Workshop on Dynamic Random Environments

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CATALYTIC BRANCHING: REACTION NETWORKS AND SPATIAL MODELS

Don Dawson

Review based on joint work with Ted Cox, Alison Etheridge, Klaus Fleischmann, Andreas Greven, Frank den Hollander, Zenghu Li, Leonid Mytnik, Ed Perkins, Rongfeng Sun, Jan Swart, Jie Xiong, Lihu Xu, Iljana Zähle

OUTLINE:

A class of random environments, dynamics random environments (DRA) and DRA with feedback

- The role of catalysis - chemical engineering, metabolism, respiration, prebiotic evolution
- Catalytic branching: chains and networks
  particle models and diffusion process limits
- Spatial models
  - lattice systems
  - continuum limits
    - euclidean continuum limits
    - ultrametric continuum limits.
- Invariance principles.

Introduction

Molecular Level

- CATALYTIC REACTIONS:
  Example: species A catalyzes production of copies of species B:
  $$A + B \to 2B$$

EXAMPLES from BIOCHEMISTRY:

- Glycolysis - metabolism
- The Krebs Cycle (citric acid cycle) is a series of enzyme-catalysed chemical reactions, which is essential to all living cells that use oxygen as part of cellular respiration. (Modelled as 17 ODE.)

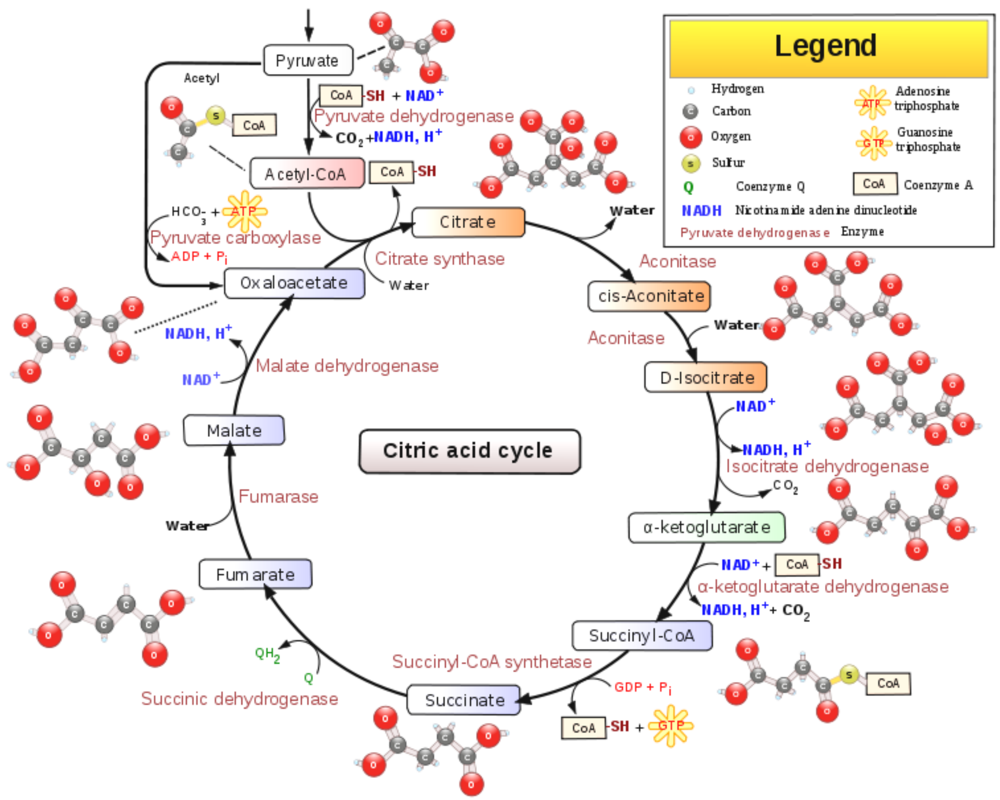

Figure 1: Krebs cycle

Catalytic chains and networks

- Closed Catalytic Chains: M types where type i serves as catalyst for production of type $(i + 1) \pmod{M}$.

Pre-biotic evolution - *autocatalytic sets and networks*

Eigen and Schuster (1977), Kauffman (1986), Jain and Krishna (2002), D. Segré, D. Ben-Eli, D. Lancet (2000)

The deterministic hypercycle was introduced by Eigen and Schuster as a model of cooperation of replicators:

$$\frac{dx_i(t)}{dt} = x_i(t)[x_{i-1}(t) - \sum_j x_j(t)x_{j-1}(t)], \quad i \in \mathbb{Z}_M$$

Recent developments: Problem of parasitic mutations and development of spatial models.

