Random walks in dynamic random environment: LLN and LDP

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joint work with Frank den Hollander and Frank Redig

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Outline of the talk

- Introduction and Model.

- Law of large numbers.

- Large deviations and slow-down phenomenon.

RWRE in a Nutshell

A Random Walk in a Random Environment (RWRE) on $\mathbb{Z}^d$ is a Markov chain/process with random transition probabilities / rates , namely

$$p(x,y \mid \xi) = \text{probability} \quad \text{to go from } x \text{ to } y$$

depends on a random field or random process ξ called Random Environment (RE).

Goal: to extend the theory of ”classical” random walk, i.e., LLN, CLT, LDP, etc...

***Static* RE:** when ξ is chosen at random at time 0 and kept fixed throughout the time evolution.
In $d = 1$ completely solved, in $d \geq 2$ many open problems.

***Dynamic* RE:** when ξ changes in time according to some stochastic dynamics.
Very little known, even in $d = 1$.

Model: dynamic random environment

Let
$$\xi = \left\{ \xi(x,t) : \ x \in \mathbb{Z}, t \geq 0 \right\}$$

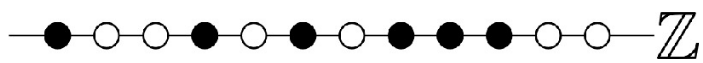
be a one-dimensional Interacting Particle System (IPS) on the integer lattice, namely, a Markov process with state space $\Omega = \{0,1\}^{\mathbb{Z}}$.

$$\xi(x,t) = 1 \ \leftrightarrow \textit{Particle at site } x \textit{ at time } t,$$

$$\xi(x,t) = 0 \ \leftrightarrow \textit{Vacancy at site } x \textit{ at time } t.$$

Examples: Independent spin-flips , stochastic Ising model, contact process, exclusion process, voter model, majority vote model.

For $\eta \in \Omega = \{0,1\}^{\mathbb{Z}}$, P^{η} represents the law of ξ starting from η.



Given a measure μ on $\Omega = \{0,1\}^{\mathbb{Z}}$, we denote by

$$P^{\mu}(\cdot) = \int_{\Omega} P^{\eta}(\cdot)\,\mu(d\eta).$$

the law of ξ starting from μ.

Assumption: ξ has an equilibrium measure μ that is shift-invariant and shift-ergodic.

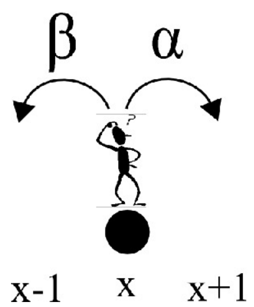
Model: random walk

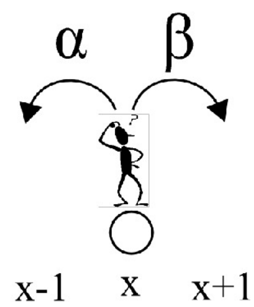
Given a realization of ξ, let $X = \{X_t\}_{t\geq 0}$, $X_0 = 0$, be the continuous-time nearest-neighbor random walk with transition rates

$$x \to x+1 \quad \text{at rate} \quad \alpha\xi(x,t) + \beta[1-\xi(x,t)]$$

$$x \to x-1 \quad \text{at rate} \quad \beta\xi(x,t) + \alpha[1-\xi(x,t)],$$

where $\alpha, \beta > 0$. Without loss of generality we may assume $\alpha > \beta$.





local drift to the right on particles

local drift to the left on holes.

Quenched Law

$P_0^{\xi}(X_t \in \cdot)$= law of X_t starting from 0 conditional on the environment ξ.

Annealed Law

$$\mathbb{P}_{\mu,0}(X_t \in \cdot) = \int_{D_\Omega[0,\infty)} P^{\mu}(d\xi)\, P_0^{\xi}(X_t \in \cdot)$$

= law of X_t averaged over the environment ξ,

where $D_\Omega[0,\infty)$ is the set of càdlàg paths in $\Omega = \{0,1\}^{\mathbb{Z}}$.

