Phase transition for the vacant set left by a random walk on random d-regular graph

Jiří Černý (ETHZ)

joint work with
Augusto Teixeira (ETHZ), David Windisch (Weizmann Institute)

Eurandom, 16 December 2009

General Problem

Model.

- $G_n = (V_n, \mathcal{E}_n)$ – sequence of finite connected graphs, $|V_n| \to \infty$.
- $(X_t^n)_{t \geq 0}$ – cont. time RW on G_n started from its invariant distribution
- $u > 0$ – parameter
- $\mathcal{V}^u = V_n \setminus \{X_t : t \in [0, u|V_n|]\}$ – vacant set.

Question: Percolative properties of the random set $\mathcal{V}^u$.

Is there $u_c = u_c(G) \in (0, \infty)$ such that

- for $u > u_c$, $\mathcal{V}^u$ has only 'small' components (say $O(\log n)$).
- for $u < u_c$, $\mathcal{V}^u$ has a 'giant' component (a component with volume $O(|V_n|)$).
- is the giant component unique
- how does $\mathcal{V}^u$ look at/close to u_c.

Known results

The problem was studied only on d-**dimensional torus** $(\mathbb{Z}/n\mathbb{Z})^d$, $d \geq 3$:

- Benjamini-Sznitman (JEMS '08): If u is small enough, then $\mathcal{V}^u$ has a giant component. This component is unique in some weak sense.
- improved slightly by D. Windisch (EJP '08)
- D. Windisch (ECP '08): Local picture on the torus converges to *Random interlacement* on $\mathbb{Z}^d$.
- Recent results on Random Interlacement can be used to prove:
  - For $u < u_{\star\star} < u_\star(\mathbb{Z}^d)$, there is a giant component
  - For $u > u_{\star\star\star} > u_\star(\mathbb{Z}^d)$, there are only small components.

  Here $u_\star(\mathbb{Z}^d)$ is the percolation threshold for Random Interlacement.
- It is believed that $u_c((\mathbb{Z}/n\mathbb{Z})^d) = u_\star(\mathbb{Z}^d)$.

Our setting

Consider graphs that are simpler for Bernoulli percolation:

- random d-regular graph
  (graph uniformly chosen from all d-regular graphs on n vertices)
- d-regular large-girth expanders

Both these classes of graphs are *good finite approximations of d-regular tree*

Bernoulli percolation on such graphs studied by Alon-Benjamini-Stacey '04, Nachmias-Peres '09, Pittel '09.

More precisely, we consider graphs satisfying:

(A0) $G_n = (V_n, \mathcal{E}_n)$ is d-regular, $|V_n| = n$.

(A1) *Local almost tree-like property:*
There exists $\alpha_1 \in (0,1)$ such that for all n and $x \in V_n$

$$\text{the ball } B(x, \alpha_1 \log n) \text{ contains at most one cycle}$$

(A2) *Uniform spectral gap:*
There exists $\alpha_2 > 0$ such that for all n

$$\lambda_1(G_n) \geq \alpha_2$$

Remarks

- random d-regular graph satisfies (A0)–(A2) with high probability as $n \to \infty$.
- (A1) $\implies$ typical point $x \in V_n$ has *tree-like* neighbourhood of radius $\frac{\alpha_1}{2} \log n$.
- (A2) is equivalent (via Cheeger's inequality) to expansion property.

$$\frac{|\partial A|}{|A|} \geq \alpha_2', \qquad \forall n,\ \forall A \subset V_n, |A| < |V_n|/2.$$

Result

Theorem

Let G_n satisfy (A0)–(A2). Then there exists explicitly computable constant $u_c = u_c(d)$ such that

1. *when $u > u_c$, there is $K = K(u, d, \alpha_1, \alpha_2)$ such that*

$$\mathbb{P}[|\mathcal{C}_{\max}(\mathcal{V}^u)| \geq K \log n] \xrightarrow{n\to\infty} 0.$$

2. *when $u < u_c$, there is $\rho = \rho(u, d, \alpha_1, \alpha_2)$ such that*

$$\mathbb{P}[|\mathcal{C}_{\max}(\mathcal{V}^u)| \geq \rho n] \xrightarrow{n\to\infty} 1.$$

3. *when $u < u_c$, for every $\varepsilon > 0$,*

$$\mathbb{P}[|\mathcal{C}_{\sec}(\mathcal{V}^u)| \geq \varepsilon n] \xrightarrow{n\to\infty} 0.$$

Here, $\mathcal{C}_{\max}(\mathcal{V}^u)$ ($\mathcal{C}_{\sec}(\mathcal{V}^u)$) is the largest (second largest) component of $\mathcal{V}^u$.

