)}$, $H(x) = \lambda x(x - e^{\lambda(x-1)})$, and we recover the both the classical threshold $\lambda = 1$ and the standard equation $\xi = e^{\lambda(\xi-1)}$ for the size of the giant component when $\lambda > 1$.
- In $G(n, p)$ with $p = (1 + \epsilon_n)/n$ where $\epsilon_n \to 0$ in the critical case, we have $\alpha_n/\epsilon_n \xrightarrow{\text{p}} 1$ by the second moment method as soon as $n\epsilon_n \to \infty$. So we need $n^{1/3}\epsilon_n \to \infty$ in order to apply the theorem.
- It is also well known that if $n^{1/3}\epsilon_n = O(1)$, then $v(\mathcal{C}_1)$ and $v(\mathcal{C}_2)$ are both of the same order $n^{2/3}$ and Theorem 3 fails, which shows that the condition $n^{1/3}\alpha_n \to \infty$ in the critical case is best possible.

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# Finding the components: standard procedure

- Pick an arbitrary vertex $v$ and determine the component of $v$ as follows: include all the neighbours of $v$ in an arbitrary order; then add in the neighbours of neighbours, and so on, until no more vertices can be added.
- The vertices included until this moment form the component of $v$.
- If there are still vertices left in the graph, pick any such vertex $w$, and repeat the above to determine the second component (the component of vertex $w$).
- Carry on in this manner until all the components have been found.

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# Finding components - equivalent procedure

- Regard each edge as consisting of two half-edges, each half-edge having one endpoint.
- Label the vertices as *sleeping* or *awake* (= used) and the half-edges as *sleeping*, *active* or *dead*; the sleeping and active half-edges are also called *living*.
- We start with all vertices and half-edges sleeping.
- Pick a vertex and label its half-edges as active. Then take any active half-edge, say $x$ and find its partner $y$ in the graph; label these two half-edges as dead; further, if the endpoint of $y$ is sleeping, label it as awake and all other half-edges there as active. Repeat as long as there is any active half-edge.
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# Finding components in a random multigraph

- Apply this to a random multigraph $G^*(n,(d_i)_1^n)$ with a given degree sequence, revealing its edges during the process.
- Thus observe initially only the vertex degrees and the half-edges, but not how they are joined to form edges. Hence, each time we need a partner of an half-edge, it is uniformly distributed over all other living half-edges. (The dead half-edges are the ones that already are paired into edges.)
- These random choices are made by giving the half-edges i.i.d. random maximal lifetimes $\tau_x$ with the distribution $\mathrm{Exp}(1)$; i.e. each half-edge dies spontaneously with rate 1 (unless killed earlier).
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# Algorithm constructing $G^*(n,(d_i)_1^n)$ and exploring its components simultaneously

- Start with all vertices and half-edges sleeping.
- C1: If there is no active half-edge, select a sleeping vertex (by choosing a half-edge uniformly at random among all sleeping half-edges), declare it awake and its half-edges active. If no sleeping half-edge left, stop; the remaining sleeping vertices are all isolated and we have explored all components.
- C2: Pick an active half-edge (which one does not matter) and kill it, i.e., change its status to dead.
- C3: Wait until the next half-edge dies (spontaneously). This half-edge is joined to the one killed in the previous step C2 to form an edge of the graph. If the vertex it belongs to is sleeping, we change this vertex to awake and all other half-edges there to active. Repeat from C1.

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The components are created between the successive times C1 is performed; the vertices in the component created during one of these intervals are the vertices that are awakened during the interval.

Note also that a component is completed and C1 is performed exactly when the number of active half-edges is 0 and a half-edge dies at a vertex where all other half-edges (if any) are dead.

