Directionally convex ordering of random measures, shot-noise fields and some applications to wireless networks

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Motivation

Stochastic Geometric Modelling of Wireless Networks

- Nodes as point process.
- Performance as certain functionals of the point process

Goal: Comparison of certain class of functionals of point processes

Why ?

- Comparison of performances of two different networks.
- Closed form expressions are hard to obtain for many networks.
- Tighter bounds than achieved by coupling.

$\rightsquigarrow$ Results shall be generic and possible applications in various models which use point processes.

Preliminaries

Definition

We say a function $f : \mathbb{R}^d \to \mathbb{R}$ *is directionally convex(dcx) if for every* $x, y, p, q \in \mathbb{R}^d$ *such that* $p \leq x, y \leq q$ *and* $x + y = p + q$,

$$f(x) + f(y) \leq f(p) + f(q).$$

Similarly dcv and $f : \mathbb{R}^d \to \mathbb{R}^n$. *A function is directionally linear (dl) if it is dcx and dcv.*

Definition

- $X \leq_{\mathfrak{F}} Y$ *if* $\mathsf{E}(f(X)) \leq \mathsf{E}(f(Y))$ *for all* $f \in \mathfrak{F}$.
- *Let* S *be a set. Suppose* $\{X(s)\}_{s \in S}$ *and* $\{Y(s)\}_{s \in S}$ *are two real-valued random fields, we say that* $\{X(s)\} \leq_{\mathfrak{F}} \{Y(s)\}$ *if for every* $n \geq 1$ *and* $s_1, \ldots, s_n$, $(X(s_1), \ldots, X(s_n)) \leq_{\mathfrak{F}} (Y(s_1), \ldots, Y(s_n))$.

$\mathfrak{F} = dcx/idcx/idcv$. Negation gives $dcv, ddcv, ddcx$.

Definition

Standard framework (Kallenberg, O. , 1983)

- $\mathbb{E}$ - Polish space. $\mathtt{B}_b(\mathbb{E})$ - σ-ring of bounded, Borel subsets(bbs).
- $\mathbb{M} = \mathbb{M}(\mathbb{E})$ - space of Radon measures on $\mathbb{E}$. $\bar{\mathbb{N}} = \bar{\mathbb{N}}(\mathbb{E})$ space of Radon counting measures.
- Random measure (rm) $\Lambda : \Omega \to \mathbb{M}$. Point process (pp) $\Phi : \Omega \to \bar{\mathbb{N}}$.
- Λ - a random field $\{\Lambda(B)\}_{B \in \mathtt{B}_b(\mathbb{E})}$

Definition

$\Lambda_1 \leq_{dcx} \Lambda_2$ *if* $(\Lambda_1(I_1), \ldots, \Lambda_1(I_n)) \leq_{dcx} (\Lambda_2(I_1), \ldots, \Lambda_2(I_n))$, *for* $I_1, \ldots, I_n$ *bbs.*

For any bbs $B_1, \ldots, B_n$, one can choose disjoint bbs $I_1, \ldots, I_m$ such that $\Lambda(B_j) = \sum_{i \in I_j} \Lambda(I_j)$. Hence definition is equivalent to the condition on disjoint bbs $I_1, \ldots, I_n$.

Point Processes and Ordering

- $\Phi = \{X_i\}$ pp. $\{Y_i\}$ i.i.d rvs. $\tilde{\Phi} = \{(X_i, Y_i)\}$ independently marked pp (impp).
- $Y_i \in \{0,1\}$, $\widehat{\Phi}$ is the independently thinned pp (ithpp).

LEMMA (MEESTER & SHANTHIKUMAR, 1993) : $\{S_j^i\}_{j>0}$ for $i = 1, \ldots, m$ independent i.i.d. sequences of non-negative rvs. If f is $dcx(idcx, idcv)$, then $g(n_1, \ldots, n_m) = \mathsf{E}\left(f(\sum_{j=1}^{n_1} S_j^1, \ldots, \sum_{j=1}^{n_m} S_j^m)\right)$ also is $dcx(idcx, idcv)$.

