On the degree distributions of random networks with concave preferential attachment

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joint work with Peter Mörters

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Classical Preferential Attachment Model

Model specification via *attachment rule* $f : \mathbb{Z}_+ \to (0, \infty)$

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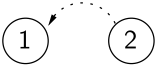

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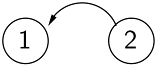

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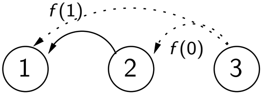

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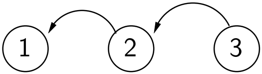

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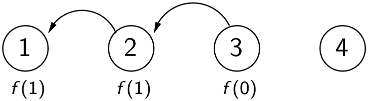

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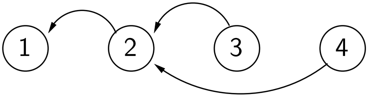

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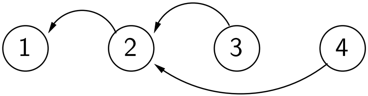


linear regime, $f(k) = \alpha(k+1)$	• model introduced in Barabási, Albert '99 (Bollobás, Riordan, Spencer, Tusnady '01) • most studied regime • maximal degree grows polynomially • degree distribution converges to a power law
superlinear regime, $f(k) = (k+1)^{\alpha} \gg k$	• in the limit only one vertex has degree ∞ (Oliveira, Spencer '05)
sublinear regime, $f(k) = (k+1)^{\alpha} \ll k$	• degree distribution has typically stretched exponential tails (Krapivski, Redner '01)

Modified Preferential Attachment Model

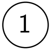
Model specification via *attachment rule* $f : \mathbb{Z}_+ \to (0, \infty)$ (f concave, $f(k) \le k+1$).

Model evolution: $(\mathcal{G}_n)_{n \in \mathbb{N}}$ sequence of growing random graphs such that at time $n = 1$, we start with a single vertex (labeled 1), and in each time step $n \to n+1$

- a new vertex (labeled $n+1$) is added and
- for each $m \le n$ a new edge $n+1 \to m$ is added independently with probability

$$\frac{f(\text{indegree of } m \text{ at time } n)}{n}.$$

Ex.: $f(m) = \sqrt{m+1}$



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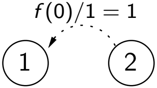

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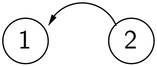

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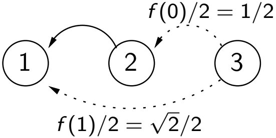

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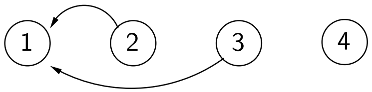

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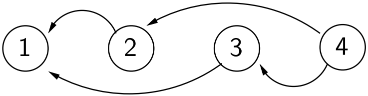


Objects of interest

(Empirical) Degree distribution

Outdegree of new vertices

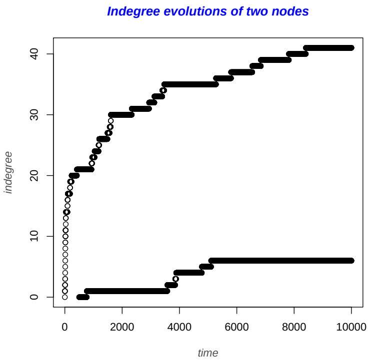
Typical behaviour of degree evolutions (of single vertices)

Exceptional behaviour of degree evolutions

Vertex with maximal degree (*hub*)

