Random dynamical generation of the TAP equations and the high-temperature solution of the SK model

Erwin Bolthausen, University of Zürich

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Classical mean-field models: X_i i.i.d. S-valued. Hamiltonian, $\mathcal{L}(X_i) = \mu$.

$$H_N \stackrel{\text{def}}{=} \frac{1}{N} \sum_{i,j} V(X_i, X_j) = N \int_{S \times S} V(x, y) L_N(dx) L_N(dy),$$

$$L_N \stackrel{\text{def}}{=} \frac{1}{N} \sum_{i=1}^{N} \delta_{X_i}.$$

$$\lim_{N \to \infty} \frac{1}{N} \log E e^{H_N} = \sup_{\nu} \left[\int V(x, y) \nu(dx) \nu(dy) - I(\nu|\mu) \right].$$

The **Curie-Weiss** case: $X_i = \sigma_i \in \{-1, 1\}, V = 1.$

Fine properties by transforming the distribution of the X_i to ν.

Spin glasses: The same type, but *random* Hamiltonians.
One attempt to apply large deviation ideas: "Perceptron version of the GREM", (with Nicola Kistler): $y_{\sigma,i}$ centered Gaussians, $\#\sigma = 2^N$, $1 \leq i \leq N$. $.\mathbb{E}\left(y_{\sigma,i} y_{\sigma',i'}\right) = \delta_{i,i'} q\left(\sigma,\sigma'\right)$. q hierarchical with a finite number of levels.

$$H_{N,\omega}\left(\sigma\right) \stackrel{\text{def}}{=} \sum_i f\left(y_{\sigma,i}\right).$$

Evaluation of the free energy by a **quenched LDP** for

$$L_{N,\sigma} \stackrel{\text{def}}{=} \frac{1}{N} \sum_{i=1}^{N} \delta_{y_{\sigma,i}}.$$

We give a "quenched" rate function $J : \mathcal{M}_1^+\left(\mathbb{R}\right) \to [0,\infty]$, such that

$$\limsup_{N\to\infty} \frac{1}{N} \log \#\left\{\sigma : L_{N,\sigma} \in A\right\} - \log 2 \ \leq \ -\inf_{\mu\in A} J\left(\mu\right), \ \text{ a.s., } A \text{ closed,}$$
$$\liminf_{N\to\infty} \frac{1}{N} \log \#\left\{\sigma : L_{N,\sigma} \in A\right\} - \log 2 \ \geq \ -\inf_{\mu\in A} J\left(\mu\right), \ \text{ a.s., } A \text{ open.}$$

and arrive at a Parisi type formula.

Sherrington-Kirkpatrick model:: $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_N) \in \Sigma_N \stackrel{\text{def}}{=} \{-1, 1\}^N$
Random interactions: Independent centered Gaussians $g_{ij},\ i < j$, with variance $1/N$, $g_{ji} = g_{ij},\ g_{ii} \stackrel{\text{def}}{=} 0$, defined on $(\Omega, \mathcal{F}, \mathbb{P})$.
Inverse temperature $\beta > 0$, strength $h > 0$ of the external field.
Hamiltonian:

$$-H_{N,\omega}(\boldsymbol{\sigma}) \stackrel{\text{def}}{=} \frac{\beta}{2} \sum_{i,j=1}^{N} g_{ij}(\omega)\, \sigma_i \sigma_j + h \sum_{i=1}^{N} \sigma_i .$$

Partition function: $Z_{N,\omega} \stackrel{\text{def}}{=} \sum_{\boldsymbol{\sigma}} 2^{-N} \exp\left[-H_{N,\omega}(\boldsymbol{\sigma})\right]$.
Gibbs measure:

$$\mathcal{G}_{N,\omega}(\boldsymbol{\sigma}) \stackrel{\text{def}}{=} \frac{2^{-N} \exp\left[-H_{N,\omega}(\boldsymbol{\sigma})\right]}{Z_{N,\omega}} .$$

Free energy:

$$f(\beta, h) \stackrel{\text{def}}{=} \lim_{N \to \infty} \frac{1}{N} \log Z_N = \lim_{N \to \infty} \frac{1}{N} \mathbb{E} \log Z_N .$$