Remark. $3.$ is equivalent with: $\exists f_n = o(n)$ such that

$$\mathbb{P}[|\mathcal{C}_{\sec}(\mathcal{V}^u)| \geq f_n] \xrightarrow{n\to\infty} 0.$$

We are far from $f_n = O(\log n)$.

Relation to Random Interlacement model

Let $\mathbb{G} = (\mathbb{V}, \mathcal{E}_{\mathbb{G}})$ be infinite, transient, connected graph, which is local limit of G_n:
$\exists r_n \to \infty$ such that for typical $x \in V_n$, the ball $B_{G_n}(x, r_n) \overset{\phi_n^x}{\simeq} B_{\mathbb{G}}(\emptyset, r_n)$.

Examples: Torus $\to \mathbb{Z}^d$, our $G_n \to d$-regular tree $\mathbb{T}_d$.

Random Interlacement is a dependent percolation on $\mathbb{G}$ which is local limit of $\mathcal{V}^u$ (should be proved).

Local definition of RI: Let $A \subset \mathbb{V}$ finite.

- *equilibrium measure:* $e_A(x) = \mathbb{1}_A(x) P_x$[RW never returns to A].
- at every point x start Poisson($u e_A(x)$) independent random walks
- RI$|_A$ has the same law as the set of vertices in A not visited by these random walks.
- *notation:* $\mathbb{C}_x$ cluster of RI containing x.

Conjecture. For $u_\star(\mathbb{G}) = \sup\{u : \mathbb{P}[|\mathbb{C}_x| = \infty] > 0\}$, the threshold of RI on $\mathbb{G}$,

$$u_c(G) = u_\star(\mathbb{G}).$$

Random Interlacement on tree $\mathbb{T}_d$

Random Interlacement on the tree has a particularly simple description in terms of branching process (Teixeira '09):

Lemma

Given $\emptyset \in RI$, the cluster $\mathbb{C}_x$ has the law of branching process whose offspring distribution is binomial with parameters $d-1$ (resp. d in the first generation) and p_u, where

$$p_u = \exp\Big\{ -\frac{u(d-2)^2}{d(d-1)} \Big\}.$$

Consequence. RI on $\mathbb{T}_d$ exhibits a phase transition on $u_\star(\mathbb{T}_d)$ given by $p_{u_\star(\mathbb{T}_d)}(d-1) = 1$.

Theorem

If G_n satisfy (A0)–(A2), then

$$u_c(G) = u_\star(\mathbb{T}_d).$$

Local convergence to Random Interlacement

RI on the tree $\mathbb{T}_d$ is indeed a good local model for the vacant set on G_n:

Lemma

There is $\beta \in (0, \alpha_1/5)$, such that for all x with tree-like neighbourhood of radius $5\beta \log n$, for all $u > 0$, $\varepsilon > 0$, there exists a coupling $\mathbb{P}$ of RW on G and RI's on $\mathbb{T}^d$ such that

$$\mathbb{P}[\mathbb{C}_\emptyset^{u-\varepsilon} \supset \phi_n^x(\mathcal{C}_x(\mathcal{V}^u \cap B(x, \beta \log n))) \supset \mathbb{C}_\emptyset^{u+\varepsilon}] \xrightarrow{n\to\infty} 1.$$

Consequence. In every tree like ball of radius $\beta \log n$ we have a good control of $\mathcal{C}_x(\mathcal{V}^u)$ by a branching process.

THE LOCAL CONTROL IS NOT SUFFICIENT!

- In the super-critical phase, the giant component cannot be contained in a ball of of radius $\beta \log n$.
- In the sub-critical phase, the largest cluster is $\sim K \log n$, but $K \to \infty$ as $u \downarrow u_c$. (In particular, since $\operatorname{diam} G = \log n(1 + o(1))$, we have $|\mathcal{C}_{\max}(\mathcal{V}^u)| \geq \operatorname{diam} G$.)

Segments-Bridges measure

Random walk is not suitable for our techniques (there is a dependence between its positions at various times).

We replace the RW trajectory by a different trajectory which possesses more independence and approximates RW well.

Segments-Bridges construction.

