Statistical models for exceedances under covariate information

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Overview

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Basic references

Falk, M., Hüsler, J. and Reiss, R.-D., Laws of Small Numbers: Extremes and Rare Events. Birkhäuser, Basel (1st ed. (1994), 2nd ed. (2004), 3rd ed. (2009/2010))

Reiss, R.-D. and Thomas, M., Statistical Analysis of Extreme Values, Birkhäuser, Basel (1st ed. (1997), 2nd ed. (2001), 3rd ed. (2007))

In SA, 3rd ed. (2007), Chapter 15: Environmental Sciences (co-authored by Rick Katz)

Univariate maxima and EVDs

The df of the maximum of iid random variables $X_1, \ldots, X_n$ with common df F is given by

$$P\left\{\max\{X_1, \ldots, X_n\} \leq x\right\} = F^n(x).$$

Max-Stability: A df F is max-stable if

$$F^n(d_n + c_n x) = F(x).$$

The max–stable dfs constitute the parametric family of *extreme value distributions (EVDs)* (consisting of Gumbel, Fréchet, Weibull dfs) with shape parameter γ. We have

$$G_\gamma(x) = \exp\left(-(1+\gamma x)^{-1/\gamma}\right).$$

EVDs are limiting dfs of sample maxima.

Univariate Exceedances and GPDs

The exceedance df at the threshold u of a random variable X with df F is given by

$$P(X \leq x | X > u) = \frac{F(x) - F(u)}{1 - F(u)} = F^{[u]}(x) \, .$$

POT-Stability: A df F is pot-stable if

$$F^{[u]}(b_u + a_u x) = F(x).$$

The possible pot–stable dfs constitute the parametric family of *generalized Pareto dfs (GPDs)* (consisiting of exponential, Pareto, beta dfs) with shape parameter γ. We have

$$W_\gamma(x) = 1 - (1 + \gamma x)^{-1/\gamma}, \quad x \geq 0.$$

The following relationship between EVDs G and GPDs W holds:

$$W = 1 + \log G, \quad \text{if } \log G > -1.$$

GPDs are limiting dfs of exceedance dfs.

The univariate Poisson process of exceedances

A full statistical model for exceedances above the threshold u is provided by the Poisson counting process given by

- a Poisson rv τ (the random number of exceedances above the threshold u)
- iid GP rvs $X_1, X_2, X_3, \ldots$ up to the number τ,

or, alternatively, given by a Poisson point process

$$\sum_{i=1}^{\tau} \varepsilon_{X_i}$$

Aims: we study

- multivariate EVDs, GPDs, and dfs deviating from GPDs,
- multivariate GPDs in complex stochastic systems.

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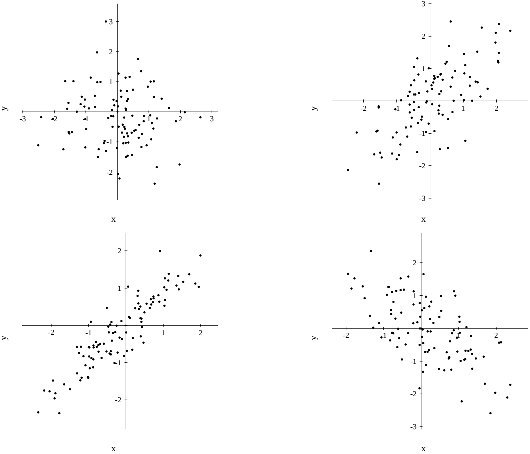
The notion of tail dependence

We say that there is upper **tail dependence** in data (x, y) if x and y are simultaneously large.

Discussion of bivariate normal samples
$(\rho = 0, 0.7, 0.9, -0.7)$



A measure for tail dependence

We deal with this question within the copula framework, that is, we study $[0,1]$-uniform rvs U and V.
As a measure of tail dependence consider the conditional probability

$$P(V > u | U > u) := P(U > u, V > u)/P(U > u)$$

Tail dependence parameter:

$$\chi = \lim_{u \uparrow 1} P(V > u | U > u).$$

We have **tail independence** if $\chi = 0$. In that case, we also study **rates of tail independence:**

$$P(V > u | U > u) \simeq (1-u)^{\beta}, \quad u \uparrow 1, \; \beta > 0. \qquad (1)$$

We call the exponent $\beta > 0$ in (1) the **residual tail dependence parameter**.

