RESOLVENT OF LARGE RANDOM GRAPHS

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TWO CLASSICAL OPERATORS ON GRAPHS

Let $G_n = (V_n, E_n)$ be a simple graph on $V_n = \{1, \cdots, n\}$. We define

$$A(G_n) = \text{ Adjacency matrix of } G_n = (\mathbb{1}((i,j) \in E_n))_{1 \le i,j \le n}$$

$$D(G_n) = \text{ Degree diagonal matrix of } G_n = \mathrm{diag}\,(\deg(G_n, 1), \cdots, \deg(G_n, n))$$

and, with $\alpha \in \{0, 1\}$,

$$\Delta(G_n) = A(G_n) - \alpha D(G_n).$$

$\Longrightarrow \Delta$ is either the adjacency operator or minus the Laplacian operator.

SPECTRAL MEASURE OF FINITE GRAPHS

Let

$$\lambda_n(G_n) \leq \cdots \leq \lambda_1(G_n)$$

denote the real eigenvalues of the symmetric matrix $\Delta(G_n)$.

The spectral measure of $\Delta(G_n)$ is

$$\mu_{G_n} = \frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_i(G_n)}.$$

$\Longrightarrow$ μ_{G_n} is the distribution of a uniformly drawn eigenvalue of $\Delta(G_n)$.

OUTLINE OF THE TALK

Let $(G_n), n \in \mathbb{N},$ be a sequence of graphs on $V_n = \{1, \cdots, n\}$ such that G_n "converges" to a limit graph G.

1. *Does μ_{G_n} converge to a measure μ_G for the usual weak convergence topology ?*
2. *Do we have a formula for μ_G in some cases ?*

MOTIVATING EXAMPLE

Let $d \geq 3$ and G_n is a random graph drawn uniformly on the set of d-regular graphs on n vertices (where dn is even). .

Theorem 1 (McKay 1981)**.** *The spectral measure μ_{G_n} converges weakly as n goes to infinity to the deterministic measure μ_{KM} supported on $[-2\sqrt{d-1}, 2\sqrt{d-1}]$*

$$\mu_{KM}(dx) = \frac{d}{2\pi} \frac{\sqrt{4(d-1) - x^2}}{d^2 - x^2} dx.$$

$\Longrightarrow$ μ_{KM} is the Kesten-McKay measure, it first appeared in (Kesten 1959) in the context of simple random walks on groups.

THE METRIC SPACE OF ROOTED GRAPHS

(Benjamini & Schramm 2001, Aldous & Steele 2004)

A rooted graph (G, o) is a graph $G = (V, E)$ with a distinguished vertex $o \in V$. Two rooted graphs are rooted isomorphic if there exists an isomorphism between the two graphs that takes the root of one to the root of the other.

We define $(G, o)[n]$ as the subgraph (G, o) spanned by the vertices within (graph)-distance at most n from the root o.

$$\begin{aligned} &\mathrm{distance}((G_1, o_1), (G_2, o_2)) \\ &= 1/\sup\{n \in \mathbb{N} : (G_1, o_1)[n] \text{ and } (G_2, o_2)[n] \text{ are rooted isomorphic}\}. \end{aligned}$$

$\Longrightarrow$ *The space $\mathcal{G}^*$ of rooted isomorphic classes of rooted locally finite graphs is a complete metric separable space*.

LOCAL WEAK CONVERGENCE OF GRAPHS

For a finite graph G, let $U(G)$ denote the distribution on $\mathcal{G}^*$ obtained by choosing a uniform random vertex as root.

For a sequence of graphs $(G_n), n \in \mathbb{N}$, on n vertices, we say that the local weak limit of G_n is $[G, o]$ with measure ρ on $\mathcal{G}^*$ if $U(G_n) \Rightarrow \rho$.

$\Longrightarrow$ *This convergence is ”local”, it formalizes the convergence of the local structure of a graph around a typical vertex.*

CONVERGENCE OF THE SPECTRAL MEASURE

Let $(G_n), n \in \mathbb{N}$, be a sequence of finite graphs and

$$\Delta(G_n) = A(G_n) - \alpha D(G_n).$$

Assume that $\deg(G_n, o)^{1+2\alpha}$ is uniformly integrable.

Theorem 2. *If $U(G_n)$ converges to $[G, o]$ with measure ρ then there exists a measure μ such that*

$$\lim_{n\to\infty} \mu_{G_n} = \mu.$$

IDEA OF PROOF

(i) *Assume first that the degree at the root is bounded by a constant.*

$\Longrightarrow$ The operator $\Delta = A - \alpha D$ of the limit graph (G, o) is self-adjoint on $L^2(\mathbb{N})$ and its spectral measure is properly defined via its resolvent

$$R(z) = (\Delta - zI)^{-1}$$

$\longrightarrow$ We may apply classical results on the convergence of bounded self-adjoint operators.