Indegree evolutions

$\sqrt{\cdot}$-attachment, $f(k) = (k+1)^{1/2}$



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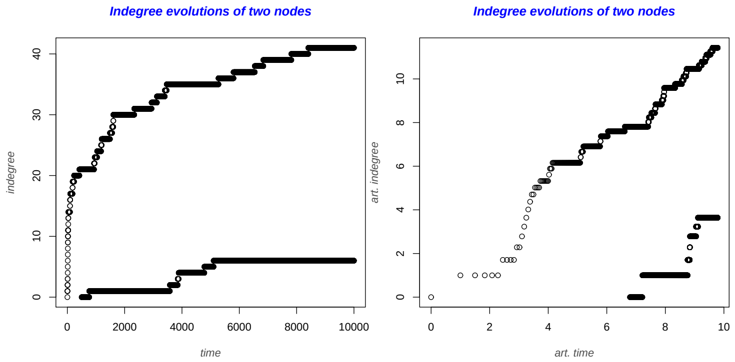

Result: The indegree evolution $Z^s(\cdot)$ of a vertex s in the artificial scaling can be represented as

$$Z^s(t) = t - s + M^s_{t-s}.$$

Moreover, $\lim_{t\to\infty} Z^s(t)/t = 1$, a.s.

Indegree evolutions

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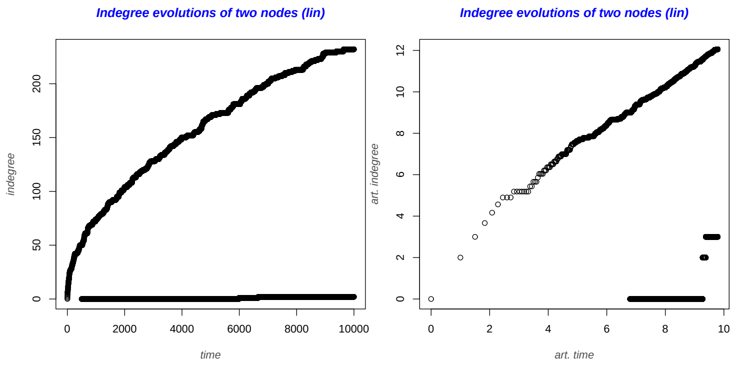

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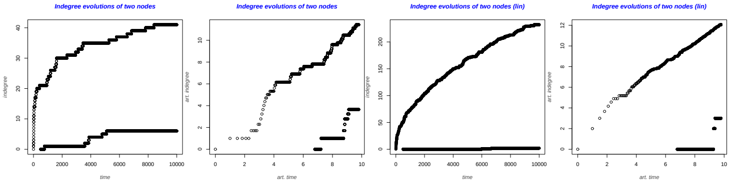
The scaling



Time Scaling (logarithmic)

natural time $n \rightsquigarrow$ artificial time $t = \Psi(n) = \sum_{m=1}^{n-1} 1/m \sim \log n$

(In-)degree Scaling

natural indegree $k \rightsquigarrow$ artificial indegree $u = \Phi(k) = \sum_{l=0}^{k-1} 1/f(k)$

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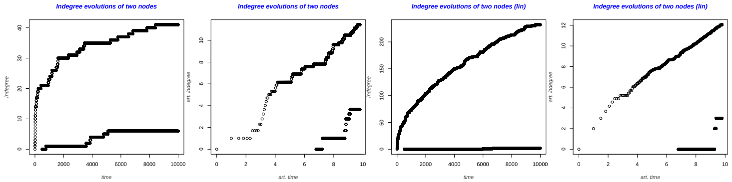

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Typical indegree evolutions behave like $\Phi^{-1}(\Psi(n))$

$$f(k) \sim c\, k^{\alpha}, \alpha \in [0,1) \quad \rightsquigarrow \quad \Phi^{-1}(\Psi(n)) \sim (c(1-\alpha)\log n)^{1/(1-\alpha)}$$

$$f(k) = \alpha\, k + o(1), \alpha \in [0,1) \quad \rightsquigarrow \quad \Phi^{-1}(\Psi(n)) \sim \mathrm{const}\; n^{\alpha}$$