Residual tail dependence parameter

There is a relationship of the residual tail dependence parameter β to the **coefficient of tail dependence**

$$\bar{\chi} = \lim_{u \uparrow 1} \frac{2 \log P\{U > u\}}{\log P\{U > u, V > u\}} - 1.$$

We have

$$\beta = \frac{1 - \bar{\chi}}{1 + \bar{\chi}} \geq 0.$$

Example: Consider a copula normal random vectors $(U, V) = (\Phi(X), \Phi(Y))$ with correlation coefficient ρ. We have

$$\bar{\chi} = \rho\,.$$

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A representation of EVDs and GPDs

For EVDs G with univariate, exponential margins the **Pickands representation** is valid:

$$G_D(x,y) = \exp\left((x+y)D\left(\frac{x}{x+y}\right)\right), \quad (x,y) \leq 0,$$

where D is the **Pickands dependence function**.

For (X,Y) with EVD df G_D and Pickands dependence function D we have

- if $D(t) = 1$: independence of X, Y
- if $D(t) = \max(t, 1-t)$: total dependence of X, Y.

Again we study the pertaining **generalized Pareto distributions (GPDs)** which are given by

$$W_D = 1 + \log G_D, \quad \text{if } \log G_D > -1,$$

or modifications on appropriate supports $S(\boldsymbol{u})$.

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A spectral decomposition

We decompose a bivariate df H, defined on $(-\infty, 0) \times (-\infty, 0)$, into an array of certain univariate dfs by using the angular and radial components

$$z = x/(x+y) \quad \text{and} \quad c = x + y.$$

Rewriting

$$H(x,y) = H(cz, c(1-z)) =: H_z(c)$$

one gets a df in c for each fixed angle z (called **spectral decomposition** of H). Consider the **spectral densities**

$$h_z(c) = \frac{\partial}{\partial c} H_z(c).$$

Remark: (i) If $H = W_D$, then $h_z(c) = D(z)$.
(ii) If $H = G_D$, then $h_z(c) = D(z) + cD^2(z) + 0(c)$.

DFs deviating from GPDs

Condition 1:

Assume that the spectral densities h_z satisfy

$$h_z(c) = D(z) + B(c)A(z) + o(B(c)), \quad c \uparrow 0,$$

for some regularly varying B with exponent $\beta > 0$.

Remark: (i) β is again the residual tail dependence parameter if $D = 1$.
(ii) Roughly speaking, $B(c) = |c|^\beta$ in Condition 1.
(iii) If $D(z)$ is replaced by $a(z)$ then $a(z) = D(z)$.

Overview

Current research, Part I

(1) Testing tail dependence against residual tail dependence under Condition 1 (M. Frick, E. Kaufmann, R.-D. Reiss (2008), M. Frick and R.-D. Reiss (2009))

(2) Discriminant analysis in GPD models with particular emphasis laid on truncated multivariate normal distributions and limiting GPDs (with B.G. Manjunath, M. Frick)

(3) Piecing-together-methods for multivariate GPDs and a lot of other topics (M. Falk and his group, University of Würzburg)

Current research, Part II

(4) Under Condition 1 one gets for bivariate dfs H_{β_n} with

$$\beta_n \to 0 \quad \text{as } n \to \infty, \tag{2}$$

a limit theorem for maxima

$$H^n_{\beta_n}\left(\frac{x}{n}, \frac{x}{n}\right) \to \exp\left((x+y)(1+\lambda A\left(\frac{x}{x+y}\right)\right),$$

where λ depends on the speed of the convergence in (2), and a related result for exceedances (with M. Frick).

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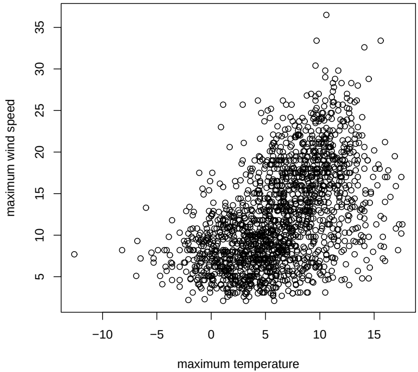
Data Example

Maximum daily winter wind speed and temperature in Aachen, Germany from 1991 to 2008.
We want to model the conditional distribution of the wind speed given the temperature in the tails.