Law of Large Numbers (LLN)

Theorem 1 (LLN) Suppose that ξ is cone-mixing. Then there exists a deterministic speed $v \in \mathbb{R}$ such that

$$\lim_{t \to \infty} \frac{X_t}{t} = v \qquad P^{\mu} - a.s.$$

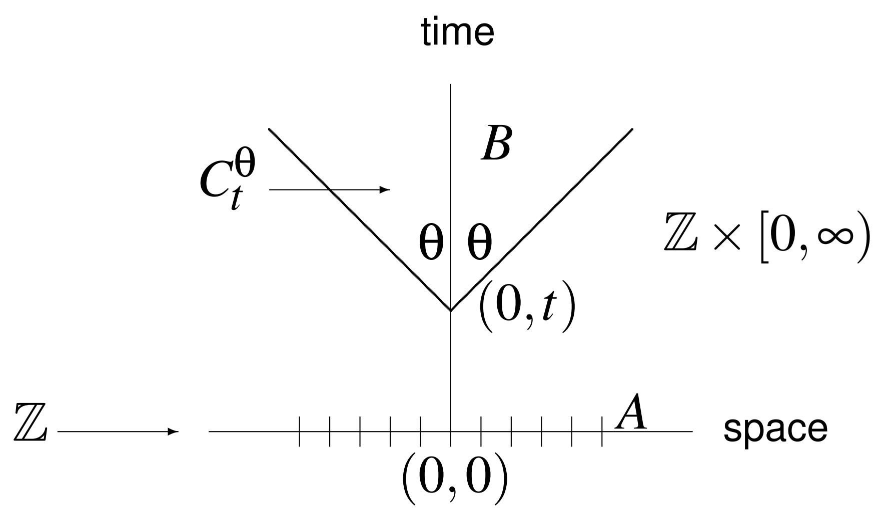
Remark 1: For small local drift $\alpha - \beta$, and exponentially mixing environments, we have an explicit formula for the speed (see F. Redig's talk).

Remark 2: Theorem 1 can be extended to higher dimension, provided the total jump rate of the walker is the same on top of particles and holes.

Definition: A random environment ξ is called cone-mixing if, for all $\theta \in (0, \frac{1}{2}\pi)$,

$$\lim_{t \to \infty} \sup_{\substack{A \in \mathcal{F}_0, B \in \mathcal{F}_t^\theta \\ \mu(A) > 0}} \left| P^\mu(B \mid A) - P^\mu(B) \right| = 0,$$

where $\mathcal{F}_0 = \sigma\{\xi_0(x)\colon\ x \in \mathbb{Z}\}$ and $\mathcal{F}_t^\theta = \sigma\{\xi_s(x)\colon\ (x,s) \in C_t^\theta\}$, with C_t^θ the cone in the picture:



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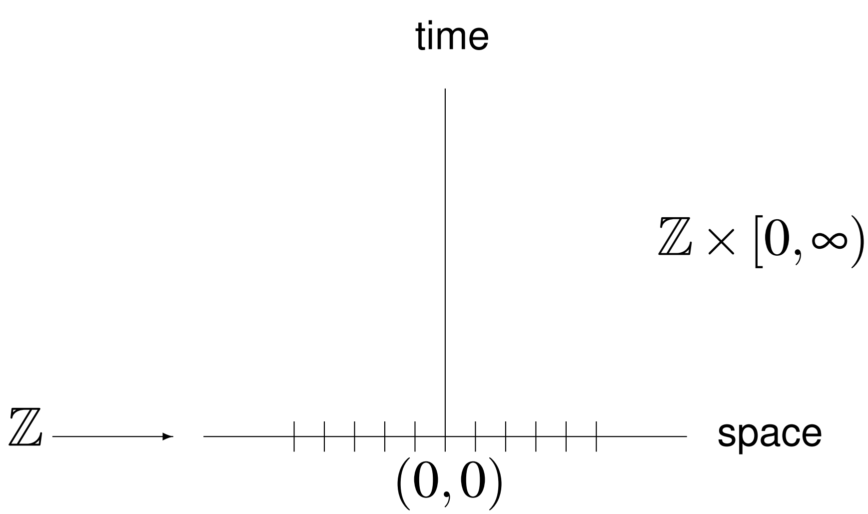

