On the dynamics of trap models in $\mathbb{Z}^d$

Luiz Renato Fontes

joint with Pierre Mathieu

Table of contents

Trap model

Embedded random walk

$\xi_1, \xi_2, \ldots$ i.i.d. $\mathbb{Z}^d$-valued:

$$X_0 = 0, \quad X_n = \sum_{i=1}^{n} \xi_i, \quad n \geq 1.$$

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Let

$$\mathcal{R}_n = \{z \in \mathbb{Z}^d : X_i = z \text{ for some } i \leq n\}$$

$$R_n = |\mathcal{R}_n|, \quad \rho_n = \mathbb{E}(R_n), \quad r_n = \mathbb{P}(X_1 \neq 0, \ldots, X_n \neq 0)$$

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$$(r_n) \text{ slowly varying}$$
$$\frac{R_n}{\rho_n} \to 1 \quad \text{in probability as } n \to \infty$$

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(include all transient, all planar, some recurrent 1d 1-stable cases)

Trap model

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$\tau = \{\tau_x,\, x \in \mathbb{Z}^d\}$ i.i.d. positive independent of X:

$$\mathbb{P}(\tau_0 > t) = \ell(t)\, t^{-\alpha},\ t \geq 0, \quad \ell \text{ slowly varying, } \alpha \in (0,1).$$

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Let

$$C(n) = \sum_{i=0}^{n} \tau_{X_i} T_i, \quad n \geq 0, \quad I_t = C^{-1}(t),$$

with $T_0, T_1, T_2, \ldots$ i.i.d. mean 1 exponentials, and

$$Y_t = \tau_{X_{I_t}}.$$

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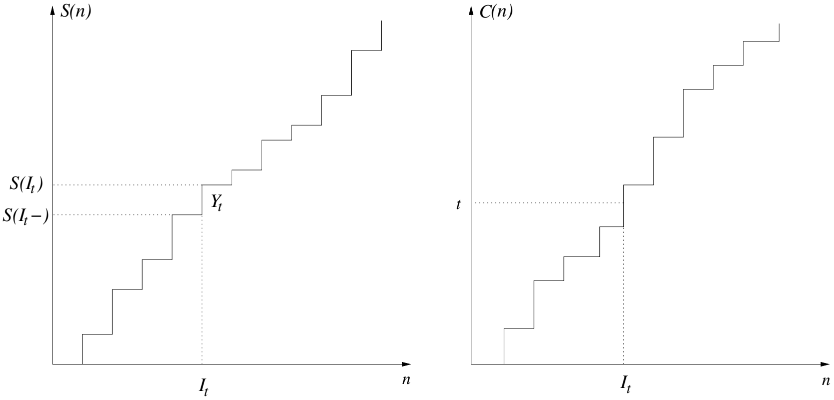
Note: $Y_t = S(I_t) - S(I_t-)$, where $S(n) = \sum_{i=0}^{n} \tau_{X_i}$.

Trap model

S, C and Y



Continuum model

$U = (U_t)_{t \geq 0}$: an α-stable subordinator

$\{T_t;\, t \geq 0\}$: iid mean 1 exponentials

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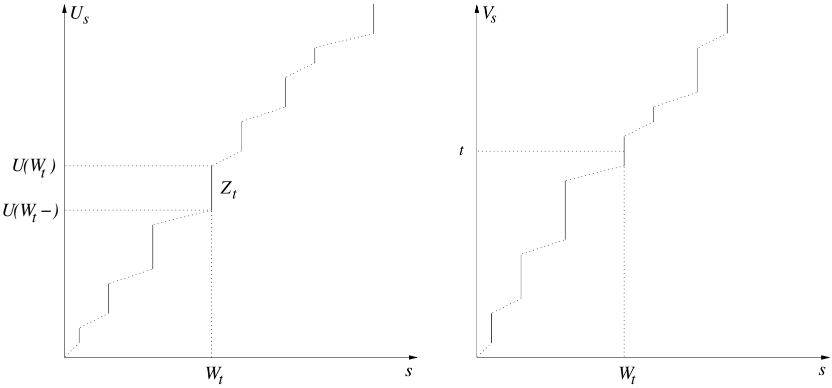
U, V and Z



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- $V \sim U$ is also an α-stable subordinator
- Given U: Z is Markovian
- Z is self-similar of index 1

Scaling limit

Topology

D, D_T: càdlàg functions on $[0, \infty)$, $[0, T]$, resp.
d_T: L_1 distance in D_T, and $d = \sum_{n=1}^{\infty} 2^{-n}(d_n \wedge 1)$.