The **replica symmetric "solution"**, given by SK:

$$\mathrm{RS}\left(\beta,h\right)=\inf_{q\geq 0}\left\{E\log\cosh\left(h+\beta\sqrt{q}Z\right)+\frac{\beta^2}{4}\left(1-q\right)^2\right\},$$

Z standard Gaussian. The minimizing $q=q\left(\beta,h\right)$ satisfies:

$$q=E\tanh^2\left(h+\beta\sqrt{q}Z\right).$$

$h>0,\ \forall\beta$, and $h=0,\ \beta\leq 1$: unique q. $h=0,\ \beta>1$, there is positive solution.

Theorem 1 (Talagrand, Guerra-Toninelli) For small enough β :

$$f\left(\beta,h\right)=\mathrm{RS}\left(\beta,h\right)$$

Remark 1 The equation is believed to be correct for β below the ***de Almayda–Thouless-line***

$$\mathrm{AT}\left(h\right)\stackrel{\mathrm{def}}{=}\sup\left\{\beta:\beta^2E\cosh^{-4}\left(h+\beta\sqrt{q}Z\right)\leq 1\right\}$$

$h = 0$: $\mathrm{AT}(0) = 1$. $f(\beta, 0) = \mathrm{RS}(\beta, 0) = \beta^2/4$ for $\beta \le 1$ by a 2nd moment computation:

$$\mathbb{E}Z_N = \exp\left[\frac{\beta^2 (N-1)}{4}\right],$$

and one easily checks for $\beta < 1$ that $\mathbb{E}Z_N^2 \le C(\beta)(\mathbb{E}Z_N)^2$

$$\begin{aligned}\mathbb{E}Z_N^2 &= \sum_{\boldsymbol{\sigma},\boldsymbol{\sigma}'} 2^{-2N} \mathbb{E}\exp\left[\beta \sum\nolimits_{i<j} g_{ij}\left(\sigma_i\sigma_j + \sigma'_i\sigma'_j\right)\right] \\ &= \sum_{\boldsymbol{\sigma},\boldsymbol{\sigma}'} 2^{-2N} \exp\left[\frac{\beta^2}{2N}\sum\nolimits_{i<j}\left(\sigma_i\sigma_j + \sigma'_i\sigma'_j\right)^2\right] \\ &= (\mathbb{E}Z_N)^2 \sum_{\boldsymbol{\sigma},\boldsymbol{\sigma}'} 2^{-2N} \exp\left[\frac{\beta^2}{2N}\left(\sum\nolimits_i \sigma_i\sigma'_i\right)^2 - \frac{\beta^2}{2}\right] \le C(\beta)(\mathbb{E}Z_N)^2\end{aligned}$$

by Curie-Weiss. By Gaussian isoperimetry:

$$f(\beta, 0) = f_{\mathrm{an}}(\beta, 0) \stackrel{\mathrm{def}}{=} \lim_{N\to\infty} \frac{1}{N}\log \mathbb{E}Z_N = \frac{\beta^2}{4}$$

However: For $h > 0, \ \forall\beta > 0$

$$\mathrm{RS}\,(\beta, h) \neq f_{\mathrm{an}}\,(\beta, h)\,.$$

Talagrand's first proof: Based on

$$\mathbb{E}\left(\mathrm{cov}_{\mathcal{G}}(\sigma_i, \sigma_j)^2\right) \leq C/N, \ i \neq j$$

proved by induction on N. From this, it is still a long way to prove the result.
Guerra-Toninelli: Use of the Guerra interpolation method and a clever "replica coupling".
Simplest proof: Recent one by **Talagrand** (PTRF 2009). Clever extension of the class of considered models + interpolation, and recursion.