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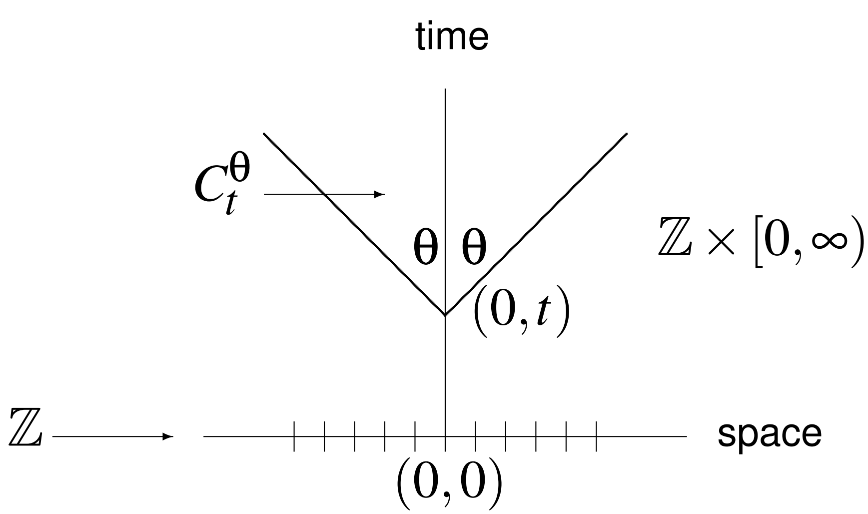

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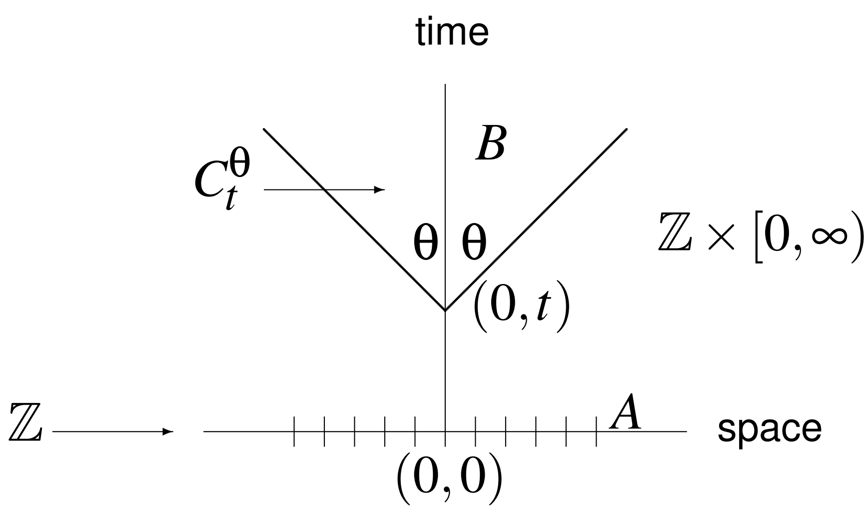