The indegree distribution

Not.: $X_k(n) =$ rel. number of vertices in $\mathcal{G}_n$ with indegree k

Result: $X_k(n)$ converges a.s. to μ_k, where

$$\mu_k = \frac{1}{1+f(0)} \prod_{l=1}^{k} \frac{f(l-1)}{1+f(l)}$$

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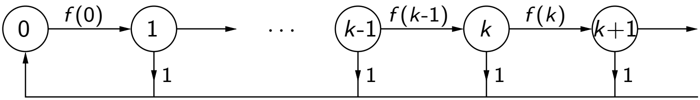
(μ_k) is the invariant distribution of the Markov chain described by:



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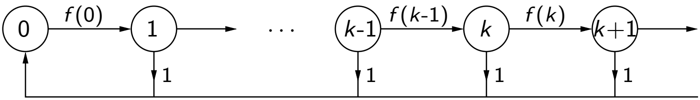
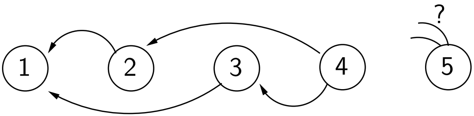


Asymptotic behaviour:

$$\begin{aligned} f(k) &= (k+1)^\alpha & &\rightsquigarrow & &\log \mu_k \sim -\tfrac{1}{1-\alpha}\, k^{1-\alpha} \\ f(k) &= \alpha(k+1) & &\rightsquigarrow & &\mu_k \sim \text{const}\; k^{-1-1/\alpha} \end{aligned}$$

The outdegree



Result:

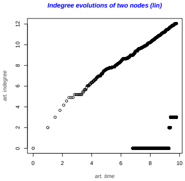
$$\mathcal{L}(\text{outdegree of vertex } n+1|\mathcal{G}_n) \Rightarrow \mathrm{Poiss}(\langle\mu, \mathrm{f}\rangle), \text{ a.s.}$$

In other words, the system stabilises!

Two phases

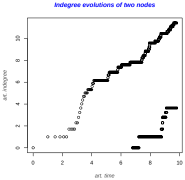
Strong preference ($\sum_{k=0}^{\infty} 1/f(k)^2 < \infty$)

- e.g., $f(k) \succ k^{1/2}(\log k)^{1/2+\varepsilon}$
- (M_t^s) converges a.s. for every vertex s
- persistent hub
  (vertex with maximal indegree)



Weak preference ($\sum_{k=0}^{\infty} 1/f(k)^2 = \infty$)

- e.g., $f(k) \prec k^{1/2}(\log k)^{1/2}$
- (M_t^s) satisfies a CLT for each s
- indegree evolutions overtake each other
  infinitely often



The indegree evolution of the hub for weak preference

Ass.: $f(k) \sim c\, k^{\alpha}$, $\alpha < 1/2$ (weak preference)

Not.:
- $M(n)$ maximal indegree at time n
- $I(n)$ index of a vertex with maximal indegree at time n

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Result: As n tends to infinity, one has in probability that

$$\log I(n) \sim \frac{1}{2}\,\frac{(1-\alpha)^{(1-2\alpha)/(1-\alpha)}}{(1-2\alpha)c^{1/(1-\alpha)}} \times (\log n)^{(1-2\alpha)/(1-\alpha)}$$

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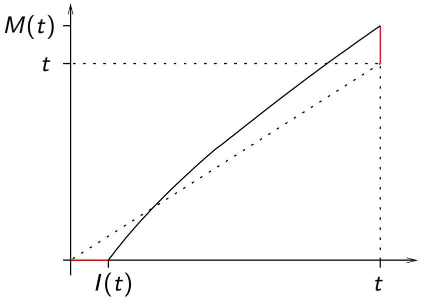
and

$$M(n) = \underbrace{(c(1-\alpha)\log n)^{1/(1-\alpha)}}_{\text{typ. evol. of the 1st vertex}} + (1+o(1))\,\frac{1}{2}\,\frac{1-\alpha}{1-2\alpha}\log n.$$

The indegree evolution of the hub - 2

(in the artificial scaling)