- Fix $L = n^\gamma$, $\ell = \log^2 n$, where $\gamma \in (0,1)$. Set $m = un/(L+\ell)$.
- Let $(Y^i)_{i<m}$ be an i.i.d. family with values in $D([0,L],V)$ and marginal P_π^L (RW started from invariant distribution, killed after L steps) = **segments**
- given (Y^i), let $(Z^i)_{i<m}$ (**bridges**) be independent family of $D([0,\ell],V)$ valued random variables,

$$\text{law of } Z^i = P^\ell_{Y^i_L, Y^{i+1}_0} = \text{RW bridge of length } \ell \text{ from } Y^i_L \text{ to } Y^{i+1}_0$$

Lemma

Law of concatenation $Y^0 Z^0 Y^1 \dots Y^m Z^m$ approximates well $(X_t : t \le un)$.

Proof: Due to spectral gap assumption, the mixing time of the RW is $O(\log n)$.

Sub-critical regime

We want to show: For some $K \geq 0$,

$$P[\mathcal{C}_{\max}(\mathcal{V}^u) \geq K \log n] \xrightarrow{n\to\infty} 0.$$

Here $\mathcal{V}^u = V \setminus \cup_{i<m}(Y^i \cup Z^i)$.

We are looking for an upper bound. It is sufficient to control $\bar{\mathcal{V}}^u = V \setminus \cup_{i<m} Y^i$:

$$\mathbb{P}[|\mathcal{C}_x(\bar{\mathcal{V}}^u)| \geq K \log n] \leq n^{-1-\varepsilon}.$$

Construct $\mathcal{C}_x(\bar{\mathcal{V}}^u)$ sequentially via a breath-first-search algorithm.

PICTURE!

The algorithm discovers some information on segments Y^i:

- $\mathcal{A}_k$ – information discovered during exploration of first $k-1$ vertices.
- y_k, k-th explored vertex.

To control the algorithm one need to control the probability that the vertex y_k is vacant given $\mathcal{A}_k$.

$$\mathbb{P}[y_k \in \bar{\mathcal{V}}^u | \mathcal{A}_k] < \tfrac{1}{d-1}.$$

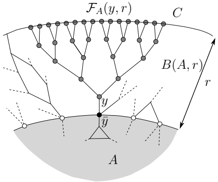
Control of the BFS algorithm

We have

$$\mathbb{P}[y_k \in \bar{\mathcal{V}}^u | \mathcal{A}_k] = \mathbb{P}[y_k \notin \cup_{i<m} Y^i | \mathcal{A}_k] \le \mathbb{P}[y_k \notin \cup_{i \text{ non-marked}} Y^i | \mathcal{A}_k].$$

On $\mathcal{A}_k$, $(Y^i)_{i \text{ non-marked}}$ are i.i.d. with the distribution $P_\pi^L[\cdot | H_{A_k} > L]$, where $A_k = \{y_1, \dots, y_{k-1}\}$ is the set of already explored points.

We need to compute $P[H_{y_k} > L | H_{A_k} > L] = \frac{P[H_{y_k \cup A_k} > L]}{P[H_{A_k} > L]}$.
This can be done, but only in a specific situation ($r = 7 \ln \ln n$):



Why:

- By Aldous-Brown (1993) $P[H_A > L] \sim e^{-L/\mathbb{E}[H_A]}$
- $\mathbb{E}[H_A] \sim C_{\text{eff}}(A, B(A,r))^{-1}$
- We need to control $C_{\text{eff}}(A, B(A,r)) - C_{\text{eff}}(A \cup y, B(A,r))$.

Deterministic estimate on bad event

In this specific situation one gets for non-marked Y^i,

$$\mathbb{P}[y_k \notin Y^i | \mathcal{A}_k] = 1 - Ln^{-1}(1 + o(1)) = 1 - n^{\gamma-1}(1 + o(1)).$$

and thus, when $u > u_c$,

$$\mathbb{P}[y_k \in \bar{\mathcal{V}}^u | \mathcal{A}_k] = \mathbb{P}[y_k \notin \cup_{i \text{ non-marked}} Y^i | \mathcal{A}_k] = (1 - n^{\gamma-1})^m < 1/(d-1).$$

Recall, $m = un/(L + \ell) \sim un^{1-\gamma}$.