IDEA OF PROOF

(ii) *If the degree is not necessarily bounded.* We use a truncation argument and set

$$\Delta^K(G_n)_{ij} = \begin{cases} 0 & \text{if } \max(\deg(G_n,i),\deg(G_n,j)) > K \\ \Delta(G_n)_{ij} & \text{otherwise} \end{cases}$$

$\longrightarrow$ We apply the inequality

$$L^3(\mu_{G_n},\mu_{G_n}^K) \leq \frac{1}{n}\mathrm{tr}(\Delta(G_n) - \Delta^K(G_n))^2$$

where L is the Lévy distance for probability measures on $\mathbb{R}$.

$\Longrightarrow$ As K goes to infinity, the spectral measure of $\Delta^K(G_n)$ converges to the spectrum of $\Delta(G_n)$ uniformly in n.

RANDOM GRAPHS

Often, we do not consider a sequence of graphs G_n but a sequence of distributions on graphs, i.e. a sequence of random graphs. For a finite random graph G, let $U_2(G)$ denote the distribution on $\mathcal{G}^* \times \mathcal{G}^*$ obtained by choosing two uniform random vertices as roots.

Assume again that $\deg(G_n, o)^{1+2\alpha}$ is uniformly integrable.

Theorem 3. *If $U_2(G_n)$ converges to a measure $\rho \otimes \rho$ then there exists a measure μ such that*

$$\lim_{n\to\infty} \mathbb{E} L(\mu_{G_n}, \mu) = 0.$$

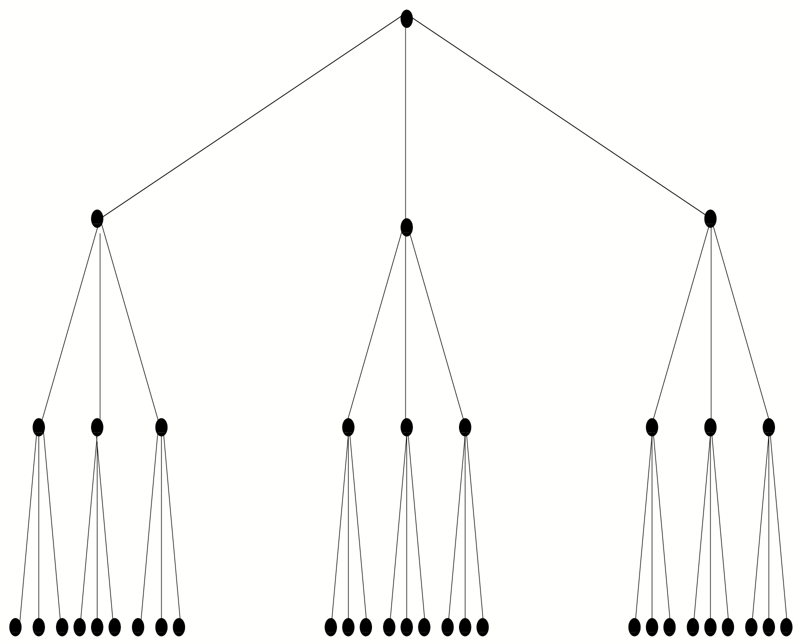
ROOTED RANDOM TREES

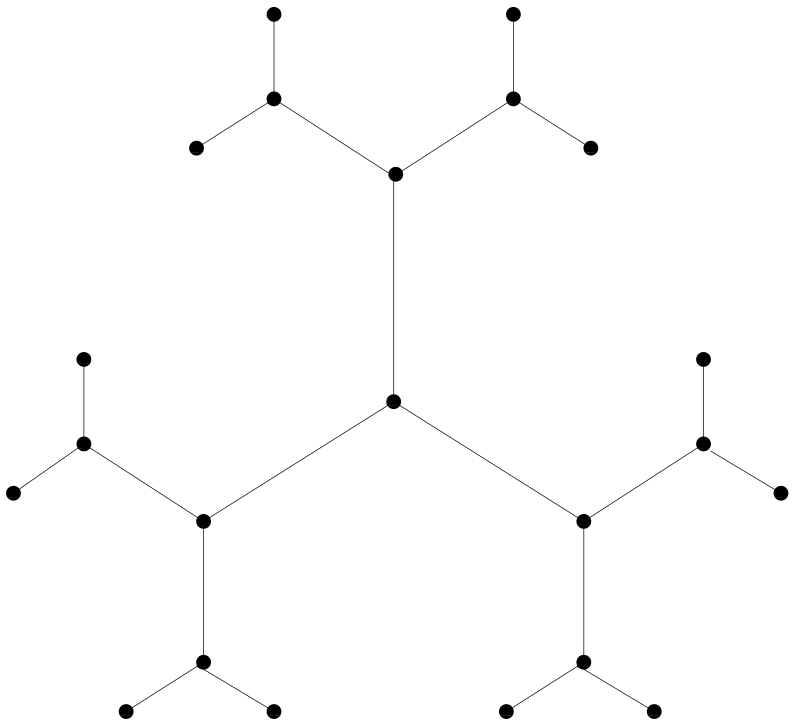
- A Galton-Watson tree (GWT) with *offspring distribution* F is the rooted random tree obtained by a Galton-Watson branching process with offspring distribution F.