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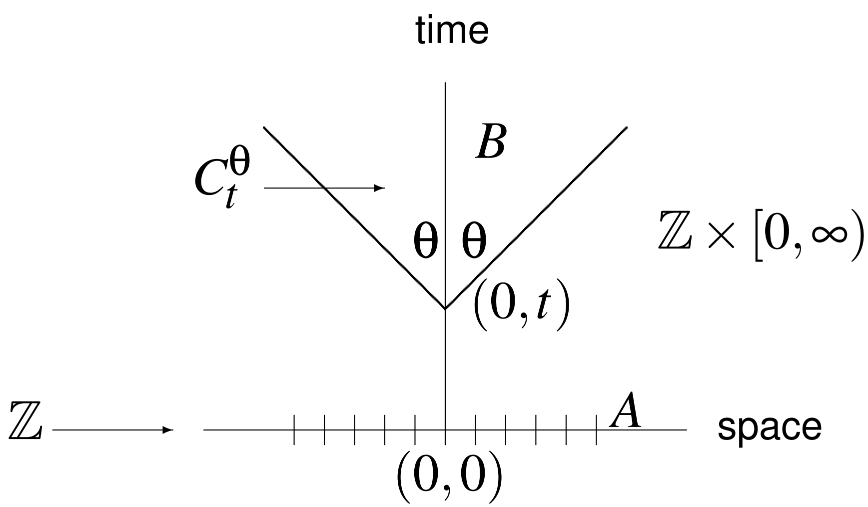

Examples of cone-mixing environments: Independent spin-flips, uniquely ergodic attractive spin systems, IPS under the $M < \varepsilon-$condition, two-dimensional Gibbs random field embedded in space-time.

Ingredients to prove the LLN

We adapt and simplify a regeneration-time argument originally developed by F. Comets and O. Zeitouni for static Random Environment (RE).

$\star$ we consider a discrete-time version X_n of our walker on $\mathbb{Z}$ in *dynamic* RE, and show that it is equivalent to a random walk $Y_n = (X_n, n)$ on $\mathbb{Z}^2$ in a *static* RE.

$\star$ the cone-mixing property of the environment allows us to construct a sequence of regeneration times, i.e., a sequence of times at which the walker Y sees the environment almost as freshly sampled from the equilibrium distribution μ.

$\star$ the increments of Y_n along this sequence of times can be treated as an i.i.d. sequence plus some small extra noise. This allows us to prove the LLN for Y_n, which in turn implies the LLN for X_n.

$\star$ we pass from discrete to continuous time afterwards, via a limiting argument.

Large Deviation Principle (LDP): attractive spin-flip systems

Definition: An IPS ξ is called a spin-flip system if only one coordinate changes at each transition, i.e., is a Markov process on $\{0,1\}^{\mathbb{Z}}$ with generator

$$(Lf)(\eta) = \sum_{x \in \mathbb{Z}} c(x,\eta)[f(\eta^x) - f(\eta)],$$

where

$\eta \in \{0,1\}^{\mathbb{Z}}$, $f =$ local function, $\eta^x = \eta$ flipped at x,
$c(x,\eta) =$ flip rate at site x in the configuration η.

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$$(Lf)(\eta) = \sum_{x\in\mathbb{Z}} c(x,\eta)[f(\eta^x) - f(\eta)].$$

Definition: A spin-flip system ξ is called attractive if, whenever $\eta \le \zeta$,

$$\begin{aligned} c(x,\eta) &\le c(x,\zeta) \quad \text{if} \quad \eta(x) = \zeta(x) = 0, \\ c(x,\eta) &\ge c(x,\zeta) \quad \text{if} \quad \eta(x) = \zeta(x) = 1, \quad \text{for all } x \in \mathbb{Z}. \end{aligned}$$

Attractive Idea: The evolution tends to make a coordinate agree with its neighborhood.

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Attractive Idea: The evolution tends to make a coordinate agree with its neighborhood.

Examples of attractive spin-flip systems: Independent spin-flips, ferromagnetic stochastic Ising model, contact process, voter model, majority vote model.