Proposition

If $\Phi_1 \leq_{dcx} (idcx, idcv)\Phi_2$ then

- $\tilde{\Phi}_1 \leq_{dcx} (idcx, idcv)\tilde{\Phi}_2$ *and*
- $\widehat{\Phi}_1 \leq_{dcx} (idcx, idcv)\widehat{\Phi}_2$.

SKETCH OF PROOF : $\tilde{\Phi}_j(A \times B) = \sum_{i=1}^{\Phi_j(A)} 1[Y_i \in B]$. Condition on pp & use Me-Sh Lemma.

Intensity measures and fields

- Φ_Λ - Cox process with intensity measure Λ.
- Intensity field : $\lambda(x)$ is density of $\Lambda \in \mathbb{M}(\mathbb{R}^d)$.

Proposition

- *If* $\Lambda_1 \leq_{dcx} (idcx, idcv)\Lambda_2$, *then* $\Phi_{\Lambda_1} \leq_{dcx} (idcx, idcv)\Phi_{\Lambda_2}$.
- *Suppose that* $\Lambda_i(.)$,$i = 1, 2$ *a.s have locally Riemman integrable densities Then* $\{\lambda_1(x)\}_{x\in\mathbb{R}^d} \leq_(dcx)(idcx, idcv)\{\lambda_2(x)\}_{x\in\mathbb{R}^d}$ *implies that* $\Lambda_1(.) \leq_(dcx)(idcx, idcv)\Lambda_2(.)$.

Sketch of Proof : Proof by approximation of integrals via middle Riemann sums and the below result. □

Lemma (Muller & Stoyan, 2002) : $(X^{(k)})$ and $(Y^{(k)})$ - sequences of random vectors with $X^{(k)} \leq_{dcx} Y^{(k)}$. If $X^{(k)} \to X$ and $Y^{(k)} \to Y$ in distribution and if moreover $\mathrm{E}\left(X^{(k)}\right) \to \mathrm{E}(X)$ and $\mathrm{E}\left(Y^{(k)}\right) \to \mathrm{E}(Y)$, then $X \leq_{dcx} Y$.

Ordering of Palm Version of Measures

- f measurable and $0 < \mathsf{E}\left(\int_{\mathbb{E}} f(x)\Lambda(dx)\right) < \infty$. The mixed Palm version $\Lambda_f \in \mathbb{M}$ of rm Λ is defined for $M \in \mathcal{M}$ as

$$\mathrm{Prob}\left(\Lambda_f \in M\right) = \frac{\mathsf{E}\left(\int_{\mathbb{E}} f(x)\Lambda(dx)1[\Lambda \in M]\right)}{\mathsf{E}\left(\int_{\mathbb{E}} f(x)\Lambda(dx)\right)},$$

- $\Lambda \in \mathbb{M}(\mathbb{R}^d)$ with a stationary density field $\{\lambda(x)\}$, the Palm version Λ_s is defined as follows :

$$\mathrm{Prob}\left(\Lambda_x \in M\right) = \frac{\mathsf{E}\left(\lambda(x)1[\Lambda \in M]\right)}{\mathsf{E}\left(\lambda(x)\right)},$$

Proposition

$\Lambda^i, i = 1, 2$- two random measures. Λ_f^i -the corresponding mixed Palm versions and Λ_s^i the Palm versions.

- *If $\Lambda^1 \leq_{dcx} \Lambda^2$, then $\Lambda_f^1 \leq_{idcx} \Lambda_f^2$.*
- *If $\{\lambda^1(x)\} \leq_{dcx} \{\lambda^2(x)\}$ then $\Lambda_s^1 \leq_{idcx} \Lambda_s^2$.*

Integral Shot-Noise fields

- S, a set. $V_\Lambda(y) = \int_{\mathbb{E}} h(x,y)\Lambda(dx)$, $y \in S$, h is $\bar{\mathbb{R}}^+$-valued and measurable in x.

Theorem

If $\Lambda_1 \leq_{dcx} (idcx, idcv)\Lambda_2$, then $\{V_{\Lambda_1}(y)\}_{y\in S} \leq_{dcx} (idcx, idcv)\{V_{\Lambda_2}(y)\}_{y\in S}$.