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$$a_\varepsilon = v_{n(\varepsilon^{-1})}^{-1} \approx \frac{\varepsilon}{r_{n(\varepsilon^{-1})}}.$$

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Theorem 1

Let $Y_t^{(\varepsilon)} = a_\varepsilon Y_{\varepsilon^{-1} t}$. Then

$$Y^{(\varepsilon)} \to Z \text{ in distribution on } (D, d).$$

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Theorem 2

$$\lim_{t\to\infty} \bar{R}(t, t+\theta t) = R(\theta),$$
$$\lim_{t\to\infty} \bar{\Pi}(t, t+\theta v_{n(t)}) = \Pi(\theta),$$
$$\lim_{t\to\infty} Q(t, t+\theta t) = Q(\theta).$$

Aging results

Limiting aging functions

- $$\Pi(\theta) = R(\theta) = \mathbb{P}(Z_1 = Z_{1+\theta}) = \mathbb{P}(\text{Range of } V \cap [1, 1+\theta] = \emptyset)$$
$$= \frac{\sin(\pi\alpha)}{\pi} \int_{\theta/(1+\theta)}^{1} s^{-\alpha}(1-s)^{\alpha-1}\, ds$$

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Scaling limit and aging results for $\hat{X}_t = X_{I_t}$, X simple symmetric

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- (Complete Graph) Bouchaud 95; Bovier-Faggionato 05; Mathieu-F 08; Gayrard 09 (asymmetric); Bezerra-Gava-Gayrard-Mathieu-F 10

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- (Tori) Ben Arous-Černý 06, Jara-Landim et al 08+ (interacting particles)

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- (Sinai's RWRE) Bovier-Faggionato 08

Stronger convergence

Results of Ben Arous-Černý-Mountford 06; Ben Arous-Černý 07 (also Barlow-Černý 09, Mourrat 09-10) hold for a.e. τ.

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In our case, need extra assumption : let X' be an independent copy of X and make

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(includes all transient, and all finite variance cases in $d \geq 2$).

Theorem 3

Under extra assumption, for every $F : (D, d) \to \mathbb{R}$ bdd unif'ly cont's

$$\mathbb{E}\left[F(Y^{(\varepsilon)})\middle|\,\tau\right] \to \mathbb{E}\left[F(Z)\right],$$

in probability as $\varepsilon \to 0$.

Stronger aging results

Theorem 4

Under extra assumption, we have

$$\lim_{t\to\infty} \mathbb{P}(Y_t = Y_{t+\theta t} | \tau) = R(\theta),$$
$$\lim_{t\to\infty} \mathbb{P}(Y_t = Y_{t+r} \text{ for all } r \in [0, \theta t] | \tau) = \Pi(\theta),$$
$$\lim_{t\to\infty} \mathbb{P}(\sup_{r\in[0,t]} Y_r < \sup_{r\in[0,t+\theta t]} Y_r | \tau) = Q(\theta).$$

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$$\tilde{C}_n = \sum_{j=0}^{\infty} \tilde{\tau}_j \sum_{i=1}^{L(\tilde{X}_j, n)} \tilde{T}_i^{(j)}, \quad n \geq 0,$$

$\{\tilde{\tau}_n,\ n \geq 0\}$ iid, $\tilde{\tau}_0 \sim \tau_0$; $\{\tilde{T}_i^{(j)}, j \geq 0, i \geq 1\}$ iid, $T_1^{(0)} \sim T_0$,
$L(x, n) = \sum_{i=0}^{n} 1\{X_i = x\}$.

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Then

$$\tilde{C} \sim C.$$

Proof of Theorem 1

Step 1 (contd.)

Let

$$\tilde{I} = \tilde{C}^{-1},$$

and consider the map $\varphi : \mathbb{N} \to \mathbb{N}$ such that

$$\varphi_n = m \quad \text{iff} \quad \tilde{X}_m = X_n.$$

Proof of Theorem 1

Step 1 (contd.)