LDP for the Empirical Speed

Informally: We say that the family $\mathbb{P}(X_t/t \in \cdot\,)$, $t > 0$, satisfies a Large Deviation Principle (LDP) with rate t and with rate function $I\colon\ \mathbb{R} \to [0, \infty)$ if

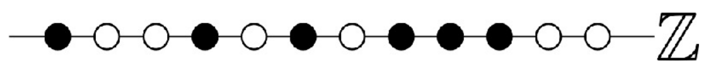
$$\exists \lim_{t\to\infty} -\frac{1}{t} \log \mathbb{P}\left(\frac{X_t}{t} \simeq \theta\right) = I(\theta) \qquad \forall \theta \in \mathbb{R},$$

$$\text{i.e. ,} \quad \mathbb{P}\left(\tfrac{X_t}{t} \simeq \theta\right) \approx e^{-tI(\theta)} \text{ as } t \to \infty.$$

Remark: The rate function for “classical“ random walks is zero at the typical speed v and positive elsewhere.

LDP for *static* Random Environment

***Static* RE:** $\xi \in \{0,1\}^{\mathbb{Z}}$ distributed according to a Bernoulli product measure ν_ρ with parameter $\rho \in (0,1)$.



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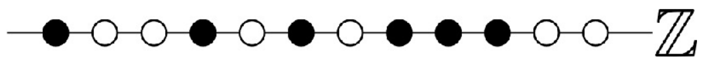

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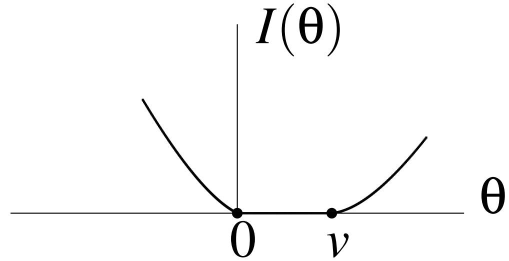
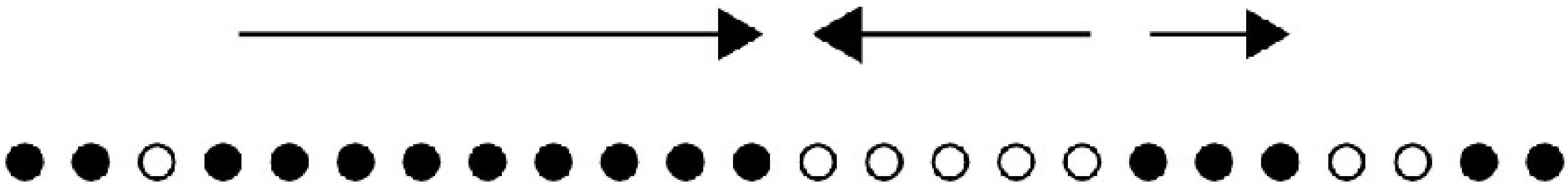
Slow-down phenomenon

When $v > 0$, the rate function is zero on $[0,v]$, namely , $\mathbb{P}\left(\frac{X_t}{t} \simeq \theta\right) \approx e^{-o(t)}$.



Traps

The slow-down phenomenon is due to the presence of traps.



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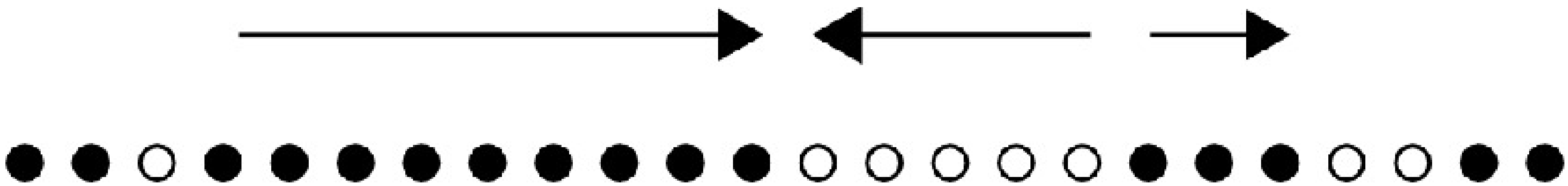

Even though globally more particles than holes, presence of long strings of holes significantly slowes down the walker.