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# Analysis of the algorithm for $G^*(n, (d_i)_1^n)$

- Let $S(t)$ and $A(t)$ be the numbers of sleeping and active half-edges at time $t$. Let $L(t) = S(t) + A(t)$ be the number of living half-edges (all assumed right-continuous).
- Consider $L(t)$ first.
- We start with $2m$ half-edges, all sleeping and thus living, but we immediately perform C1 and C2 and kill one of them; thus $L(0) = 2m - 1$.
- Afterwards, as soon as a living half-edge dies, perform C3 and then (instantly) either C2 or both C1 and C2.
- Since C1 does not change the number of living half-edges while C2 and C3 each decrease it by 1, $L(t)$ is decreased by 2 each time one of the living half-edges dies, except when the last living one dies and the process terminates.

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- We start with $2m$ half-edges, all sleeping and thus living, but we immediately perform C1 and C2 and kill one of them; thus $L(0) = 2m - 1$.
- Afterwards, as soon as a living half-edge dies, perform C3 and then (instantly) either C2 or both C1 and C2.
- Since C1 does not change the number of living half-edges while C2 and C3 each decrease it by 1, $L(t)$ is decreased by 2 each time one of the living half-edges dies, except when the last living one dies and the process terminates.

Hence we have:

**Lemma**

*As $n \to \infty$,*

$$\sup_{t \geq 0} |n^{-1} L(t) - \lambda e^{-2t}| \xrightarrow{\text{p}} 0.$$

**Proof.**

This (or rather an equivalent statement in a slightly different situation) was proved in JL (2006) as a consequence of the Glivenko–Cantelli theorem on convergence of empirical distribution functions. $\square$

# Analysis continued

- Next consider the sleeping half-edges. Let $V_k(t)$ be the number of sleeping *vertices* of degree $k$ at time $t$; thus

$$S(t) = \sum_{k=1}^{\infty} kV_k(t).$$

  Note that C2 does not affect sleeping half-edges, and that C3 implies that each sleeping vertex of degree $k$ is eliminated (i.e., awakened) with intensity $k$, independently of all other vertices. There are also some sleeping vertices eliminated by C1.

- We first ignore the effect of C1: let $\tilde{V}_k(t)$ be the number of vertices of degree $k$ such that all their half-edges have maximal lifetimes $\tau_x > t$. (I.e., none of their $k$ half-edges would have died spontaneously up to time $t$, assuming they all escaped C1.)

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# Analysis continued

**Lemma**

*As* $n \to \infty$,

$$\sup_{t \geq 0} |n^{-1} \tilde{V}_k(t) - p_k e^{-kt}| \xrightarrow{\mathrm{p}} 0 \tag{12}$$

*for every* $k \geq 0$ *and*

$$\sup_{t \geq 0} |n^{-1} \sum_{k=0}^{\infty} \tilde{V}_k(t) - g(e^{-t})| \xrightarrow{\mathrm{p}} 0, \tag{13}$$

$$\sup_{t \geq 0} |n^{-1} \tilde{S}(t) - h(e^{-t})| \xrightarrow{\mathrm{p}} 0. \tag{14}$$

**Proof.**

Once again, this follows from Glivenko-Cantelli, together with the uniform integrability of the $D_n$.

## Difference between $S(t)$ and $\tilde{S}(t)$

This is easily estimated.

**Lemma**

*If* $d_{\max} := \max_i d_i$ *is the maximum degree of* $G^*(n, (d_i)_1^n)$, *then*

$$0 \le \tilde{S}(t) - S(t) < \sup_{0 \le s \le t} \big(\tilde{S}(s) - L(s)\big) + d_{\max}.$$

Let $\tilde{A}(t) := L(t) - \tilde{S}(t) = A(t) - \big(\tilde{S}(t) - S(t)\big)$.

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By the above, we obtain further the relation

$$\tilde{A}(t) \leq A(t) < \tilde{A}(t) - \inf_{s \leq t} \tilde{A}(s) + d_{\max}.$$

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# Functional equation

We further need to study the behaviour of the function $H(x)$.