Stochastic Models of Catalytic Branching

Catalytic branching particle systems

(Li and Ma (JTP - 2008))

Stochastic equations driven by Poisson noise

Catalyst

$$\xi(t) = \xi(0) + \int_0^t \int_{\mathbb{N}} \int_0^{\alpha_1 \xi(s-)} (z-1) N_1(ds, dz, du)$$

Reactant

$$\eta(t) = \eta(0) + \int_0^t \int_{\mathbb{N}} \int_0^{\alpha_2 \xi(s-)\eta(s-)} (z-1) N_2(ds, dz, du)$$

N_j Poisson with intensity $ds\mu_j(dz)du$,

$\mu_j(\{i\}) = p_i^j, i \in \mathbb{Z}_+ \; j = 1, 2$ (offspring distributions)

Diffusion Approximation

Itô stochastic differential eqn's

Example: Finite population stochastic cyclic branching system:

$$dX^{(i)}(t) = b_i X^{(i)}(t) X^{(i-1)}(t)\Big(1 - \frac{1}{K}\sum_j X^{(j)}(t)\Big)dt$$
$$+ \sqrt{\gamma_i X^{(i)}(t) X^{(i-1)}(t)}\, dW^{(i)}(t),$$

$i \in \mathbb{Z}_M,\ b_i, \gamma_i \geq 0$

Note the non-Lipschitz coefficients!

Representation as Girsanov transform of the critical (i.e. $K = \infty$) system.

The problem of singular diffusions

- Non-Lipschitz and singular diffusion coefficients
- $M = 2$, $K = \infty$ - Mutually catalytic branching
  Mytnik Dual to establish weak uniqueness
- Duality fails if $M > 2$ or γ_i are non-constant,
- Moments: If $K = \infty$, $\gamma_i > 0$, then

$$m_n^k(t) = P_{\mathbf{a}}[(X_t^k)^n] \geq c_1 e^{c_2 n^2 t}$$

Therefore Carleman's condition:

$$\sum_n (m_{2n}^k(t))^{-1/2n} = \infty \text{ fails.}$$

Catalytic Branching Network

Finite Directed Graph $(V, \mathcal{E})$

Types: $i \in V$

- $(i, j) \in \mathcal{E}$: directed edge from i to j, $i \neq j$
  $\Leftrightarrow$ Type $i \in C$ *catalyzes* the branching of the *reactant* type $j \in R$

- (GH) each vertex i has at most one incoming edge (c_i, i)

Branching Network Martingale Problem

For $f \in C_b^2(\mathbb{R}_+^d)$, let

$$\mathcal{A}f(x) = \sum_{j \in R} \gamma_j(x) x_{c_j} x_j f_{jj}(x) + \sum_{j \notin R} \gamma_j(x) x_j f_{jj}(x)$$
$$+ \sum_{j \in V} b_j(x) f_j(x)$$

- For $i \in V$, $\gamma_i : \mathbb{R}_+^d \to (0, \infty)$, $b_i : \mathbb{R}_+^d \to \mathbb{R}$ are continuous Hölder Hypothesis: Hölder of index $\alpha \in (0,1)$ on compact subsets
- $|b_i(x)| \leq c(1 + |x|)$ on $\mathbb{R}_+^d$,
- $b_i(x) \geq 0$ if $x_i = 0$. In addition,

$$b_i(x) > 0 \text{ if } i \in C \cup R \text{ and } x_i = 0.$$

The Problem of Weak Uniqueness

Series of papers:

Athreya, Barlow, Bass, Perkins (2002) monotype

Bass, Perkins (2003) monotype

Dawson-Perkins (2006) (multitype, Hölder + (GH))

Kliem (2008) (general network, Hölder)

Bass-Perkins (2008) (GH)

Theorem

For any probability ν on $S = \mathbb{R}_+^{|V|}$, there is exactly one solution P_ν to martingale problem $MP(\mathcal{A}, \nu)$.

Proofs involve localization and L^2 semigroup perturbation arguments following the Stroock-Varadhan approach. The core of the proofs involve difficult (due to the degeneracy) analytic estimates on the resolvent of the perturbed process.

Spatial Models of Catalytic Reaction-Diffusion

Classical equation

$$\frac{\partial u(t,x)}{\partial t} = \frac{1}{2}\Delta u(t,x) - \Gamma(x)R(u(t,x))u(t,x)$$
$$u(0) = \phi \in \mathfrak{B}_+(D), \quad D \subset \mathbb{R}^d$$

$$\begin{aligned} \Gamma(x) &= \text{ catalytic density at } x \\ &= \text{ reaction rate at } x \end{aligned}$$

Chemical engineering - lower dimensional such as surface catalysts and fractal catalysts: higher efficiency of catalyst in increasing the reaction rate.