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Defining

$$\tilde{Y}_t = \tilde{\tau}_{\varphi_{\tilde{I}_t}}, \; t \geq 0,$$

we have that

$$(\tilde{Y}_t)_{t \geq 0} \sim (Y_t)_{t \geq 0}.$$

Proof of Theorem 1

Step 2: scaling

Let

$$\hat{C}_t^{(\varepsilon)} = \varepsilon \sum_{j=0}^{\infty} \tilde{\tau}_j \sum_{i=1}^{L(\tilde{X}_j, n(\varepsilon^{-1} t))} \tilde{T}_i^{(j)},$$

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$\hat{I}^{(\varepsilon)} = (\hat{C}^{(\varepsilon)})^{-1}$, and

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$$(\hat{Y}_t^{(\varepsilon)}) = (a_\varepsilon \, \tilde{Y}_{\varepsilon^{-1} t}) \sim Y^{(\varepsilon)}.$$

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For $x \in \tilde{\varepsilon}\,\mathbb{N}$, with $\tilde{\varepsilon} = \rho^{-1}_{n(\varepsilon^{-1})}$, let

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Then $\{\tau^{(\varepsilon)}_x, x \in \tilde{\varepsilon}\,\mathbb{N}\} \sim \{\tilde{\tau}_j, j \in \mathbb{N}\}$,

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Step 3: couple scaled environment to limit

For $x \in \tilde{\varepsilon}\,\mathbb{N}$, with $\tilde{\varepsilon} = \rho^{-1}_{n(\varepsilon^{-1})}$, let

$$\tau^{(\varepsilon)}_x = v_{n(\varepsilon^{-1})}\, g_\varepsilon(U_{x+\tilde{\varepsilon}} - U_x)\,,$$
$$g_\varepsilon(y) = \frac{1}{v_{n(\varepsilon^{-1})}}\, G^{-1}(\tilde{\varepsilon}^{-1/\alpha} y),$$
$$\mathbb{P}(U_1 > G(y)) = \mathbb{P}(\tau_0 > y).$$

Then $\{\tau^{(\varepsilon)}_x,\, x \in \tilde{\varepsilon}\,\mathbb{N}\} \sim \{\tilde{\tau}_j,\, j \in \mathbb{N}\}$, and $\{\hat{C}^{(\varepsilon)}_t,\, t \geq 0\} \sim$

$$\bar{C}^{(\varepsilon)}_t = \sum_{x \in \tilde{\varepsilon}\mathbb{N}} g_\varepsilon(U_{x+\tilde{\varepsilon}} - U_x) \left\{ \bar{r}_{n(\varepsilon^{-1})} \sum_{i=1}^{L(\tilde{X}_{\tilde{\varepsilon}^{-1}x},\, n(\varepsilon^{-1}t))} T^{(\tilde{\varepsilon}^{-1}x)}_i \right\},$$

with $$\bar{r}_{n(\varepsilon^{-1})} = \varepsilon\, v_{n(\varepsilon^{-1})} = \varepsilon\, \nu_{n(\varepsilon^{-1})}\, r_{n(\varepsilon^{-1})}.$$

Proof of Theorem 1

Step 3 (contd.)

Let now $\bar{I}^{(\varepsilon)} = (\bar{C}^{(\varepsilon)})^{-1}$, and make

$$\bar{Y}_t^{(\varepsilon)} = a_\varepsilon \, \tau^{(\varepsilon)}_{\tilde{\varepsilon}\varphi\left(n(\varepsilon^{-1}\bar{I}_t^{(\varepsilon)})\right)} = g_\varepsilon \left(U_{\tilde{\varepsilon}\left(\varphi\left(n(\varepsilon^{-1}\bar{I}_t^{(\varepsilon)})\right)+1\right)} - U_{\tilde{\varepsilon}\varphi\left(n(\varepsilon^{-1}\bar{I}_t^{(\varepsilon)})\right)} \right).$$

Proof of Theorem 1

Step 3 (contd.)

Let now $\bar{I}^{(\varepsilon)} = (\bar{C}^{(\varepsilon)})^{-1}$, and make

$$\bar{Y}_t^{(\varepsilon)} = a_\varepsilon\, \tau^{(\varepsilon)}_{\tilde{\varepsilon}\varphi\left(n(\varepsilon^{-1}\bar{I}_t^{(\varepsilon)})\right)} = g_\varepsilon\left(U_{\tilde{\varepsilon}\left(\varphi\left(n(\varepsilon^{-1}\bar{I}_t^{(\varepsilon)})\right)+1\right)} - U_{\tilde{\varepsilon}\varphi\left(n(\varepsilon^{-1}\bar{I}_t^{(\varepsilon)})\right)}\right).$$

Then $(\bar{Y}_t^{(\varepsilon)})_{t\geq 0} a \sim (\hat{Y}_t^{(\varepsilon)})_{t\geq 0}$.

Proof of Theorem 1

Step 3 (contd.)