- A GWT with *degree distribution* F_* is the rooted random tree obtained by a Galton-Watson branching process where the root has offspring distribution F_* and all other genitors have offspring distribution F with

$$F(k-1) = \frac{kF_*(k)}{\sum_\ell \ell F(\ell)}.$$

EXAMPLE OF REGULAR TREES





Left: the 3-ary tree is a GWT with offspring distribution δ_3.

Right: the 3-regular tree is a GWT with degree distribution δ_3.

RANDOM GRAPHS WITH TREES AS LOCAL WEAK LIMIT

Three important examples of random graph on $\{1, \cdots, n\}$,

- the uniform k-regular graph converges to the GWT with degree distribution δ_k.
- the Erdös-Reny graph with an edge between two vertices with probability p/n independently of everything else converges to the GWT with degree and offspring distribution $Poi(p)$.
- the random graphs with asymptotic prescribed degree distribution F_* converges to the GWT with degree distribution F_* (Molloy and Reed 1995).

RESOLVENT AND STIELJES TRANSFORM

Recall that μ_{G_n} is the spectral measure of $\Delta(G_n) = A(G_n) - \alpha D(G_n)$. On $\mathbb{C}_+ = \{z \in \mathbb{C} : \Im(z) > 0\}$, its Stieljes transform is defined by

$$s_{G_n}(z) = \int_{\mathbb{R}} \frac{\mu_n(dx)}{x - z} = \frac{1}{n}\sum_{i=1}^{n} \frac{1}{\lambda_i - z} = \frac{1}{n}\mathrm{tr} R_{G_n}(z) = \mathrm{E}\langle R_{G_n}(z)o, o\rangle,$$

where the *expectation is with respect to the uniformly chosen root*, and

$$R_{G_n}(z) = (\Delta(G_n) - zI)^{-1}.$$

$\longrightarrow$ $\langle R_{G_n}(z)v, v\rangle$ is the Green's function of the graph G_n at vertex v.

CONVERGENCE OF THE STIELJES TRANSFORM

Let G_n be a sequence of random graphs converging to a GWT with degree distribution F_* such that $\deg(G_n, o)^{2+\alpha}$ is uniformly integrable. Let $\mathcal{H}$ be the set of *bounded analytic functions on* $\mathbb{C}_+$.

Theorem 4. *(i) There exists a unique probability distribution on $\mathcal{H}$ such that*

$$Y(z) \stackrel{d}{=} -\left(z + \alpha(N+1) + \sum_{i=1}^{N} Y_i(z)\right)^{-1},$$

where N has distribution F and Y_i are iid copies of Y independent of N.

(ii) For all $z \in \mathbb{C}_+$, $s_{G_n}(z)$ converges in L^1 to $\mathbb{E}X(z)$ where

$$X(z) \stackrel{d}{=} -\left(z + \alpha N_* + \sum_{i=1}^{N_*} Y_i(z)\right)^{-1},$$

where N_* has distribution F_* and is independent of Y_i.

EXAMPLES FOR THE ADJACENCY OPERATOR : $\alpha = 0$

- If G_n is the uniform k-regular graph on n vertices, then

  $Y(z) =$ Stieljes transform of the semi-circle law with radius $2\sqrt{k-1}$.

  $X(z) =$ Stieljes transform of Kesten-McKay measure.

- If G_n is the Erdös Rényi graph on n vertices with parameter p/n, then

$$Y(z) \stackrel{d}{=} X(z) \stackrel{d}{=} -\left(z + \sum_{i=1}^{N} X_i(z)\right)^{-1},$$

where N is a $Poi(p)$ variable. It gives a new characterization of the spectral measure, see also (Khorunzhy, Scherbina and Vengerovsky 2004).