Does this slow-down survive in dynamic random environments?

LDP for the Empirical Speed: attractive spin-flip systems

Theorem 2 (Annealed LDP) The family $\mathbb{P}_{\mu,0}(X_t/t \in \cdot\,)$, $t > 0$, satisfies the LDP with rate t and with convex rate function $I^{\text{ann}}\colon \mathbb{R} \to [0,\infty)$ satisfying

$$I^{\text{ann}}(\theta) \begin{cases} = 0, & \text{if } \theta \in [v^-, v^+], \\ > 0, & \text{if } \theta \in \mathbb{R} \setminus [v^-, v^+], \end{cases}$$

for some $-(\alpha-\beta) \le v^- \le v \le v^+ \le \alpha - \beta$.

Moreover, for small drift $\alpha - \beta$ and exponentially mixing environments ,

$$v^- = v = v^+ \quad (\text{no slow-down phenomenon}).$$

Main ingredient of the proof: attractivness $\to$ FKG, subadditivity.

Theorem 3 (Quenched LDP) For P^{μ}-a.e. ξ, the family $P_0^{\xi}(X_t/t \in \cdot\,), t > 0$, satisfies the LDP with rate t and with deterministic convex rate function $I^{\mathrm{que}}\colon \mathbb{R} \to [0,\infty)$ satisfying

$$I^{\mathrm{que}}(\theta) \begin{cases} = 0, & \text{if } \theta \in [v^-, v^+], \\ > 0, & \text{if } \theta \in \mathbb{R} \backslash [v^-, v^+], \end{cases}$$

for some $-(\alpha - \beta) \leq v^- \leq v \leq v^+ \leq \alpha - \beta$.

Furthrtmore, $I^{\mathrm{que}} \geq I^{\mathrm{ann}}$, and the following symmetry relation holds:

$$I^{\mathrm{que}}(-\theta) = I^{\mathrm{que}}(\theta) + \theta(2\rho - 1)\log(\alpha/\beta), \qquad \theta \geq 0.$$

Moreover, for small drift $\alpha - \beta$ and exponentially mixing environments:

$$v^- = v = v^+ (\text{no slow-down phenomenon}).$$

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No, if the dynamic random environment is not ”fast mixing“, then slow-down can survive.

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We can prove this when ξ is Simple Symmetric Exclusion (SSE).

RW on Simple Symmetric Exclusion (SSE)

Definition: an IPS ξ evolves according to a SSE dynamics if each particle performs an independent simple symmetric random walk at rate 1 such that jumps to occupied sites are suppressed.

Equilibrium measures: Bernoulli product measures ν_ρ with parameter $\rho \in (0, 1)$.

RW on Simple Symmetric Exclusion (SSE)

Definition: an IPS ξ evolves according to a SSE dynamics if each particle performs an independent simple symmetric random walk at rate 1 such that jumps to occupied sites are suppressed.
Equilibrium measures: Bernoulli product measures ν_ρ with parameter $\rho \in (0,1)$.

Unfortunately: SSE is not cone-mixing and so we do not have LLN and LDP.
We performed simulations showing that LLN does hold with speed $v > 0$ as soon as $\rho > 1/2$.

We can prove subexponential decay for the probability that the displacement is sublinear.