### Lemma

*Suppose that the Condition holds and let $H(x)$ be given as above, i.e. $H(x) = \lambda x^2 - h(x)$, with $h(x) = xg'(x)$.*

1. *If $\mathbb{E}\, D(D-2) = \sum_k k(k-2)p_k > 0$, then there is a unique $\xi \in (0,1)$ such that $H(\xi) = 0$; or equivalently $g'(\xi) = \lambda\xi$; moreover, $H(x) < 0$ for $x \in (0,\xi)$ and $H(x) > 0$ for $x \in (\xi, 1)$. and $H'(\xi) > 0$.*
2. *If $\mathbb{E}\, D(D-2) = \sum_k k(k-2)p_k \leq 0$, then $H(x) < 0$ for $x \in (0,1)$.*

# Connecting all the pieces together

- Let $\xi$ be the zero of $H$ given by the above and let $\tau := -\ln \xi$. Then, by the lemma, $H(e^{-t}) > 0$ for $0 < t < \tau$, and thus $\inf_{t \le \tau} H(e^{-t}) = 0$.
- Consequently,
$$n^{-1} \inf_{t \le \tau} \tilde{A}(t) = \inf_{t \le \tau} n^{-1} \tilde{A}(t) - \inf_{t \le \tau} H(e^{-t}) \xrightarrow{\mathrm{p}} 0. \tag{17}$$
- Further, by the Condition, $d_{\max} = O(\sqrt{n})$, and thus $n^{-1} d_{\max} \to 0$, which implies
$$\sup_{t \le \tau} n^{-1} |A(t) - \tilde{A}(t)| = \sup_{t \le \tau} n^{-1} |\tilde{S}(t) - S(t)| \xrightarrow{\mathrm{p}} 0 \tag{18}$$
- Hence also
$$\sup_{t \le \tau} |n^{-1} A(t) - H(e^{-t})| \xrightarrow{\mathrm{p}} 0. \tag{19}$$

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- Also, $H(e^{-\tau-\varepsilon}) < 0$ and so $n^{-1}\tilde{A}(\tau + \varepsilon) \xrightarrow{\mathrm{p}} H(e^{-\tau-\varepsilon})$, while $A(\tau + \varepsilon) \geq 0$.
- Thus, with $\delta := |H(e^{-\tau-\varepsilon})|/2 > 0$, **whp**

$$\tilde{S}(\tau+\varepsilon) - S(\tau+\varepsilon) = A(\tau+\varepsilon) - \tilde{A}(\tau+\varepsilon) \geq -\tilde{A}(\tau+\varepsilon) > n\delta, \tag{20}$$

  while $\tilde{S}(\tau) - S(\tau) < n\delta$ **whp**.
- Consequently, **whp** $\tilde{S}(\tau+\varepsilon) - S(\tau+\varepsilon) > \tilde{S}(\tau) - S(\tau)$, so C1 is performed between $\tau$ and $\tau + \varepsilon$.
- Let $T_1$ be the last time C1 was performed before $\tau/2$ and let $T_2$ be the next time it is performed. Then **whp** $0 \leq T_1 \leq \varepsilon$ and $\tau - \varepsilon \leq T_2 \leq \tau + \varepsilon$, so $T_1 \xrightarrow{\mathrm{p}} 0$ and $T_2 \xrightarrow{\mathrm{p}} \tau$.

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# One more lemma

**Lemma**

*Let $T_1^*$ and $T_2^*$ be two (random) times when C1 are performed, with $T_1^* \le T_2^*$, and assume that $T_1^* \xrightarrow{\mathrm{p}} t_1$ and $T_2^* \xrightarrow{\mathrm{p}} t_2$ where $0 \le t_1 \le t_2 \le \tau$. Let $C^*$ be the union of all components explored between $T_1^*$ and $T_2^*$. Then*

$$v_k(C^*)/n \xrightarrow{\mathrm{p}} p_k\left(e^{-kt_1} - e^{-kt_2}\right), \qquad k \ge 0, \tag{21}$$

$$v(C^*)/n \xrightarrow{\mathrm{p}} g(e^{-t_1}) - g(e^{-t_2}), \tag{22}$$

$$e(C^*)/n \xrightarrow{\mathrm{p}} \frac{1}{2}h(e^{-t_1}) - \frac{1}{2}h(e^{-t_2}). \tag{23}$$