IDEA OF PROOF : RESOLVENT OF A TREE

Recall that $R_G(z) = (\Delta(G) - zI)^{-1}$. If $\alpha = 0$, we prove that

$Y(z) = \langle R_T(z)o, o\rangle$ where T is a GWT with offspring distribution F and root o.

$X(z) = \langle R_{T'}(z)o, o\rangle$ where T' is a GWT with degree distribution F_* and root o.

Indeed, we use the decomposition formula

$$\langle R_G(z)o, o\rangle = -\left(z + \alpha \deg(G, o) + \sum_{v,w \overset{G}{\sim} o} \langle R_{G\backslash o}(z)v, w\rangle \right)^{-1}.$$

If G is a tree and $v \neq w$ then,

$$\mathbb{1}(v \overset{G}{\sim} o)\mathbb{1}(w \overset{G}{\sim} o)\langle R_{G\backslash o}(z)v, w\rangle = 0.$$

Indeed two neighbors of o are not in the same connected component of $G\backslash o$.

IDEA OF PROOF : RESOLVENT OF A GWT

If T is a GWT with offspring distribution F then the subtrees of $T \backslash o$ are also iid GWT with offspring distribution F. We deduce the Recursive Distributional Equation (RDE) for

$$\widetilde{R}_T = (\Delta(T) - \alpha e_o e_o^t - zI)^{-1}$$

$$Y(z) := \langle \widetilde{R}_T(z) o, o \rangle \stackrel{d}{=} - \left(z + \alpha(N+1) + \sum_{i=1}^{N} Y_i(z) \right)^{-1},$$

where N has distribution F and Y_i are iid copies of Y independent of N.

$\longrightarrow$ It remains to check the unicity of the solution of this RDE...

IDEA OF PROOF : UNICITY OF THE RDE

We consider the mapping Ψ on probability measures on $\mathcal{H}$ (the bounded analytic functions on $\mathbb{C}_+$) where $\Psi(P)$ is the law of

$$z \mapsto -\left(z + \alpha(N+1) + \sum_{i=1}^{N} Y_i(z)\right)^{-1},$$

where Y_i are iid copies with law P, independent of N with law F.

We define the distance on probability measures on $\mathcal{H}$

$$W(P,Q) = \inf \mathbb{E} \int_{\Omega} |X(z) - Y(z)| dz,$$

where Ω is a bounded domain in $\mathbb{C}_+$ and the infimum is over all possible couplings of P and Q : X and Y have laws P and Q respectively.

$\longrightarrow$ A few lines of computation show that Ψ is a contraction if Ω is at distance larger than $\sqrt{\mathbb{E}N}$ from the straight line $\Im(z) = 0$ of real numbers.

COMMENTS AND EXTENSION

- The spectral measure is a local functional of the graph around a typical vertex.
- The same type of results holds for weighted graphs.
- It is also possible to prove a RDE for the limit Stieljes transform of a random bipartite graphs with given asymptotic degree distributions.
- For bi-regular uniform bipartite graphs we get an explicit solution, already found with different methods (Godsil and Mohar 1988, Mizuno and Sato 2003).
- We have assumed that $\mathbb{E}N < \infty$ to prove the unicity of the RDE

$$Y(z) \stackrel{d}{=} -\left(z + \sum_{i=1}^{N} Y_i(z)\right)^{-1}.$$

What about the case $\mathbb{E}N_* < \infty$ but $\mathbb{E}N = \mathbb{E}N_*^2/\mathbb{E}N_* - 1 = \infty$?

EXTENDED STATES IN RANDOM GRAPHS

Absolutely continuous part of the limit spectral measure of an Erdös-Rényi graph with parameter p/n *:* a phase transition is expected for $p = e$ at $x = 0$ (Bauer and Golinelli 2001). For all $a < b$ continuity points of μ

$$\mu([a,b]) = \lim_{t \to 0+} \frac{1}{\pi} \int_a^b \Im s_\mu(x+it)dx \text{ and } \mu(\{x\}) = \lim_{t \to 0+} t \Im s_\mu(x+it).$$

Is it possible to use the RDE to look at this issue ?

Hint : recall that $Y(z) \stackrel{d}{=} -\left(z + \sum_{i=1}^N Y_i(z)\right)^{-1}$, set $t = e^{-\beta}$ and $\Im Y(0+ + it) = e^{\beta h(\beta)}$, then we find

$$h(\beta) \stackrel{d}{=} -\frac{1}{\beta} \log \left(e^{-\beta} + \sum_{i=1}^N e^{\beta h_i(\beta)} \right).$$

This RDE was found by (Zdeborová and Mézard 2006) as the *cavity fields solution of the number of matchings* in random graphs...