Singular space-time catalysts:

$$\Gamma(t,dx) = \ \textit{Catalytic mass in } dx \ \ \textit{at time } t$$

Reformulation of Catalytic Reaction Diffusion Equation as mild equation:

$$u(t,x) = \int p(t,x-y)\phi(y)dy$$
$$- \int_0^t \int p(t-s,x-y)R(u(s,y))u(s,y)\Gamma(s,dy)ds$$

Function-space approach - see below.

Questions

- What is the effect of a singular catalytic medium?
- Phenomena of “hot spots” and clumping in small and large space-time scales
- Application to catalytic branching (via the Laplace transform):

$$R(u) = [cu + \sigma^2 u^2 + \int_0^\infty (e^{-zu} - 1 + zu) m(du)]$$

where $(u \wedge u^2) m(du)$ is a finite measure on $(0, \infty)$, for example

$$R(u) = [cu + \gamma u^{1+\beta}], \quad 0 < \beta \leq 1,\ c \in \mathbb{R}.$$

Lattice Models of Catalytic Branching

Catalyst: $\{\Gamma_t(x) : x \in S\}$ where S is a countable abelian group, e.g. $S = \mathbb{Z}^d$.

$$X_t(x) \;=\; X_0(x) + \int_0^t \sum_{y \neq x} a_{xy} X_s(y)\,\mathrm{d}s + \int_0^t \sqrt{\Gamma_s(x)\, X_s(x)}\,\mathrm{d}W_s(x)$$

where a_{xy} are transition rates of a random walk on S with generator A.

Log-Laplace equation:

$$E(e^{-\sum_{x\in S}\varphi(x)X_t(x)}) = e^{-\sum_{x\in S}u_t(x)X_0(x)}$$

where u is the solution of

$$\frac{\partial u_t(x)}{\partial t} = Au_t(x) - \Gamma_t(x)(u_t(x))^{1+\beta}$$
$$u_0(x) = \varphi(x) \geq 0$$

The continuum limit:
Catalytic Super-Brownian Motion in $\mathbb{R}^d$

Scaling

Speeded up random walk on $\varepsilon\mathbb{Z}^d$ to have generator

$$\Delta^\varepsilon f = \frac{1}{\varepsilon^2} \sum_{|y-x|=\varepsilon} (f(y) - f(x)).$$

We now consider the system of SDE's on $\varepsilon\mathbb{Z}^d$ given by

$$X_t^\varepsilon(x) = X_0^\varepsilon(x) + \int_0^t \frac{1}{2}\Delta^\varepsilon X_s^\varepsilon(x)ds + \int_0^t \sqrt{\Gamma_s^\varepsilon(x) X_s^\varepsilon(x)}\varepsilon^{-\frac{d}{2}} dW_s(x)$$

Continuum Limit $\varepsilon \to 0$.

Given a measure-valued catalytic process Γ_t, the expected limit is a *measure-valued process characterized the transition Laplace functional*:

$$E_{X_0}\left(\exp\left(-\int \varphi(x)X_t(dx)\right)\right) = \exp\left(-\int u(t,x)X_0(dx)\right)$$

where $u(t,x)$ is given by

<u>Dynkin's Function-space reformulation of the log-Laplace Equation</u>

$$u(t,x) = E_x\left[\varphi(W_t) - \int_0^t L_{[\Gamma,W]}(ds)u^2(t-s,W_s)\right]$$

Here $L_{[\Gamma,W]}$ is the *collision local time* between the measure-valued process process Γ_t and the Brownian particle δ_{W_t}.

Collision local time between measure-valued processes

(Barlow-Evans-Perkins (1991))

$$\int_0^\infty \int_{\mathbb{R}^d} \psi(s,x) L_{[\Gamma,X]}(ds,dx)) =$$

$$\lim_{\delta\to 0} \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \psi\left(s, \frac{x^1+x^2}{2}\right) p_\delta(x^1 - x^2) X_s(dx_1)\Gamma_s(dx_2)ds$$

Note: Can be positive even if Borel supports of X and Γ are disjoint!

The process X_t can also be characterized as the solution to the martingale problem

<u>Martingale Problem:</u>

$$M_t(\varphi) = \langle X_t, \varphi \rangle - \langle X_0, \varphi \rangle - \int_0^t \left\langle X_s, \frac{1}{2}\Delta\varphi \right\rangle ds$$

is a continuous square integrable martingale with increasing process

$$\langle\langle M(\varphi)\rangle\rangle_t = \int_0^t \int_{\mathbb{R}^d} \varphi^2(x) L_{[\Gamma,X]}(ds, dx)$$

where

$L_{[\Gamma,X]}(ds, dx) =$ Collision local time of X with the catalyst

Question: For which catalytic processes does the collision local time exist.