Sketch of Proof : Check the following approximation satisfies Mu-St Lemma.

$$V_k^j(y_i) = \sum_{n=1}^{\gamma_k} \frac{n-1}{2^k}\Lambda_j(I_{kn}^i),$$

for $j = 1, 2$ and I_{kn}^i are bbs.

- $V_k^j(y_i) \nearrow V_{\Lambda_j}(y_i)$ a.s. Hence in distribution and in L_1.
- As I_{kn}^i are bbs, by definition of ordering , V_k^j's are ordered. $\square$

$\rightsquigarrow$ Miyoshi(2004) proved Theorem when Λ_i's are three different point processes, for h lower semi-continuous and for rvs $V_i(x)$ under icx.

Remarks

- $V_{\Lambda_1+\Lambda_2} = V_{\Lambda_1} + V_{\Lambda_2}$ a increasing linear function in Λ. Not a surprising result !! The proof is easy if one can prove that $\Lambda_1 \leq_{dcx} \Lambda_2$ as random fields is equivalent to $\Lambda_1 \leq_{dcx} \Lambda_2$ as measure-valued rvs. This remains an open question.
- $h(A \times B) = 1[x \in A]F(B)$, A, B bbs in $\mathbb{E}, \mathbb{E}_1$; $F(.)$ be a probability distribution on $\mathbb{E}_1$. $V_\Lambda(A \times B)$ represents intensity measure of independently marked Cox(Λ) with mark distribution $F(.)$.
- $p(x, A)$ - a sub-probability kernel. Choose $h(x, A) = p(x, A)$, then $\{V_{\Lambda_i}(A)\}_{A \in S}$ represents the intensity measure of the Cox(Λ) whose points are independently and randomly transformed via the probability kernel $p(x, A)$. $p(x, A) = 0$ for A such that $x \notin A$ corresponds to independent thinning.
- $\{f_s\}_{s \in S}$ increasing convex(concave),then $\{f_s(V_{\Lambda_1}(s))\} \leq_{idcx} (idcv)\{f_s(V_{\Lambda_2}(s))\}$ and also their integrals under appropriate assumptions.

Extremal Shot Noise Fields

- Φ a pp, max shot-noise field $U_\Phi(y) = \max_{X_i \in \Phi}\{h(y, X_i)\}$.
- Lower orthant(lo) order for vectors : $X \leq_{lo} Y$ if $\text{Prob}\,(X \leq t) \geq \text{Prob}\,(Y \leq t)$ for every $t \in R^d$.
- For 1-dimensional vectors, *lo* is same as strong order i.e, order generated by increasing functions.

Theorem

Let $\Phi_1 \leq_{idcv} \Phi_2$. Then $\{U_{\Phi_1}(.)\} \leq_{lo} \{U_{\Phi_2}(.)\}$.

SKETCH OF PROOF :

$$\text{Prob}\,(U(x_i) \leq a_i, 1 \leq i \leq m) = \mathrm{E}\left(e^{-\sum_i \hat{U}(x_i)}\right),$$

where $\hat{U}(x_i) = \sum_n -\log 1_{[h(x_i, Y_n) \leq a_i]}$. $\rightsquigarrow$ Expectation of a *ddcx* function of an additive shot-noise field.

Monotonicity of Shot-Noise Fields

- A stationary and isotropic random field $\{X(s)\}_{s \in \mathbb{R}^d}$ is said to be *dcx* regular if for any k and $s_1, \ldots, s_k, t_1, \ldots, t_k$ such that $\|t_i - t_j\| \leq \|s_i - s_j\|$
$$(X(s_1) \ldots, X(s_k)) \leq_{dcx} (X(t_1) \ldots, X(t_k)).$$
- If $\{X(s)\}_{s \in \mathbb{R}^d}$ is *dcx* regular, $\{X_c(s) = X(cs)\}$ are *dcx* decreasing random fields in $c > 0$.
- If the intensity field $\{\lambda(s)\}$ is *dcx* regular then we get a parametric family of ordered Cox processes and hence their shot-noise fields.
- $\lim_{s \to 0} \lambda_c(s) = \lambda(0)$ - mixed Poisson pp
- If the field is ergodic, then$\lim_{s \to \infty} \lambda_c(s) = \mathrm{E}(\lambda(0))$. - Stationary Poisson pp.