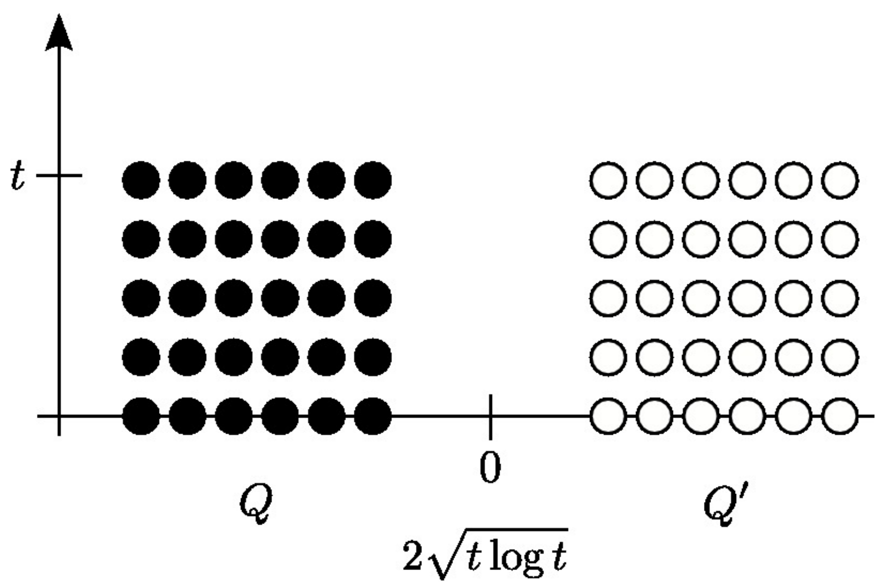
Proposition (Annealed slow-down for the SSE) For all $\rho \in (0,1)$,

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}_{\nu_\rho,0}\big(|X_t| \leq 2\sqrt{t\log t}\big) = 0.$$

Key estimate for slow-down in the SSE case

There exist $C, \delta > 0$ such that, for all intervals $Q, Q' \subset \mathbb{Z}$ separated by a distance at least $2\sqrt{t \log t}$ and all $t \geq 1$,

$$\mathbb{P}_{\nu_\rho}\Big(\xi(x,s) = 1, \xi(y,s) = 0 \;\forall x \in Q \;\forall y \in Q' \;\forall s \in [0,t]\Big) \geq \delta e^{-C(|Q|+|Q'|)\sqrt{t}}.$$