Morita type correction: Random Hamiltonian $H_{N,\omega}(\boldsymbol{\sigma})$, where $f < f_{\mathrm{an}}$: Sometimes, it is possible to subtract a (simple!) $\boldsymbol{\sigma}$-independent sequence

$$\hat{H}_{N,\omega}(\boldsymbol{\sigma}) = H_{N,\omega}(\boldsymbol{\sigma}) - \psi_{N,\omega},$$

for which:

$$\frac{1}{N}\psi_{N,\omega} \to \alpha \text{ a.s.},$$

then

$$f = \lim_{N\to\infty} \frac{1}{N} \log \sum_{\boldsymbol{\sigma}} 2^{-N} \mathrm{e}^{-H_{N,\omega}(\boldsymbol{\sigma})} = \hat{f} - \alpha.$$

Therefore, if we can prove $\hat{f} = \hat{f}_{\mathrm{an}}$, and we can evaluate the latter, and α, we are done.

Thouless-Anderson-Palmer equations: Consider the Gibbs-expectations of the σ_i :

$$m_i(\omega) \stackrel{\text{def}}{=} \langle \sigma_i \rangle_{N,\omega}.$$

$$m_i = \tanh\left(h + \beta \sum\nolimits_j g_{ij} m_j - \beta^2 (1-q) m_i \right).$$

Can be true only in an asymptotic sense $N \to \infty$.
Physicists claim:

- True for all β.
- Solutions encode for the “pure states”.
- High temperature: One solution.

Mathematical proofs only for high temperature: Talagrand, Chatterjee. Using some facts from the high-temperature regime, like Talagrand’s estimate

$$\mathbb{E}\left(\mathrm{cov}_{\mathcal{G}}(\sigma_i, \sigma_j)^2 \right) \leq C/N,$$

it is not too difficult to prove the equation in an asymptotic sense.

Direct construction of TAP without reference to the Gibbs measure: Define recursively $m_i^{(k)}$ by

$$m_i^{(0)} \stackrel{\text{def}}{=} 0,\ m_i^{(1)} \stackrel{\text{def}}{=} \sqrt{q},$$

$$h_i^{(k)} \stackrel{\text{def}}{=} h + \beta \sum_{j=1}^{N} g_{ij} m_j^{(k-1)} - \beta^2 (1-q)\, m_i^{(k-2)},\ k \geq 2$$

$$m_i^{(k)} \stackrel{\text{def}}{=} \tanh\left(h_i^{(k)}\right),\ k \geq 2.$$

Not too difficult to prove that the scheme converges for small enough β, maybe up to the AT-line.
It is also easy to implement the scheme on the computer for large N.

Proposition 1 a)

$$\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}m_i^{(k)2}=q,\ \text{a.s.},\ \forall k\geq 1.$$

b) β *small enough* $\Longrightarrow\ \exists\, C(\beta)>0,\ 0<\rho(\beta)<1,$ *s.th.*

$$\limsup_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\left(m_i^{(k)}-m_i^{(k-1)}\right)^2\leq C\rho^k.$$

c)

$$\lim_{k\to\infty}\lim_{N\to\infty}\frac{1}{N}\sum_{i,j=1}^{N}g_{ij}m_i^{(k)}m_j^{(k)}=2\beta q\,(1-q)\,.$$

Proof by an alternative representation of $h_i^{(k)}$:

$$h_i^{(2)} = h + \beta\sqrt{q}\xi_i,\ \xi_i \stackrel{\text{def}}{=} \sum_j g_{ij},$$

$$h_i^{(3)} = h + \beta \sum_j g_{ij} \tanh\left(h + \beta\sqrt{q}\xi_j\right) - \beta^2\sqrt{q}\,(1-q)\,.$$

Replace g_{ij} by ones which are independent of the ξ_i : $g_{ij}^{(2)} \stackrel{\text{def}}{=} g_{ij} - \left(\xi_i + \xi_j\right)/N$. Is $\approx$ independent of the ξ_k (up to corrections which are negligible for $N \to \infty$). Then correct $g_{ij}^{(2)}$ s.th. $\mathcal{L} = \mathcal{L}\left(\{g_{ij}\}\right)$: Choose independent $\overline{\xi}_i$ with $\mathcal{L}\left(\left\{\overline{\xi}_i\right\}\right) = \mathcal{L}\left(\{\xi_i\}\right)$, and put $\overline{g}_{ij}^{(2)} \stackrel{\text{def}}{=} g_{ij}^{(2)} + \left(\overline{\xi}_i + \overline{\xi}_j\right)/N$