Conditional quantiles

Let $(\boldsymbol{X}_i, Y_i)$ $i = 1, 2, \ldots$ iid, $(\boldsymbol{X}_1, Y_1) \sim (\boldsymbol{X}, Y)$ with values in $\mathbb{R} \times S$ and

$$F(y|\boldsymbol{x}) = P\left(Y \leq y|\boldsymbol{X} = \boldsymbol{x}\right). \tag{3}$$

Aim:
Estimation of conditional q–quantiles $F^{-1}(q|\boldsymbol{x})$ for $\boldsymbol{x} \in S$.

Estimation methods

1. non–parametric by moving sample quantiles
2. parametric model in the upper tail using GPDs

$$F(y|\boldsymbol{x}) = W_{\gamma_{\boldsymbol{x}},\mu_{\boldsymbol{x}},\sigma_{\boldsymbol{x}}}(y), \quad y > u$$

and estimating the parameters $\gamma_{\boldsymbol{x}}, \mu_{\boldsymbol{x}}, \sigma_{\boldsymbol{x}}$.

Parametric modeling

We assume that

$$F(y|\mathbf{x}) = W_{\gamma_{\theta,\mathbf{x}},\mu_{\theta,\mathbf{x}},\sigma_{\theta,\mathbf{x}}}(y), \quad y > u$$

where $\theta \in \Theta \subset \mathbb{R}^d$ is a parameter, for example if $S = \mathbb{R}$ one may choose

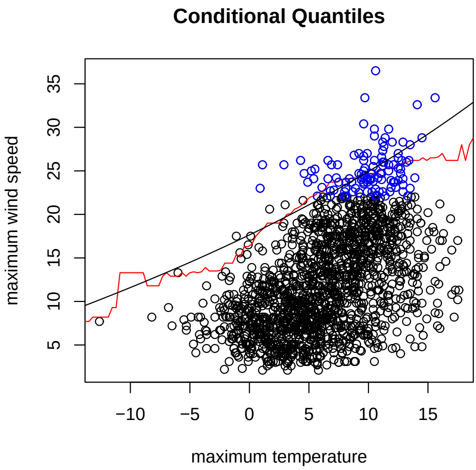
$$\gamma(x) \equiv \theta_1 \in \mathbb{R},$$

$$\sigma_x = \exp(\theta_2 + \theta_3 x), \quad \theta_2, \theta_3 \in \mathbb{R}$$

as well as

$$\mu_x = \theta_4 + \theta_5 x, \quad \theta_4, \theta_5 \in \mathbb{R}.$$

Results


Conditional Quantiles
maximum wind speed
maximum temperature
5
10
15
20
25
30
35
−10
−5
0
5
10
15

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Modeling by Poisson point processes

Let

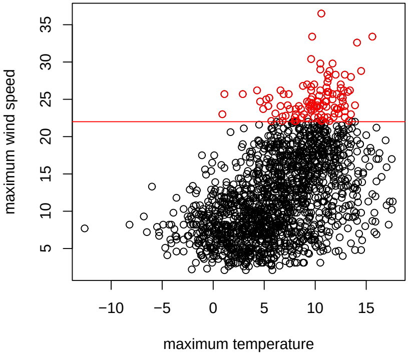
$$N = \sum_{i=1}^{\tau} \varepsilon_{(\boldsymbol{x}_i, Y_i)},$$

$\tau \sim \mathrm{P}_\lambda$ the Poisson point process of the observed data (on $T = S \times \mathbb{R}$). Define

$$N^{[S,u]} = N(\cdot \cap S \times (u, \infty))$$

the point process of exceedances over the threshold u and the pertaining covariates.

Point Process of Exceedances



The threshold is chosen as $u = 22m/s$ ($79.2km/h$) this yields 113 exceedances out of a total sample size of 1684

The Process of Exceedances

Now it holds that $N^{[S,u]}$ is again a Poisson Process

$$N^{[S,u]} \overset{d}{=} \sum_{i=1}^{\tau^*} \varepsilon_{(\boldsymbol{X}_i^*, Y_i^*)} \qquad (4)$$

where τ^* and $(\boldsymbol{X}_i^*, Y_i^*)$, $i \in \mathbb{N}$ are independent, τ^* is a Poisson random variable with parameter $\lambda^* = \lambda P\{Y > u\}$,

$$P(Y^* \leq y \,|\boldsymbol{X}^* = \boldsymbol{x}) = W^{[u]}_{\gamma_{\theta,\boldsymbol{x}},\mu_{\theta,\boldsymbol{x}},\sigma_{\theta,\boldsymbol{x}}}(u|\boldsymbol{x}) \qquad (5)$$