Fixed deterministic catalyst in $\mathbb{R}^d$ (Delmas (1996))

$$\sup_{x\in\mathbb{R}^d}\int_{\|y-x\|\leq 1}|y-x|^{-d+2-\beta}\Gamma(dy)<\infty, \qquad \beta\in(0,2)$$

guarantees the existence of collision local time between Γ and the Brownian particle δ_{W_t}.

Example: Single point in $\mathbb{R}$, surface measure on $(d-1)$-dimensional manifolds in $\mathbb{R}^d$.

Alternate approach: Mörters and Vogt (2005) - direct construction of the collision local time

A dynamical random environment

Super-Brownian Motion in a super-Brownian Catalyst

D-Fleischmann (1997)

Γ_t = super-Brownian motion in $\mathbb{R}^d$ (singular in dimensions $d \geq 2$).

- X_t – Nondegenerate existence in $d = 1, 2, 3$ only.
- $L_{[\Gamma, W]} \equiv 0$ a.s. if $d \geq 4$
- Process X_t has absolutely continuous (in space) states

Properties of Catalytic Superprocesses in R^d

- X_t Absolutely continuous OFF the catalyst Γ_t
- X_t Singular on support of absolutely continuous component of the catalyst Γ_t in dimensions $d \geq 2$.
- In $d = 2$, Γ as in Delmas, $L_{[\Gamma, W]}(ds, dx) = K_s(dx)ds$
  K_s has carrying dimension 2 (Delmas-Fleischmann (2001))

A dynamical random environment with feedback

Mutually Catalytic Branching - Lattice Models

D-Perkins (1998)

Pair of mutually catalytic reactants denoted by
$\mathbf{X} = \left(X^1, X^2\right)$.

System of stochastic differential equations:

$$X_t^j(x) \;=\; X_0^j(x) + \int_0^t \frac{1}{2}\Delta X_s^j(x)\,\mathrm{d}s + \int_0^t \sqrt{\gamma\, X_s^i(x)\, X_s^j(x)}\,\mathrm{d}W_s^j(x)$$

$j = 1, 2,\ i \neq j\ \ t \geq 0,\ x \in \mathbb{Z}^d$, and $\{W^j(x) :\ x \in \mathbb{Z}^d,\ j = 1, 2\}$ is a system of independent one-dimensional Brownian motions. $\gamma > 0$

is the *collision rate*, Δ is the discrete Laplacian.

Existence-Uniqueness

Mytnik Self Duality

Let $(X_t^1(x), X_t^2(x))_{x\in S}$ and $\tilde{X}_t^1(x), \tilde{X}_t^2(x))_{x\in S}$ be two solutions to the system of SDE   Define

$$H((X^1, X^2), (\tilde{X}^1, \tilde{X}^2))$$
$$:= \exp(-\left\langle (X^1 + X^2), (\tilde{X}^1 + \tilde{X}^2) \right\rangle + i \left\langle (X^1 - X^2), (\tilde{X}^1 - \tilde{X})^2) \right\rangle$$
$$\text{where} \left\langle Y, \tilde{Y} \right\rangle := \sum_S Y(x)\tilde{Y}(x)$$

Weak uniqueness follows from the DUALITY RELATION:

$$E(H((X_t^1, X_t^2), (\tilde{X}_0^1, \tilde{X}_0^2))) = E(H((X_0^1, X_0^2), (\tilde{X}_t^1, \tilde{X}_t^2))).$$

Large space-time scale analysis

Theorem (D-Perkins (1998)). Consider simple rw in $d \geq 3$, $X_0^1,\ X_0^2$ are uniform measures. Then

$$P((X_t^1, X_t^2) \in \cdot) \Longrightarrow P_{equil}$$

where P_{equil} is a translation invariant equilibrium measure that preserves the initial intensity.

Recurrent case: $X_0^1,\ X_0^2$ uniform measures. As $t \to \infty$, locally see one predominant type near 0. Analysis of the size and height of the "one type blocks" for t large in the case of Ω_N - Cox-D-Greven (2004).

Scaling limits of lattice systems

The Continuum Limit in $\mathbb{R}^2$

Scaling Limit

(X_t^1, X_t^2) = Solution of systems of SDE on $\mathbb{Z}^2$

Speed up both branching and RW by $\frac{1}{\varepsilon^2}$.