Ising-Cox Point Process

Let $\mathbb{Z}^{'}$ be the $\mathbb{Z}^d$ lattice shifted by a point distributed uniformly on the cube. Let $\{X(s)\}_{s \in \mathbb{Z}'}$ be i.i.d $\{+1, -1\}$-valued rvs. Let $\mu_2 \leq \mu_1$. Define $\lambda(y) = \mu_1 1_{[X(s_y)=1]} + \mu_2 1_{[X(s_y)=-1]}$ where s_y represents the unique nearest "lower left point" in $\mathbb{Z}^{'}$ to y. Suppose $p = \text{Prob}\,(X(s) = 1)$.

- $\Phi_h \sim \text{Po}(p(\mu_1 - \mu_2) + \mu_2)$
- $\Phi_c \sim \text{Cox}(\lambda(y))$
- $\Phi_m \sim \text{Cox}(\lambda(0))$

Proposition

$$\Phi_h \leq_{dcx} \Phi_c \leq_{dcx} \Phi_m$$

$\rightsquigarrow$ Proposition holds under little more general condition on the Ising Model.
$\rightsquigarrow$ Points of Φ_c occurs more in some cubes than others i.e, clustering.

Poisson-Poisson Cluster Point Process

- $h(x)$, non-negative $\int_{\mathbb{R}^d} h(x)dx = \lambda_1 < \infty$.
- $\Phi \sim \text{Po}(\lambda)$ and $\Phi_1 \sim \text{Po}(\lambda \times \lambda_1)$.
- $\lambda_2(x) = \sum_{Y_i \in \Phi} h(x - Y_i)$. $\Phi_2 \sim \text{Cox}(\lambda_2(x))$.
- $\Lambda(dy) = \lambda dy$. $\Lambda_1(dy) = \lambda \times \lambda_1 dy$.

Proposition

$\Lambda_1 \leq_{dcx} \Phi_1 \leq_{dcx} \Phi_2$.

SKETCH OF PROOF :

$$\lambda \times \lambda_1 = \int_{\mathbb{R}^d} h(x-y)\lambda dy = \int_{\mathbb{R}^d} h(x-y)\Lambda(dy), \lambda_2(x) = \int_{\mathbb{R}^d} h(x-y)\Phi(dy)$$

Jensen's inequality implies $\Lambda \leq_{dcx} \Phi$. Hence by our main theorem result follows. □

$\rightsquigarrow$ Usually $h(x) = \mathbf{1}[\|x\| \leq R]$. This pp Illustrates clustering better.

Batch Point Process

$\rightsquigarrow$ Limiting case of Poisson-Poisson Cluster point process.

- $\Phi \sim \text{Po}(\lambda)$ and $\Phi_1 \sim \text{Po}(\lambda \times \lambda_1)$.
- $\Lambda_2(A) = \sum_{X_i \in \Phi \cap A} Y_i$ where Y_i i.i.d with mean λ_1. $\Phi_2 \sim \text{Cox}(\Lambda_2(.))$
- Λ_2 does not have a density !

Proposition

$\Phi_1 \leq_{dcx} \Phi_2$.

SKETCH OF PROOF : By Jensen's $(\lambda \times \lambda_1)dy \leq_{dcx} \Lambda_2(dy)$. $\square$

Lévy based Cox processes (LCP)

Lévy Basis - $L \in \mathbb{M}(\mathbb{R}^d)$ such that

- $L(A_i)$ independent for disjoint bbs.
- $L(A)$ is infinitely divisible rv.

Definition

(Hellmund, G. et al, 2008.)Φ LCP if intensity field is $\lambda(y) = \int_{\mathbb{R}^d} h(x,y)L(dx)$ for a Lévy basis L.