$\sim$ const $t^{(1-2\alpha)/(1-\alpha)}$

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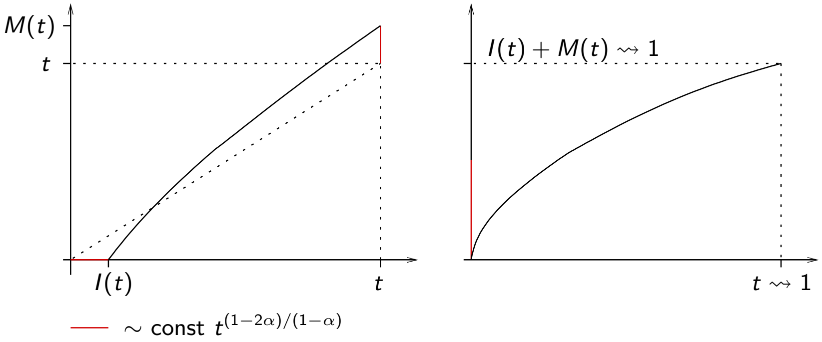

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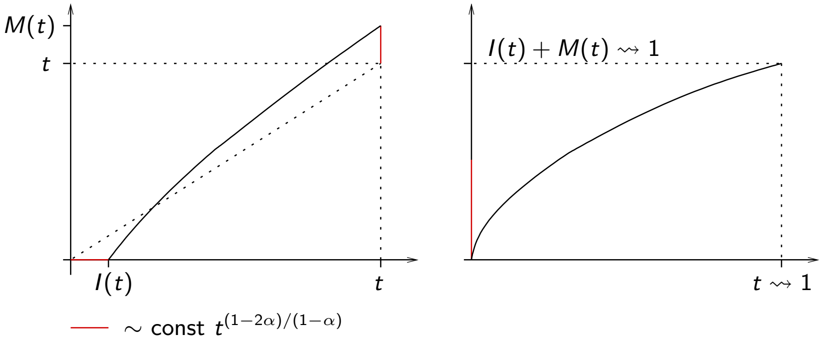

Moderate deviation principle with speed $t^{(1-2\alpha)/(1-\alpha)}$ and rate function

$$J(x) = \begin{cases} \frac{1}{2}\int_0^\infty (\dot{x}_s)^2\, s^{\frac{\alpha}{1-\alpha}}\, ds - \mathrm{const}\, x_0 & \text{if } x \text{ is abs. cont. and } x_0 \leq 0, \\ \infty & \text{otherwise.} \end{cases}$$

Ideas of the proofs

Moderate deviation principle

- Occupation times are approximately exponentially distributed (exponentially good approximation)
- Use moderate deviations for the occupation times to deduce the pathwise principle.

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Evolution of the hub

- It suffices to look at evolutions with indices of order $t^{(1-2\alpha)/(1-\alpha)}$.
- The result follows by a Borel-Cantelli argument based on the moderate deviation principle together with a Varadhan type lemma.

A recap

Empirical degree distribution

- A.s. convergence to an explicit distribution

Outdegree of new vertices

- A.s. convergence to a Poisson distribution

Typical behaviour of degree evolutions (of single vertices)

Exceptional behaviour of degree evolutions

Vertex with maximal degree (*hub*)

A recap

- Empirical degree distribution
  - A.s. convergence to an explicit distribution
- Outdegree of new vertices
  - A.s. convergence to a Poisson distribution
- Typical behaviour of degree evolutions (of single vertices)
  - Linear behaviour in the artificial picture
  - Dichotomy for persistent hubs
  - Degree distribution of the neighbors of a fixed vertex (similar as Móri '07)
- Exceptional behaviour of degree evolutions
- Vertex with maximal degree (*hub*)

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Exceptional behaviour of degree evolutions

- Moderate deviation principle and large deviation principle

Vertex with maximal degree (*hub*)

- Typical evolutions for weak preference