$({}^{\varepsilon}X_t^1, {}^{\varepsilon}X_t^2)$ = System on $\varepsilon\mathbb{Z}^2$:, $(\varepsilon > 0)$

$$ {}^{\varepsilon}X_t^j(x) := X^j_{\frac{t}{\varepsilon^2}}(\frac{x}{\varepsilon}) $$

Theorem (DEFMPX 2002, 2003) For $\gamma < \gamma_*$, as $\varepsilon \to 0$, $({}^\varepsilon X_t^1, {}^\varepsilon X_t^2)_{t \geq 0}$ converges weakly as $M_f(\mathbb{R}^2)$-valued processes to solution of MP

$$M_t^j(\varphi^j) = \left\langle X_t^j, \varphi^j \right\rangle - \left\langle X_0^j, \varphi^j \right\rangle - \int_0^t \left\langle X_s^j, \frac{1}{2}\Delta\varphi^j \right\rangle ds$$

is a continuous square integrable martingale with

$$\begin{aligned} &\langle\langle M^j(\varphi^j), M^k(\varphi^k)\rangle\rangle_t \\ &= \gamma\delta_{jk} \int_0^t \int_{\mathbb{R}^2} \varphi^j(x)\varphi^k(x) L_{[X^1,X^2]}(d(s,x)) \end{aligned}$$

where $L_{[X^1,X^2]}$ is collision local time of X^1 and X^2.

Existence of $L_{[X^1,X^2]})$ is part of statement!

Properties of Mutually Catalytic Branching on $\mathbb{R}^2$

- Absolutely continuous states:

$$X_t^j(dx) = X_t^j(x)dx, \quad a.s. \text{ for fixed } t > 0$$

- State space (energy condition)

$$M_{f,e} = \{\mu = (\mu_1, \mu_2) : \\ : \int\int \left(1 + \log^+ \frac{1}{|x_1 - x_2|}\right) \mu_1(dx_1)\mu(dx_2) < \infty\}$$

- Segregation of Types: For a.a. x the law of $(X_t^1(x), X_t^2(x))$ coincides with the exit state $B(\tau)$ of planar BM started from

$$(X_0^1 \star p_t(x), X_0^2 \star p_t(x))$$

- Blow-up at the interface:

$$\|X^1\|_U = \|X^2\|_U = \infty$$

where $\|X^i\|_U$ is the essential sup over an open subset U of $[0,\infty) \times \mathbb{R}^2$.

- Self-similarity: If $(X_0^1, X_0^2) = (\ell, \ell)$ (Lebesgue), then

$$t \to \varepsilon^2 X_{t/\varepsilon^2}(\frac{\cdot}{\varepsilon}) \approx^L t \to X_t(\cdot)$$

Open Problems

- Can we remove the restriction $\gamma < \gamma_*$?
- Does the continuum limit of mutually catalytic branching exist in $\mathbb{R}^d$, $d > 2$ and what is its structure ?
- Nature of the interface of the two types. How do we determine structural properties, for example, the Hausdorff dimension of the interface?
- Consider the k-cyclic analogue, $k \geq 3$. In $\mathbb{R}^1$ existence but not uniqueness is known. The situation in higher dimensions is unknown.

Exploration of a caricature

This is based on

- Replace the euclidean lattice with a sequence of ultrametric lattices Ω_N.
- Focus on random walks " near the critically recurrent random walk".
- Prove that in this random walk regime the multiple scale structures converge to a Markov chain in the $N \to \infty$ limit.
- Identify the structure of the resulting Markov chain.
- Identify universality classes and their domains of attraction.

This was successfully carried out for monotype systems related to Feller branching and Wright-Fisher diffusions in a series of papers of D-Greven, Baillon, Clément, Greven and den Hollander in the 1990's.

Continuum Limits in an ultrametric space

The Hierarchical Lattice

$$\Omega_N^j = \{(\xi_\ell)_{\ell\in\mathbb{Z},\ell\geq -j} : \xi_\ell \in \{0,1,\ldots,N-1\},\ \exists \ell_0, \xi_\ell = 0\ \forall\ \ell \geq \ell_0\}$$

Hierarchical Distance

$$\delta(\xi,\eta) := \min\{k \in \mathbb{Z} : \xi_m = \eta_m \quad \forall \quad m \geq k\}$$

$$\rho_{N,d}(\xi,\eta) = N^{\frac{\delta(\xi,\eta)+1}{d}} \text{ if } -j < \delta(\xi,\eta) \leq 0 \qquad \text{Ultrametric}$$

Ω_N^∞ is a locally compact abelian group.