$$h_i^{(3)} \approx h + \beta \sum_j \overline{g}_{ij}^{(2)} \left(m_j^{(2)} - \left\langle \mathbf{m}^{(2)}, \mathbf{1}\right\rangle\right) + \beta\left\langle \mathbf{m}^{(2)}, \mathbf{1}\right\rangle \xi_i,$$

where $\mathbf{m}^{(2)} \stackrel{\text{def}}{=} \left(m_1^{(2)}, \ldots, m_N^{(2)}\right)$, $\langle \mathbf{x}, \mathbf{y}\rangle \stackrel{\text{def}}{=} \frac{1}{N}\sum_{i=1}^N x_i y_i$.

General scheme: $\mathbf{m}^{(1)}, \mathbf{m}^{(2)}, \ldots$. $\Pi^{(k)} : \mathbb{R}^N \to \operatorname{span}\left(\mathbf{m}^{(1)}, \ldots, \mathbf{m}^{(k)}\right)$ projections.

$$\hat{\mathbf{m}}^{(k)} \stackrel{\text{def}}{=} \frac{\mathbf{m}^{(k)} - \Pi^{(k-1)}\left(\mathbf{m}^{(k)}\right)}{\left\|\mathbf{m}^{(k)} - \Pi^{(k-1)}\left(\mathbf{m}^{(k)}\right)\right\|}, \ \hat{\mathbf{m}}^{(1)} \stackrel{\text{def}}{=} \mathbf{1}.$$

Recursive construction of σ-fields $\mathcal{G}_1, \mathcal{G}_2, \ldots, g_{ij}^{(1)}, g_{ij}^{(2)}, \ldots$

$$\xi_i^{(k)} \stackrel{\text{def}}{=} \sum_j g_{ij}^{(k)} \hat{m}_j^{(k)}, \ \mathcal{G}_k \stackrel{\text{def}}{=} \sigma\left(\boldsymbol{\xi}^{(1)}, \ldots, \boldsymbol{\xi}^{(k)}\right),$$

$$g_{ij}^{(k+1)} \stackrel{\text{def}}{=} g_{ij}^{(k)} - \frac{\xi_i^{(k)} \hat{m}_j^{(k)} + \xi_j^{(k)} \hat{m}_i^{(k)}}{N}.$$

Conditionally on $\mathcal{G}_{k-1}$, $\left\{g_{ij}^{(k+1)}\right\}$ and $\left\{\xi_i^{(k)}\right\}$ are $\approx$ independent.
Recovering the "correct" distribution:

$$\overline{g}_{ij}^{(k+1)} \stackrel{\text{def}}{=} g_{ij}^{(k)} + \frac{\overline{\xi}_i^{(k)} \hat{m}_j^{(k)} + \overline{\xi}_j^{(k)} \hat{m}_i^{(k)}}{N},$$

where *conditionally on* $\mathcal{G}_{k-1}$, $\left\{\overline{\xi}_i^{(k)}\right\}_i$ and $\left\{\xi_i^{(k)}\right\}_i$ are independent, and have the same conditional Gaussian law.
Then

$$h_i^{(k)} \approx h + \beta \sum_j \overline{g}_{i,j}^{(k-1)} \left(m_j^{(k-1)} - \Pi^{(k-2)} \left(\mathbf{m}^{(k-1)} \right)_j \right) + \beta \sum_{r=1}^{k-2} \left\langle \hat{\mathbf{m}}^{(r)}, \mathbf{m}^{(k-1)} \right\rangle \xi_i^{(r)}.$$

Using this representation, the proposition follows.