Proposition

- *$L_1 \leq_{dcx} L_2$ iff $L_1(A) \leq_{cx} L_2(A)$ for every bbs A.*
- *Suppose $L_1 \leq_{dcx} L_2$. Then $\Phi_1 \leq_{dcx} \Phi_2$.*

EXAMPLE Let $\{x_i\}$ be a fixed locally finite point configuration. $\{X_i^1\}$ be i.i.d $(exp(1/2) + exp(1/2))$ rvs. $\{X_i^2\}$ be i.i.d exp(1) rvs. For A a bbs of $\mathbb{R}^d$ and $j = 1, 2$, define

$$L_j(A) = \sum_{x_i \in A} X_i^j; \qquad \Rightarrow \quad L_1 \leq_{dcx} L_2.$$

Log Cox Point Process

Definition

- *(Hellmund, G. et al, 2008) Φ Log-Lévy driven Cox Process(LLCP) if intensity field is* $\lambda(x) = \exp\left(\int_{\mathbb{R}^d} k(x,y)L(dy)\right)$
- *Φ is Log-Gaussian Cox Process(LGCP) if intensity field is* $\lambda(x) = \exp\{X(x)\}$ *for $X(x)$ is a Gaussian field.*

Remarks

$\rightsquigarrow$ Exponential is convex function, hence ordering of the arguments imply the ordering of point processes.

$\rightsquigarrow$ For LLCP ordering of inside terms was studied in last example.

$\rightsquigarrow$ For LGCP, ordering of Gaussian fields are well-known in literature. As expected, sufficient and necessary conditions on expectation vector and covariance matrix of the fields.

Generalized Shot-Noise Cox Point Process

Definition

(Møller, 2005) Φ is GNSCP if the intensity field $\lambda(x) = \sum_j \gamma_j k_{b_j}(c_j, x)$ where $(c_j, b_j, \gamma_j) \in \Psi$, a point process on $\mathbb{R}^d \times (0, \infty) \times (0, \infty)$.

- $\Psi_1 \leq_{dcx} \Psi_2$ implies that $\lambda_1(x) \leq_{dcx} \lambda_2(x)$ and $\Phi_1 \leq_{dcx} \Phi_2$.
- This class includes Matern-Cluster processes, Neyman-Scott Processes, Thomas Processes.
- If b_j's are constants and (c_j, γ_j) is Poisson pp, Shot-Noise Cox Process. This is also a LCP.

Applications

SIGNAL-TO-INTERFERENCE NOISE RATIO(SINR) NETWORKS (Baccelli, F. et al, 2001)

- $\{x_i\}$ emitters; $\{y_i\}$ receivers; Φ pp of interferers;
- x_i successfully transmits to y_i if $\frac{S_i l(\|x_i - y_i\|)}{\sum_{X_j \in \Phi} S_j l(\|X_j - y_i\|) + W} \geq T_i$
- If S_i are i.i.d; exponential, Probability $\forall i, x_i$ successfully transmits to y_i is $p = \mathrm{E}\left(exp\{-\sum_i c_i V(y_i)\}\right)$, where $V(y_i) = \sum_{X_j \in \Phi} S_j l(\|X_j - y_i\|)$. Hence, $\Phi_1 \leq_{ddcx} \Phi_2$ implies that $p_1 \leq p_2$.

COVERAGE PROCESSES

- Φ a point process. $U(x) = \max_{X_i \in \Phi}\{1[\|x - X_i\| \leq R]\}$
- $U(x)$ is the indicator fn. of x being covered by a point process and is the key quantity of analysis in the theory of coverage processes. (Hall, P., 1988)
- Comparison exists (Hall, P., 1988) for certain class of Cox processes with Poisson pp
- $\Phi_1 \leq_{idcv} \Phi_2$ implies that $\{U_1(x)\} \leq_{lo} \{U_2(x)\}$.

Further Questions

- Greater in *dcx* implies greater attraction for a point process. Notions of attractions aren't fully developed ??
- Better understanding of Palm versions ??
- Ordering induced by random measures as measure-valued rvs. This implies ordering as random fields. Converse ? No. May be something weaker ?? For convex ordering by Scarsini et al, 1991 ; Strong order - Equivalence was proved by Rolski et al, 1991.
- Comparison of Hard-Core point processes with Poisson processes ?? More generally dependent vs independent thinning/marking ??
- Cluster processes with differing base intensites and differing radii ?? Some other order possibly ??

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