One further needs:

- the set of marked segments is much smaller m.
- (A1) $\implies$ The good situation does not occur at most crK^2 times during the run of the algorithm (deterministically)

Super-critical regime

Strategy for Bernoulli percolation. $p > p_c$

1. Fix $p' \in (p_c, p)$.
2. Use local branching-process control to construct many components of size n^β, $\beta \in (0,1)$:
$$\mathbb{P}_{p'}[|\{x : |\mathcal{C}_x| \geq n^\beta\}| \geq \rho n] \xrightarrow{n\to\infty} 1.$$
3. Sprinkle additional edges over G (increase from p' to p).
These edges will with high probability joint the components constructed in 2.
(local $\to$ global control)
4. For 3, the expansion property is very useful.
(It implies $\forall A, B \subset V$, $|A|, |B| \geq cn$, there exist $\Theta(n)$ disjoint paths from A to B all having length smaller than C.)

It is not clear how to implement the sprinkling for the random walk trajectory!

First idea of sprinkling

1. In super-critical regime $u < u_c$. So, fix $u' \in (u, u_c)$.
2. Consider $\mathcal{V}^{u'}$ and use the local branching-process comparison to construct many large components:

$$\mathbb{P}[|\{x : |\mathcal{C}_x(\mathcal{V}^{u'})| \geq n^\beta\}| \geq \rho n] \xrightarrow{n\to\infty} 1.$$

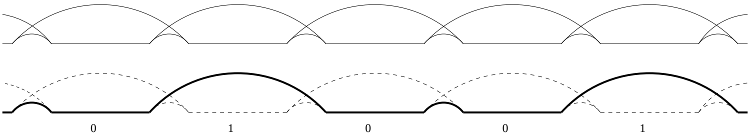
3. Erase the last part of the trajectory, that is $\{X_t : t \in [un, u'n]\}$.
4. BUT! What is the law of this part given $\mathcal{V}^{u'}$??
5. We were not able to control it.

Second idea of sprinkling

1. Use Segment-Bridges measure as before for $u' \in (u, u_c)$. That is set $m' = u'n/(L+\ell)$
2. By local branching-process comparison, construct many large components as before.
3. Consider sequence $(R_i)_{i<m'}$ of i.i.d. Bernoulli random variables with parameter $(u'-u)/u'$, independent of Y^i's and Z^i's.
4. Erase segments Y^i with $R^i = 1$.
5. PROBLEM: We obtain non-nearest-neighbour path.

Solution. Put even more bridges into the picture, so that erasing some segments does not disconnect the trajectory:
Long range bridges Z^{ij}, $i < m'$, $j < \log n$.

$$\text{law of } Z^{ij} = P^{\ell}_{Y^i_L, Y^{i+j}_0}$$



Technical complications

The union of segments and all long range bridges is not n.n. path, but quite complicated object $\implies$ it is hard to obtain the local control by branching process.

1. Establish control by the branching process for complement of segments only.
2. Use this control to prove:

$$\mathbb{P}\Big[\big|\{x \in V : x \text{ is proper in } \bar{\mathcal{V}}^{u'}\}\big| \geq cn\Big] \xrightarrow{n\to\infty} 1,$$

   where x is proper if

   - $|\mathcal{C}_x(\bar{\mathcal{V}}^{u'} \cap B(x, \beta \log n))| \geq c((d-1)p_{u'+\varepsilon})^{\beta \log n}$.
   - $|\mathcal{C}_x(\bar{\mathcal{V}}^{u'}) \cap \partial B(x,k)| \leq ((d-1)p_{u'-\varepsilon})^k$ for all $k \in [c' \log\log n, \beta \log n]$.

3. Show that if number of proper points is larger than cn, then fliping $n^{1-\gamma} \log^5 n$ vertices does not destroy a proportion of large components (deterministic statement):
   After adding long range bridges:

$$|\{x : \mathcal{C}_x \geq c((d-1)p_{u'+\varepsilon})^{\beta \log n}\}| \geq cn/8.$$

Uniqueness

The proof follows the same lines as the proof of existence of giant component.

Sprinkling shows that any two giant components at level u' join at level u.

Problem: the sprinkling can also create new giant component at level u, that was not there at level u'

It is actually not possible, since any giant component at level u must contain a positive proportion of vertices that were in n^{β} components at level u'.

By sprinkling again, we know that these are joined together.

Open problems

1. Density of giant cluster for $u < u_c$.
2. Stronger uniqueness result. Is $|\mathcal{C}_{\mathrm{sec}}| = O(\log n)|$.
3. What about $u = u_c$. Is there the double jump as in the Bernoulli case: $|\mathcal{C}_{\max}| \sim n^{2/3}$? Critical window?

The end

Thank you for your attention.