Random Walks on Ω_N^j

Make a jump with hierarchical distance $\delta = k$ at rate $\frac{c^{k-1}}{N^{\frac{2}{d}(k-1)}}$, $k \geq -j$ and then choose a point at random from the ball $B_k^j(\xi)$

$$a^j(\xi,\eta) = \sum_{k=-j}^{\infty} \left(\frac{c^{k-1}}{N^{\frac{2}{d}(k-1)}} \cdot \frac{1_{B_k^j(\xi)}(\eta)}{|B_k^j(\xi)|} \right)$$

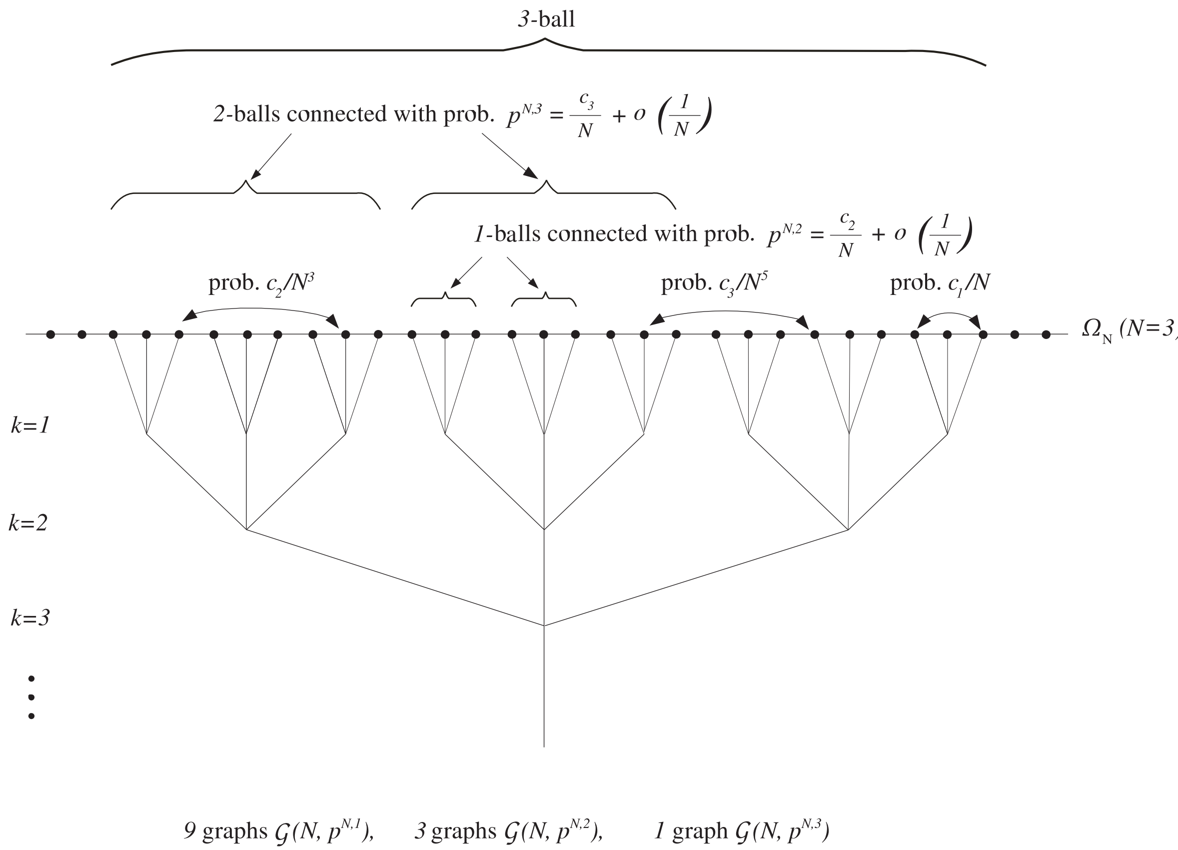

Figure 2: Tree representation of Ω_3

Scaling Limit of the RW:

$j \to \infty$ is the Evans-Lévy process on Ω_N^∞.

Green's function

$$G_N(\eta) = \text{const} \frac{1}{\rho_{N,d}(0,\eta)^{(d-2)}}, \qquad d > 2,\ c = 1$$

Above the critically recurrent case, $d = 2+,\ c > 1$,

$$G_{N,c}(\eta) \to \frac{c}{c-1} \frac{1}{c^{\delta(0,\eta)}} \quad \text{as } N \to \infty.$$

Exploration of the 2+ dimensional analogue on Ω_∞^∞

Dimension $d = 2+$**,** $c > 1$**:**

Approximate densities over small balls in Ω_N^j - Block averages

$$(X_k^{1,j}, X_k^{2,j}) = \frac{\sum_{\xi \in B_k(\xi)} (X_\xi^1, X_\xi^2)}{|B_k|_j}$$

where $|B_k|_j$ denotes the number of Ω_N^j points in the k-ball with center 0.