*In particular, if $t_1 = t_2$, then $v(C^*)/n \xrightarrow{\mathrm{p}} 0$ and $e(C^*)/n \xrightarrow{\mathrm{p}} 0$.*

- Let $\mathcal{C}'$ be the component created at $T_1$ and explored until $T_2$.
- By the above lemma, with $t_1 = 0$ and $t_2 = \tau$,

$$v_k(\mathcal{C}')/n \xrightarrow{\mathrm{p}} p_k(1 - e^{-k\tau}), \tag{24}$$

$$v(\mathcal{C}')/n \xrightarrow{\mathrm{p}} g(1) - g(e^{-\tau}) = 1 - g(\xi), \tag{25}$$

$$e(\mathcal{C}')/n \xrightarrow{\mathrm{p}} \tfrac{1}{2}\big(h(1) - h(e^{-\tau})\big) = \tfrac{1}{2}\big(h(1) - h(\xi)\big) = \frac{\lambda}{2}(1 - \xi^2), \tag{26}$$

using the fact that $H(1) = H(\xi) = 0$.
- We have found one large component $\mathcal{C}'$ with the claimed numbers of vertices and edges. It remains to show that there is **whp** no other large component.

- Let $\mathcal{C}'$ be the component created at $T_1$ and explored until $T_2$.
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- However, let $T_3$ be the first time after $T_2$ that C1 is performed. One can show that $T_3 \xrightarrow{\mathrm{p}} \tau$.
- Thus if $\mathcal{C}''$ is the component created between $T_2$ and $T_3$, then $v(\mathcal{C}'')/n \xrightarrow{\mathrm{p}} 0$ and $e(\mathcal{C}'')/n \xrightarrow{\mathrm{p}} 0$.
- Also, if $\eta > 0$, then the total number of vertices and edges in all components found before $\mathcal{C}'$, i.e, before $T_1$, is $o_P(n)$, because $T_1 \xrightarrow{\mathrm{p}} 0$; hence

$$\mathbb{P}(\text{a component } \mathcal{C} \text{ with } e(\mathcal{C}) \geq \eta m \text{ is found before } \mathcal{C}') \to 0. \tag{27}$$

- On the other hand, conditioning on the final graph $G^*(n,(d_i)_1^n)$ that is constructed by the algorithm, if there exists a component $\mathcal{C} \neq \mathcal{C}'$ in $G^*(n,(d_i)_1^n)$ with at least $\eta m$ edges that has not been found before $\mathcal{C}'$, then with probability at least $\eta$, the vertex chosen at random by C1 at $T_2$ starting the component $\mathcal{C}''$ belongs to $\mathcal{C}$, and thus $\mathcal{C} = \mathcal{C}''$.

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- On the other hand, conditioning on the final graph $G^*(n, (d_i)_1^n)$ that is constructed by the algorithm, if there exists a component $\mathcal{C} \neq \mathcal{C}'$ in $G^*(n, (d_i)_1^n)$ with at least $\eta m$ edges that has not been found before $\mathcal{C}'$, then with probability at least $\eta$, the vertex chosen at random by C1 at $T_2$ starting the component $\mathcal{C}''$ belongs to $\mathcal{C}$, and thus $\mathcal{C} = \mathcal{C}''$.

- Consequently,

$$\mathbb{P}(\text{a component } \mathcal{C} \text{ with } e(\mathcal{C}) \geq \eta m \text{ is found after } \mathcal{C}') \leq \eta^{-1}\,\mathbb{P}(e(\mathcal{C}'') \geq \eta m) \to 0. \quad (28)$$

- Hence **whp** there is no component except $\mathcal{C}'$ with at least $\eta m$ edges.
- Take $\eta$ small to deduce that **whp** $\mathcal{C}' = \mathcal{C}_1$, the largest component, and further $e(\mathcal{C}_2) < \eta m$.
- Hence also $v(\mathcal{C}_2)/n \xrightarrow{\mathrm{P}} 0$ because $m = O(n)$ and $v(\mathcal{C}_2) \leq e(\mathcal{C}_2) + 1$.

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