The transformation: New i-th spin distribution

$$p_i\left(\sigma_i\right) \stackrel{\text{def}}{=} \frac{1}{2} \frac{\mathrm{e}^{h_i \sigma_i}}{\cosh\left(h_i\right)},$$

$$p\left(\boldsymbol{\sigma}\right) \stackrel{\text{def}}{=} \prod_{i=1}^{N} p_i\left(\sigma_i\right).$$

h_i should be $h_i^{(k)}$ for large k such that that $h_i^{(k)}$ is close to $h_i^{(k-1)}$.

$$\begin{aligned} Z_N &= \sum_{\boldsymbol{\sigma}} 2^{-N} \exp\left[-H_N\left(\boldsymbol{\sigma}\right)\right] \\ &= \exp\left[\sum_i \log\cosh\left(h_i\right)\right] \sum_{\boldsymbol{\sigma}} p\left(\boldsymbol{\sigma}\right) \exp\left[-H\left(\boldsymbol{\sigma}\right) - \sum_i h_i \sigma_i\right]. \end{aligned}$$

$$\sum_i \log\cosh\left(h_i\right) \approx N \int \log\cosh\left(h + \beta\sqrt{q} z\right) \phi\left(dz\right).$$

Centering the σ_i by $\hat{\sigma}_i \stackrel{\mathrm{def}}{=} \sigma_i - m_i$, we get after some computations and the Proposition:

$$-H(\boldsymbol{\sigma}) - \sum\nolimits_i h_i \sigma_i \approx \frac{\beta^2 N}{4}(1-q)^2 + \frac{\beta}{2}\sum_{i,j} g_{ij}\hat{\sigma}_i\hat{\sigma}_j - \frac{\beta^2}{4N}\sum_{i,j}\hat{\sigma}_i^2\hat{\sigma}_j^2 + \frac{\beta^2}{N}\left(\sum\nolimits_i m_i\hat{\sigma}_i\right)^2,$$

The last term is annoying, but harmless for small β :CW-type term. Leaving it out:

$$f(\beta,h) = \mathrm{RS}(\beta,h) + \lim_{N\to\infty}\frac{1}{N}\log\sum_{\boldsymbol{\sigma}} p(\boldsymbol{\sigma})\exp\left[\frac{\beta}{2}\sum_{i,j} g_{ij}\hat{\sigma}_i\hat{\sigma}_j - \frac{\beta^2}{4N}\sum_{i,j}\hat{\sigma}_i^2\hat{\sigma}_j^2\right].$$

Claim: The second summand vanishes by a 2nd moment computation, if β is small.
Problem: $p(\boldsymbol{\sigma})$ depends on $\{g_{ij}\}$.
Solution: Conditional second moment, given $\mathcal{G}_k$, by a replacement of $\{g_{ij}\}$ by $\left\{\overline{g}_{ij}^{(k)}\right\}$.
The replacement introduces CW-type summands which are handled quenched. The CW-computation is quite messy here.

Hope: **SK-type perceptron model:** $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_N) \in \{-1, 1\}^N$, $y_{\sigma,i}$ with

$$\mathbb{E}\left(y_{\sigma,i} y_{\sigma',i'}\right) = \delta_{i,i'} q\left(\sigma, \sigma'\right),$$

with $q\left(\sigma, \sigma'\right) \stackrel{\text{def}}{=} N^{-1} \sum_{i=1}^{N} \sigma_i \sigma_i'$. Let

$$L_{N,\sigma} \stackrel{\text{def}}{=} \frac{1}{N} \sum_{i=1}^{N} \delta_{y_{\sigma,i}}.$$

We search for a "quenched" rate function $J : \mathcal{M}_1^+(\mathbb{R}) \to [0, \infty]$, such that

$$\#\left\{\sigma : L_{N,\sigma} \sim \mu\right\} \sim 2^N \exp\left[-NJ(\mu)\right], \text{ a.s.}$$

The "annealed" rate function is $I(\mu|\phi)$, ϕ standard normal:

$$\mathbb{E}\#\left\{\sigma : L_{N,\sigma} \sim \mu\right\} = 2^N \mathbb{P}\left(L_{N,\sigma} \sim \mu\right) \approx 2^N \exp\left[-NI(\mu|\phi)\right]$$

Up to now, we have not been able to find the "correct" TAP equation.
I expect that $J(\mu) = I(\mu|\phi)$ on a hypersurface in $\mathcal{M}_1^+(\mathbb{R})$ of codimension 1, and μ close to ϕ. For μ close to ϕ, but not on this surface, there should be a kind of RS-solution. For μ further away from ϕ, there is probably a full Parisi picture.