Approximate collision measures

$$L_{\xi,t}^{j,k}(N) = \int_0^t \left(X_j^{1,j}(N^{j-k}s) X_j^{2,j}(N^{j-k}s) \right)_k ds$$

Continuum Hierarchical Mean Field Limit

Theorem: Existence - Cox-D-Greven (2004)

1. HMF Limit. For fixed $t > 0$, as $N \to \infty$

$$\mathcal{L}(\left(X_j^{1,j}, X_j^{2,j}\right)_{k=0,-1,\dots,-j} \Longrightarrow \mathcal{L}\left((M_k^j)_{k=0,\dots,-j}\right)$$

where $\left((M_k^j)_{k=0,\dots,-j}\right)$ is a Markov chain on $(\mathbb{R}_+)^2$ and

$$(L_\xi^{j,k}(N))_{t\geq 0} \Longrightarrow (L^{j,k})_{t\geq 0}$$

2. HMF Continuum Limit. As $j \to \infty$

$$(M_k^j)_{k=0,-1,\dots,-j} \Longrightarrow (M_k^{*,\infty})_{k\in\mathbb{Z}_-}$$

3. **Singularity - absolute continuity dichotomy**

(a) $, d = 2-,\ c < 1$

$$M_\infty^\infty \in \mathrm{int}(\mathbb{R}^+)^2$$

with positive probability.

(b) $, d = 2+,\ c > 1$

$$M_\infty^\infty \in \partial(\mathbb{R}^+)^2$$

4. **The Interface and Hot Spots**.

Size-biasing: h-transform where $h(x, y) = xy$ is a harmonic function for the Markov chain.

h-transform: $\widehat{M}_j^j = (\widehat{U}_j^j, \widehat{V}_j^j)$

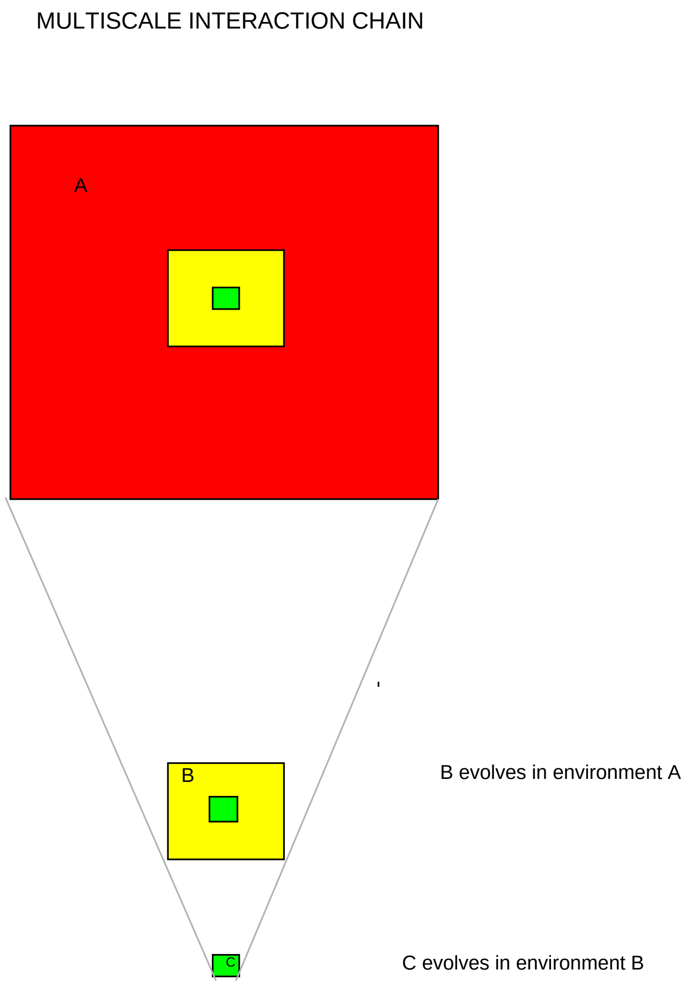

Figure 3: Microscopic zoom with size-biasing

Theorem (D-Greven-Zähle (in progress))

Hot spots: critical case

If $c = 1$, then the xy-biased chain $\widehat{M}_j^j$ satisfies: (a) the growth of the components satisfies:

$$\left(\frac{1}{j}\log \widehat{U}_j^j, \frac{1}{j}\log \widehat{V}_j^j\right) \to_{j\to\infty} (g^\gamma, g^\gamma) > 0$$

with fluctuations given by the CLT

$$\mathcal{L}[((\log \widehat{U}_j^j) - g^\gamma j, (\log \widehat{V}_j^j) - g^\gamma j))/\sqrt{j}] \to_{j\to\infty} \mathcal{N}(0, \sigma^2)$$

(b) Asymptotic type distribution. As $j \to \infty$,

$$\mathcal{L}\left[\left(\frac{\widehat{U}_j^j}{\widehat{U}_j^j + \widehat{V}_j^j}, \frac{\widehat{V}_j^j}{\widehat{U}_j^j + \widehat{V}_j^j}\right)\right] \to_{j\to\infty} \mathcal{L}[(\widehat{Z}_\infty^\infty, 1 - \widehat{Z}_\infty^\infty)],$$

$\widehat{Z}_\infty^\infty$ is in law symmetric around $1/2$ and $\widehat{Z}_\infty^\infty(0,1) = 1$.