and

$$P\{\boldsymbol{X}^* \in B\} = P(\boldsymbol{X} \in B \,|Y > u)\,. \qquad (6)$$

A conditional approach

Let N and $N^{[S,u]}$ be as before. Let π_1, be the projection mapping

$$\pi_1\left(\sum_{i=1}^{n} \varepsilon_{(\boldsymbol{x}_i, y_i)}\right) = \sum_{i=1}^{n} \varepsilon_{\boldsymbol{x}_i}$$

and define

$$N_1 = \pi_1(N).$$

Likelihoods

- Let $(\boldsymbol{x}_1, y_1), \ldots, (\boldsymbol{x}_n, y_n)$ be the observed data
- $(\boldsymbol{x}_1^*, y_1^*), \ldots, (\boldsymbol{x}_k^*, y_k^*)$ the pertaining "exceedances"
- $\tilde{\boldsymbol{x}}_1, \ldots, \tilde{\boldsymbol{x}}_{n-k}$ the covariates belonging to y-values smaller then u.
- let $\boldsymbol{\eta} = \sum_{i=1}^k \varepsilon_{(\boldsymbol{x}_i^*, y_i^*)}$ and $\mu = \sum_{i=1}^n \varepsilon_{\boldsymbol{x}_i}$

First approach: based on a density of $\mathcal{L}\left(N^{[S,u]}\right)$

$$\begin{aligned} l_{\boldsymbol{\eta}}(\theta) &= \prod_{i=1}^k w_{\gamma_{\boldsymbol{x}_i^*,\theta}, \mu_{\boldsymbol{x}_i^*,\theta}, \sigma_{\boldsymbol{x}_i^*,\theta}}(y_i^*) \\ &\quad \cdot \exp\left(\lambda - \lambda \int W_{\gamma_{\boldsymbol{x},\theta}, \mu_{\boldsymbol{x},\theta}, \sigma_{\boldsymbol{x},\theta}}(u) \, d\mathcal{L}(\boldsymbol{X})(\boldsymbol{x})\right). \end{aligned}$$

Second approach: based on a density of $P\left(N^{[S,u]} \in \cdot | N_1 = \mu\right)$

$$l_{\boldsymbol{\eta},\mu}(\theta) = \prod_{i=1}^{n-k} W_{\gamma_{\theta,\tilde{\boldsymbol{x}}_i}, \mu_{\theta,\tilde{\boldsymbol{x}}_i}, \sigma_{\theta,\tilde{\boldsymbol{x}}_i}}(u) \prod_{i=1}^k w_{\gamma_{\theta,\boldsymbol{x}_i^*}, \mu_{\theta,\boldsymbol{x}_i^*}, \sigma_{\theta,\boldsymbol{x}_i^*}}(y_i^*)$$

Simulations

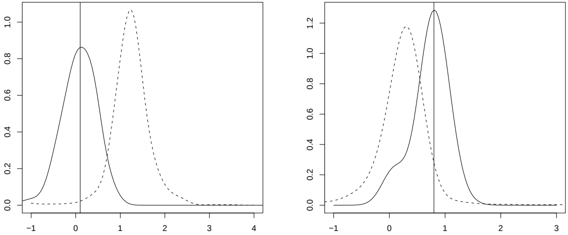
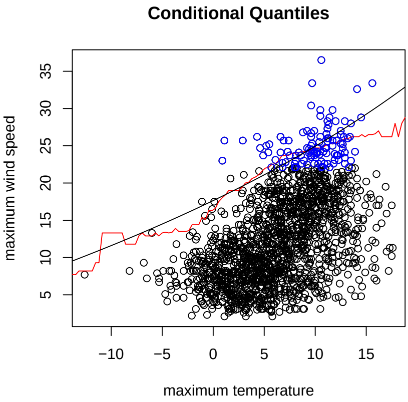


Figure: Kernel densities of simulated estimators for θ_2 and θ_3, first approach (dashed) and second approach (solid).

95 % conditional Quantiles

Parameter estimates: $\gamma_{\boldsymbol{x}} = -0.12$, $\sigma_{\boldsymbol{x}} = \exp(1.12 + 0.04\boldsymbol{x})$, $\mu_{\boldsymbol{x}} = 9.93 + 0.35\boldsymbol{x}$



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