Scaling exponents for a one-dimensional directed polymer

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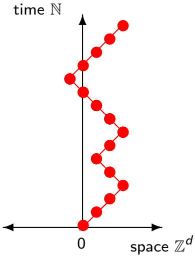
Directed polymer in a random environment



simple random walk path $(x(t), t)$, $t \in \mathbb{Z}_+$

space-time environment $\{\omega(x,t) : x \in \mathbb{Z}^d,\ t \in \mathbb{N}\}$

inverse temperature $\beta > 0$

partition function $Z_n = \sum_{x(\cdot)} \exp\Big\{\beta \sum_{t=1}^{n} \omega(x(t),t)\Big\}$

summed over all n-paths

quenched path measure $\quad Q_n\{x(\cdot)\} = \dfrac{1}{Z_n} \exp\Big\{\beta \sum_{t=1}^{n} \omega(x(t),t)\Big\}$

$\mathbb{P}$ probability distribution of ω, often $\{\omega(x,t)\}$ i.i.d.

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General question: Behavior of model as $\beta > 0$ and dimension d vary. Especially whether $x(\cdot)$ is diffusive or not, that is, does it scale like standard RW.

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1988 Imbrie and Spencer: $n^{-1} E^Q(|x(n)|^2) \to c \quad \mathbb{P}$-a.s.

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This talk is about scaling exponents in $d = 1$.

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Our result: these exact exponents for a model with a particular weight distribution and properly chosen boundary conditions.

Earlier results for $d = 1$ exponents

Past rigorous bounds give $3/5 \leq \zeta \leq 3/4$ and $\chi \geq 1/8$:

- Brownian motion in Poissonian potential: Wüthrich 1998, Comets and Yoshida 2005.
- Gaussian RW in Gaussian potential: Petermann 2000 $\zeta \geq 3/5$, Mejane 2004 $\zeta \leq 3/4$
- Piza 1997: in general, fluctuations of $\log Z_n$ diverge
- Licea, Newman, Piza 1995-96: corresponding results for first passage percolation

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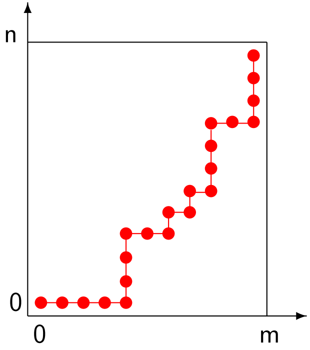
Continuum model: $\zeta = 2/3$ and $\chi = 1/3$ proved for "Hopf-Cole solution of KPZ equation" (Balázs, Quastel, S. 2009)

Polymer in first quadrant with fixed endpoints



For our purposes, we turn the picture 45 degrees so that polymer becomes an up-right path from $(0,0)$ to (m,n) in $\mathbb{Z}_+^2$.

Polymer in first quadrant with fixed endpoints

- $\beta > 0$
- environment $\omega = (\omega(i,j) : (i,j) \in \mathbb{Z}_+^2)$
- $\Pi_{m,n} = \{$ up-right paths (x_k) from $(0,0)$ to (m,n) $\}$
- $Z_{m,n} = \sum_{x_\centerdot \in \Pi_{m,n}} \exp\Big\{\beta \sum_{k=1}^{m+n} \omega(x_k)\Big\}$
- $Q_{m,n}(x_\centerdot) = \frac{1}{Z_{m,n}} \exp\Big\{\beta \sum_{k=1}^{m+n} \omega(x_k)\Big\}$

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- $Z_{m,n} = e^{\beta\omega(m,n)}(Z_{m-1,n} + Z_{m,n-1})$

Last-passage percolation

$\beta \nearrow \infty$ limit: Q concentrates on paths that maximize $\sum_{k=1}^{m+n} \omega(x_k)$.

Analogue of $\log Z_{m,n}$ is last-passage time

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Correct exponents and Tracy-Widom limits proved for $\omega(i,j) \sim$ Exp or Geom, and for a handful of other last-passage models. [Baik, Balázs, Ben-Arous, Borodin, Cator, Deift, Ferrari, Gravner, Groeneboom, Johansson, Péché, Prähofer, Spohn, Tracy, Widom, ...]