Present proof requires $\gamma < \gamma_*$.

Interplay of Anderson and modified Wright-Fisher dynamics

$$
\begin{aligned}
dz_t \quad &= \tfrac{\varrho(\theta_1+\theta_2)}{r_t}\left(\tfrac{\theta_1}{\theta_1+\theta_2} - z_t\right)dt + \gamma[(2z_t - 1)z_t(1 - z_t)]dt \\
&\quad +(1 - z_t)\sqrt{\gamma z_t(1 - z_t)}dw_t^1 - z_t\sqrt{\gamma z_t(1 - z_t)}dw_t^2 \\
dr_t \; &= \varrho(1 - r_t)dt + \sqrt{2\gamma r_t^2 z_t(1 - z_t)}d\bar{w}_t \quad , \quad \bar{w}_t = w_t^1 + w_t^2. \qquad (1)
\end{aligned}
$$

Conclusions

In this simplified setting we have obtained information on the nature of the interface.

- Exponential growth of the densities of both types at the interface as a function of the radius of the ball.
- Both types grow at the same order but with random proportions.
- Conjecture: In the case $c > 1$, the densities of both types grow super-exponentially.

Open problems

- Nature of the interfaces and their dynamics.
- Non-constant coefficients - universality class of mutually catalytic branching
- Extension to cyclical catalytic branching
- Role of spatial structure in the emergence and evolution of catalytic branching networks.

Block average dynamics

$$(X_k^1(t), X_2^2(t)) = \frac{1}{N^k} \sum_{d(0,\xi)\leq k} (X_\xi^1(t), X_\xi^2(t))$$

$$dX_k^i(t) = \sum_{j\geq 1} \frac{c_{k+j-1}}{N^{j-1}} \left[X_{k+j}^i(t) - X_k^i(t)\right] d\left(\frac{t}{N^k}\right) + \sqrt{g_i^{N,k}(X_k^i(t))} \cdot dW_k^i\left(\frac{t}{N^k}\right)$$

HMF Limit Dynamics

Multiple Space and Time Scales

$$X_k^i(tN^k) \Rightarrow \mathcal{X}_k^i(t)$$

$$d\mathcal{X}_k^i(t) = c_k \left[\theta - \mathcal{X}_k^i(t)\right] dt + \sqrt{(\mathcal{F}^k g_i)(\mathcal{X}_k^i(t))} dW_k^i(t)$$

$$\mathcal{F}^k g = (\mathcal{F}_{c_{k-1}} \circ \mathcal{F}_{c_{k-2}} \circ \cdots \circ \mathcal{F}_{c_0})g$$

The renormalization map

- Assume that g is in a some regularity class $\mathcal{H}$ uniquely determines a diffusion and this diffusion has a *unique equilibrium* ${\Gamma_\theta}^{c,g}$.

  We define the *nonlinear map* $\mathcal{F}_c$ by

$$\mathcal{F}_c(g)(\theta) = E^{{\Gamma_\theta}^{c,g}}(g) = \int_{(\mathbb{R}^+)^2} g(\widetilde{\theta})\, {\Gamma_\theta}^{c,g}(d\widetilde{\theta}), \quad \theta \in (\mathbb{R}^+)^2.$$

$$\mathcal{F}_c(g) : \mathcal{H} \longrightarrow \mathcal{H}$$

- Iterates $(\mathcal{F}_c(g))^k$: a sequence of diffusions where the $(k+1)st$ diffusion matrix g_{k+1} arises from the McKean-Vlasov limit $N \to \infty$ of an exchangeable sequence of interacting diffusions with the diffusion matrix g_k.

Fixed Points $\sim$ Universality Classes

Theorem (D-Greven-den Hollander-Sun,Swart (2008))

The set of all the fixed points of the map $\mathcal{F}_c$ is given by:

$$g_i(x_1, x_2) = b_i x_i + c_i x_1 x_2, \quad i = 1, 2,$$

$$b_i, c_i \geq 0, c_i \geq 0, b_i + c_i > 0.$$

Open Problems

- Describe the orbit followed by iterates $(\mathcal{F}_c)^n$. Does this always converge to the fixed point?
  Partial results in D-Greven-den Hollander-Sun,Swart (2008)
- Determine the fixed points and their domains of attraction for cyclic catalytic branching with $M > 2$.