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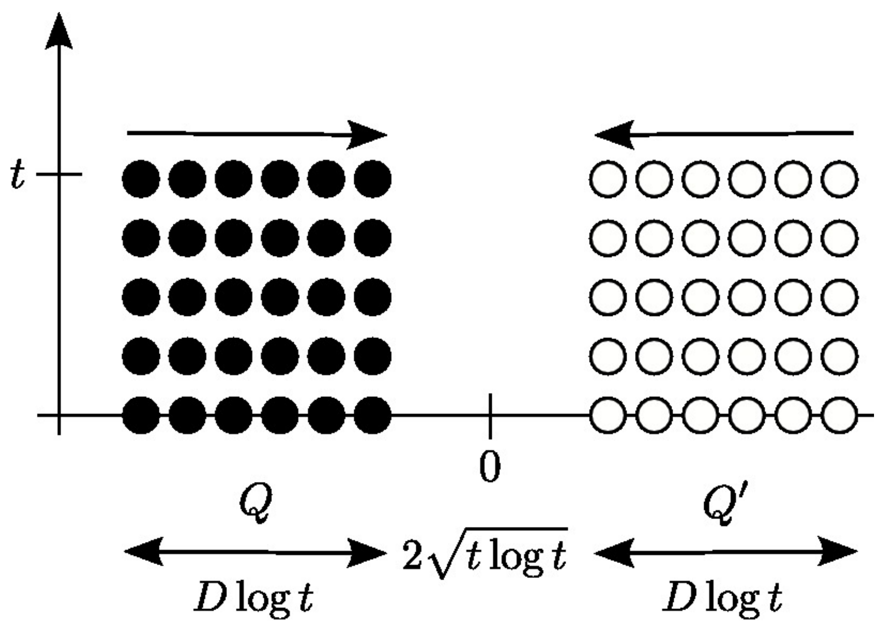
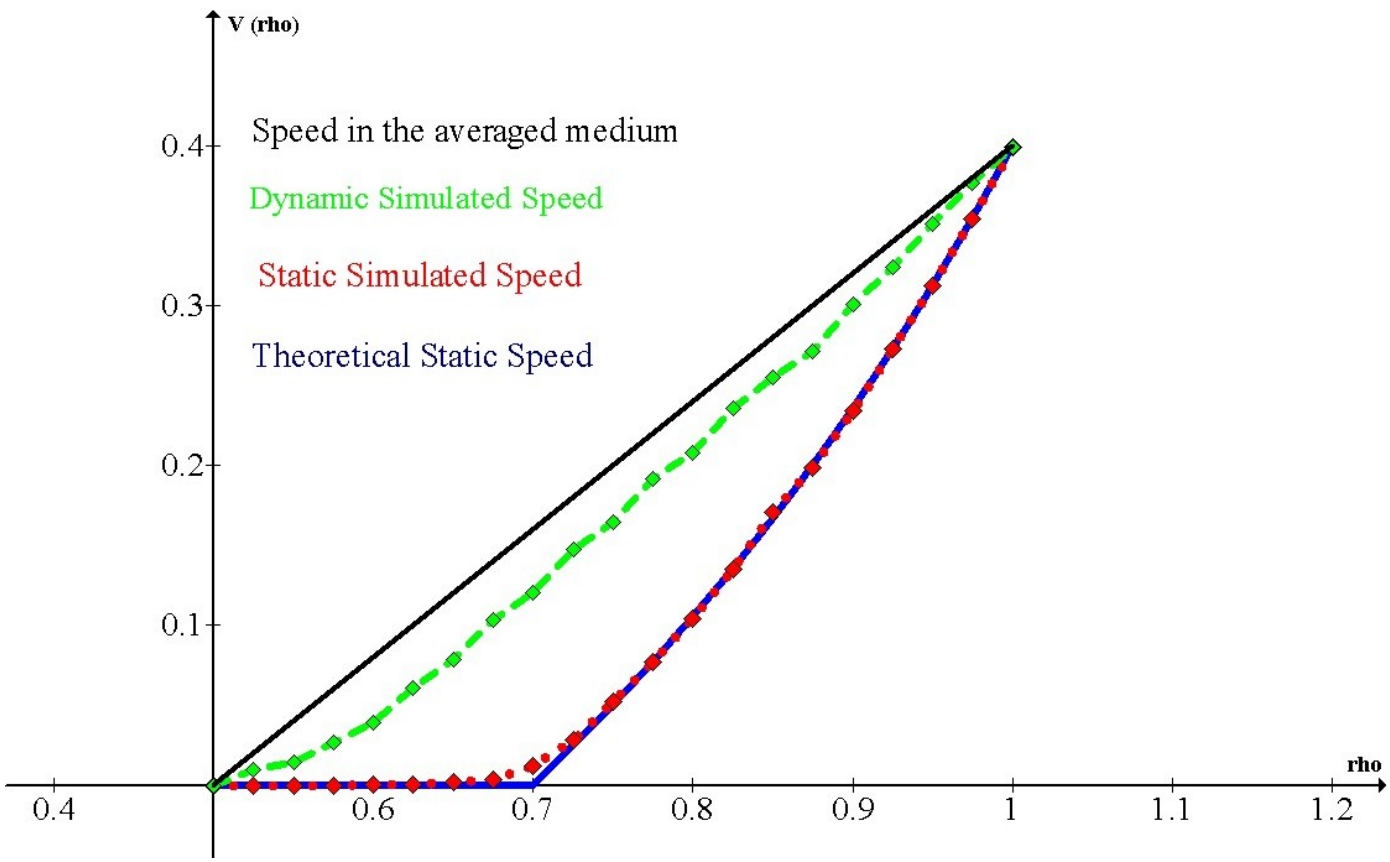


Simulations: SSE



$$\alpha/(\alpha+\beta) = 0.7, \quad \alpha+\beta = 1.$$

Essential references

⋆ L. Avena, F. den Hollander and F. Redig, *Law of large numbers for a class of random walks in dynamic random environments*, EURANDOM Report 2009-032.

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Thanks

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