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This observation informs the work. We look for a polymer model that satisfies an analogue of Burke's Theorem for queues, and then build on that to confirm the exponents, as was done for last-passage percolation in Balázs-Cator-S (EJP, 2006).

Polymer with multiplicative weights

The notation for our results:

- Fix $\beta = 1$. $Y_{i,j} = e^{\omega(i,j)}$ independent.
- Environment $\omega = (Y_{i,j} : (i,j) \in \mathbb{Z}_+^2)$ with distribution $\mathbb{P}$.

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- $Z_{m,n} = \sum_{x_\bullet \in \Pi_{m,n}} \prod_{k=1}^{m+n} Y_{x_k} \qquad Q_{m,n}(x_\bullet) = \frac{1}{Z_{m,n}} \prod_{k=1}^{m+n} Y_{x_k}$
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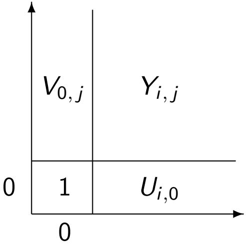
- $P_{m,n}(x_\bullet) = \mathbb{E} Q_{m,n}(x_\bullet)$
- Boundary weights also $U_{i,0} = Y_{i,0}$ and $V_{0,j} = Y_{0,j}$.

Polymer with boundary, Gamma^{-1} multiplicative weights



Initial weights $(i, j \in \mathbb{N})$:

$U_{i,0}^{-1} \sim \text{Gamma}(\theta, 1)$

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- parameters $0 < \theta < \mu$
- Gamma$(\theta, 1)$ density: $\Gamma(\theta)^{-1} x^{\theta-1} e^{-x}$ on $\mathbb{R}_+$

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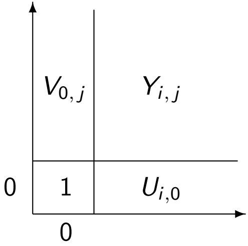

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- parameters $0 < \theta < \mu$
- Gamma(θ, 1) density: $\Gamma(\theta)^{-1} x^{\theta-1} e^{-x}$ on $\mathbb{R}_+$
- $\mathbb{E}(\log U) = -\Psi_0(\theta)$ and $\mathbb{V}\text{ar}(\log U) = \Psi_1(\theta)$
- $\Psi_n(s) = (d^{n+1}/ds^{n+1}) \log \Gamma(s)$, $\quad \Psi_1 > 0$

Main results with boundary: variance bounds for $\log Z$

With $0 < \theta < \mu$ fixed and $N \nearrow \infty$ assume

$$|\, m - N\Psi_1(\mu - \theta)\,| \le CN^{2/3} \quad \text{and} \quad |\, n - N\Psi_1(\theta)\,| \le CN^{2/3} \quad (1)$$

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Theorem

For (m, n) as in (1), $\quad C_1 N^{2/3} \le \mathbb{V}\mathrm{ar}(\log Z_{m,n}) \le C_2 N^{2/3}$.

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Theorem

Suppose $n = \Psi_1(\theta)N$ and $m = \Psi_1(\mu - \theta)N + \gamma N^{\alpha}$ with $\gamma > 0$, $\alpha > 2/3$. Then

$$N^{-\alpha/2}\Big\{\log Z_{m,n} - \mathbb{E}\big(\log Z_{m,n}\big)\Big\} \;\Rightarrow\; \mathcal{N}\big(0, \gamma\Psi_1(\theta)\big)$$

Main results with boundary: fluctuation bounds for path

$v_0(j)$ = leftmost, $v_1(j)$ = rightmost point of $x_\cdot$ on horizontal line:

$$v_0(j) = \min\{i \in \{0, \dots, m\} : \exists k : x_k = (i, j)\}$$

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Theorem

Assume (m, n) as in (1), $0 < \tau < 1$. Then

(a) $P\Big\{ v_0(\lfloor \tau n \rfloor) < \tau m - bN^{2/3} \ \text{ or } \ v_1(\lfloor \tau n \rfloor) > \tau m + bN^{2/3} \Big\} \le \dfrac{C}{b^3}$

(b) $\forall \varepsilon > 0 \ \exists \delta > 0$ such that

$$\varlimsup_{N \to \infty} P\{ \ \exists k \text{ such that } |\, x_k - (\tau m, \tau n)| \le \delta N^{2/3} \ \} \le \varepsilon.$$

Main results without boundary

Start paths from $(1, 1)$ to use only bulk weights, no boundary.

- $\Pi_{(1,1),(m,n)} = \{$ up-right paths from $(1, 1)$ to (m, n) $\}$
- $Z_{(1,1),(m,n)} = \sum_{x_\cdot \in \Pi_{(1,1),(m,n)}} \prod_{k=1}^{m+n} Y_{x_k}, \qquad Y^{-1} \sim \text{Gamma}(\mu, 1)$

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Define $f_{s,t}(\mu) = -\big(s\Psi_0(\theta) + t\Psi_0(\mu-\theta)\big)$.

Upper bounds extend to model without a boundary.

Main results without boundary

Theorem

(a) Law of large numbers:

$$\lim_{N\to\infty} N^{-1} \log Z_{(1,1),(\lfloor Ns\rfloor,\lfloor Nt\rfloor)} \;=\; f_{s,t}(\mu) \quad \mathbb{P}\text{-a.s.}$$

(b) Upper bound for $\log Z$: for $b \geq 2$ and $N \geq N_0$,

$$\mathbb{P}\Big\{ \,|\log Z_{(1,1),(\lfloor Ns\rfloor,\lfloor Nt\rfloor)} - Nf_{s,t}(\mu)| \geq bN^{1/3} \Big\} \leq Cb^{-3/2}.$$

(c) Upper bound for path: for $0 < \tau < 1$.

$$P\Big\{v_0(\lfloor \tau Nt\rfloor) < \tau Ns - bN^{2/3} \;\text{ or }\; v_1(\lfloor \tau Nt\rfloor) > \tau Ns + bN^{2/3}\Big\} \leq \frac{C}{b^3}$$

Proving results without boundary

The choice of θ s.t. $\Psi_1(\mu-\theta)/\Psi_1(\theta) = s/t$

- makes the $Ns \times Nt$ rectangle characteristic for θ, μ
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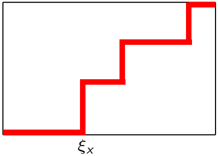

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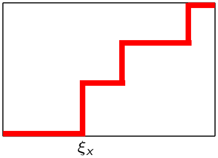

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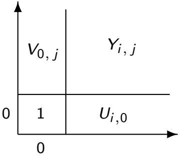
As a consequence $Z((1,1),(\lfloor Ns \rfloor, \lfloor Nt \rfloor))$ without boundary is approximated well enough by $Z(\lfloor Ns \rfloor, \lfloor Nt \rfloor)$ with $(\theta, \mu - \theta)$ parameters on the boundary.

Burke property for model with boundary



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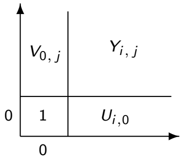

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Compute $Z_{m,n}$ for all $(m, n) \in \mathbb{Z}_+^2$ and then define

$$U_{m,n} = \frac{Z_{m,n}}{Z_{m-1,n}} \qquad V_{m,n} = \frac{Z_{m,n}}{Z_{m,n-1}} \qquad X_{m,n} = \left(\frac{Z_{m,n}}{Z_{m+1,n}} + \frac{Z_{m,n}}{Z_{m,n+1}} \right)^{-1}$$

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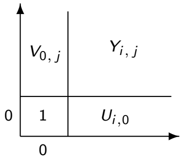

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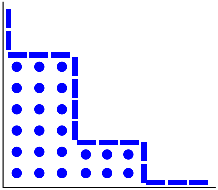
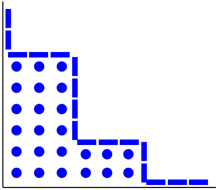
For an undirected edge f: $$T_f = \begin{cases} U_x & f = \{x - e_1, x\} \\ V_x & f = \{x - e_2, x\} \end{cases}$$



down-right path (z_k) with edges $f_k = \{z_{k-1}, z_k\}$, $k \in \mathbb{Z}$

interior points $\mathcal{I}$ of path (z_k)



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Theorem

Variables $\{T_{f_k}, X_z : k \in \mathbb{Z},\, z \in \mathcal{I}\}$ are independent with marginals
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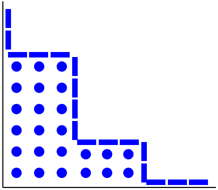
and $X^{-1} \sim \text{Gamma}(\mu, 1)$.



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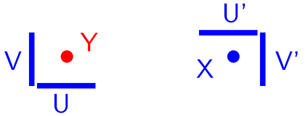
"Burke property" because the analogous property for last-passage is a generalization of Burke's Theorem for M/M/1 queues, via the last-passage representation of M/M/1 queues in series.

Proof of Burke property

Induction on $\mathcal{I}$ by flipping a growth corner:

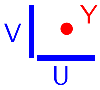


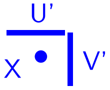
$$U' = Y(1 + U/V) \qquad V' = Y(1 + V/U)$$

$$X = \left(U^{-1} + V^{-1}\right)^{-1}$$

Proof of Burke property

Induction on $\mathcal{I}$ by flipping a growth corner:





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Lemma. Given that (U, V, Y) are independent positive r.v.'s, $(U', V', X) \stackrel{d}{=} (U, V, Y)$ iff (U, V, Y) have the gamma distr's.

Proof "if" part by computation, "only if" part from a characterization of gamma due to Lukacs (1955). □

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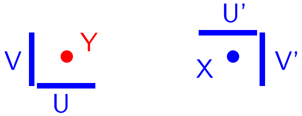

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This gives all (z_k) with finite $\mathcal{I}$. General case follows.

Proof of off-characteristic CLT

Recall that $\begin{cases} n = \Psi_1(\theta)N \\ m = \Psi_1(\mu-\theta)N + \gamma N^{\alpha} \end{cases}$ $\gamma > 0$, $\alpha > 2/3$.

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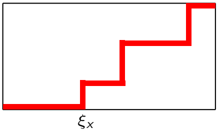
Set $m_1 = \lfloor \Psi_1(\mu-\theta)N \rfloor$. Since $Z_{m,n} = Z_{m_1,n} \cdot \prod_{i=m_1+1}^{m} U_{i,n}$

$$N^{-\alpha/2}\,\overline{\log Z_{m,n}} \;=\; N^{-\alpha/2}\,\overline{\log Z_{m_1,n}} \;+\; N^{-\alpha/2}\sum_{i=m_1+1}^{m}\overline{\log U_{i,n}}$$

First term on the right is $O(N^{1/3-\alpha/2}) \to 0$. Second term is a sum of order N^α i.i.d. terms. $\square$

Variance identity



Exit point of path from x-axis

$$\xi_x = \max\{k \geq 0 : x_i = (i, 0) \text{ for } 0 \leq i \leq k\}$$

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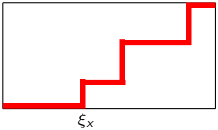

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For $\theta, x > 0$ define positive function

$$L(\theta, x) = \int_0^x \big(\Psi_0(\theta) - \log y\big) x^{-\theta} y^{\theta-1} e^{x-y}\, dy$$

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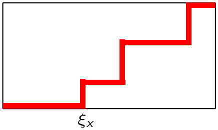

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Theorem. For the model with boundary,

$$\mathbb{V}\mathrm{ar}\big[\log Z_{m,n}\big] = n\Psi_1(\mu - \theta) - m\Psi_1(\theta) + 2\, E_{m,n}\Big[\sum_{i=1}^{\xi_x} L(\theta, Y_{i,0}^{-1})\Big]$$

Sketch of proof:

$$N = \log Z_{m,n} - \log Z_{0,n}$$

$$W = \log Z_{0,n} \qquad E = \log Z_{m,n} - \log Z_{m,0}$$

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$$\begin{aligned}
\mathbb{V}\mathrm{ar}\big[\log Z_{m,n}\big] &= \mathbb{V}\mathrm{ar}(W + N) \\
&= \mathbb{V}\mathrm{ar}(W) + \mathbb{V}\mathrm{ar}(N) + 2\,\mathbb{C}\mathrm{ov}(W, N) \\
&= \mathbb{V}\mathrm{ar}(W) + \mathbb{V}\mathrm{ar}(N) + 2\,\mathbb{C}\mathrm{ov}(S + E - N, N) \\
&= \mathbb{V}\mathrm{ar}(W) - \mathbb{V}\mathrm{ar}(N) + 2\,\mathbb{C}\mathrm{ov}(S, N) \qquad (E, N \text{ ind.}) \\
&= n\Psi_1(\mu - \theta) - m\Psi_1(\theta) + 2\,\mathbb{C}\mathrm{ov}(S, N).
\end{aligned}$$

To differentiate w.r.t. parameter θ of S while keeping W fixed, introduce a separate parameter ρ $(= \mu - \theta)$ for W.

$$-\mathbb{C}\mathrm{ov}(S, N) = \frac{\partial}{\partial\theta}\mathbb{E}(N) = \widetilde{\mathbb{E}}\Big[\frac{\partial}{\partial\theta}\log Z_{m,n}(\theta)\Big]$$

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Differentiate: $\quad \frac{\partial}{\partial\theta}\log Z_{m,n}(\theta) \;=\; -\,E^{Q_{m,n}}\Big[\sum_{i=1}^{\xi_x} L(\theta, Y_{i,0}^{-1})\Big].$

Together:

$$\begin{aligned}\mathbb{V}\mathrm{ar}\big[\log Z_{m,n}\big] &= n\Psi_1(\mu-\theta) - m\Psi_1(\theta) + 2\,\mathbb{C}\mathrm{ov}(S,N)\\ &= n\Psi_1(\mu-\theta) - m\Psi_1(\theta) + 2\,E_{m,n}\Big[\sum_{i=1}^{\xi_x} L(\theta, Y_{i,0}^{-1})\Big].\end{aligned}$$

This was the claimed formula. □

Sketch of upper bound proof

The argument develops an inequality that controls both $\log Z$ and ξ_x simultaneously. Introduce an auxiliary parameter $\lambda = \theta - bu/N$. The weight of a path $x_\centerdot$ such that $\xi_x > 0$ satisfies

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Since $H_\lambda(\eta) \le H_\theta(\eta)$,

$$Q^{\theta,\omega}\{\xi_x \ge u\} = \frac{1}{Z(\theta)} \sum_{x_\bullet} \mathbf{1}\{\xi_x \ge u\} W(\theta) \;\le\; \frac{Z(\lambda)}{Z(\theta)} \cdot \prod_{i=1}^{\lfloor u \rfloor} \frac{H_\lambda(\eta_i)}{H_\theta(\eta_i)}.$$

For $1 \le u \le \delta N$ and $0 < s < \delta$,

$$\mathbb{P}\Big[Q^\omega\{\xi_x \ge u\} \ge e^{-su^2/N}\Big] \le \mathbb{P}\bigg\{ \prod_{i=1}^{\lfloor u \rfloor} \frac{H_\lambda(\eta_i)}{H_\theta(\eta_i)} \ge \alpha \bigg\} + \mathbb{P}\bigg(\frac{Z(\lambda)}{Z(\theta)} \ge \alpha^{-1} e^{-su^2/N} \bigg).$$

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Choose α right. Bound these probabilities with Chebychev which brings $\mathbb{V}\mathrm{ar}(\log Z)$ into play. In the characteristic rectangle $\mathbb{V}\mathrm{ar}(\log Z)$ can be bounded by $E(\xi_x)$. The end result is this inequality:

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Handle $u \ge \delta N$ with large deviation estimates. In the end, integration gives the moment bounds.