Let now $\bar{I}^{(\varepsilon)} = (\bar{C}^{(\varepsilon)})^{-1}$, and make

$$\bar{Y}_t^{(\varepsilon)} = a_\varepsilon \, \tau^{(\varepsilon)}_{\tilde{\varepsilon}\varphi\left(n(\varepsilon^{-1}\bar{I}_t^{(\varepsilon)})\right)} = g_\varepsilon\left(U_{\tilde{\varepsilon}\left(\varphi\left(n(\varepsilon^{-1}\bar{I}_t^{(\varepsilon)})\right)+1\right)} - U_{\tilde{\varepsilon}\varphi\left(n(\varepsilon^{-1}\bar{I}_t^{(\varepsilon)})\right)}\right).$$

Then $(\bar{Y}_t^{(\varepsilon)})_{t\geq 0} a \sim (\hat{Y}_t^{(\varepsilon)})_{t\geq 0}$. Enough now to prove:

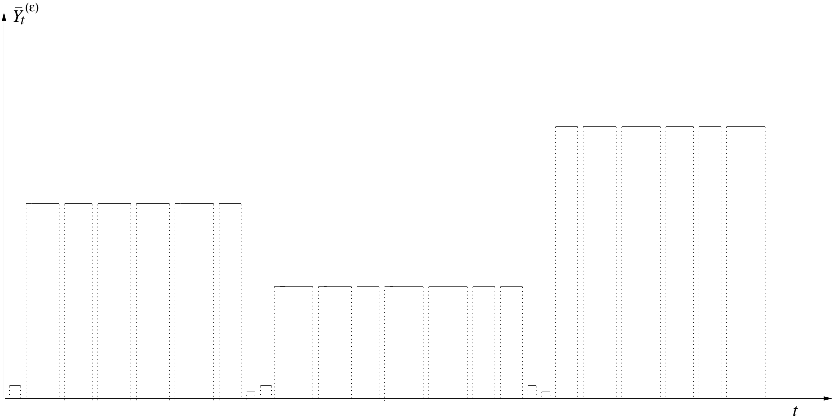
Lemma

For almost every U

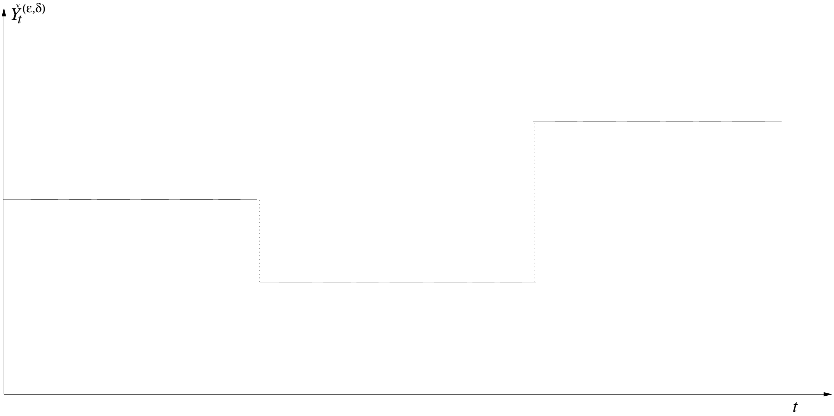
$$\bar{Y}^{(\varepsilon)} \to Z$$

as $\varepsilon \searrow 0$ in distribution on (D, d).

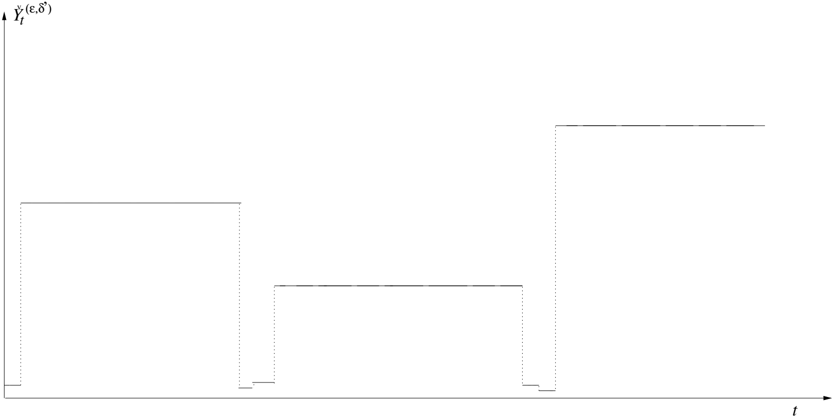
$\bar{Y}^{(\varepsilon)}$



$\check{Y}^{(\varepsilon,\delta)}$



$\check{Y}^{(\varepsilon,\delta')}$, $\delta' < \delta$



Conclusion of proof of lemma

For a.e. U,

$$\lim_{\delta \to 0} \limsup_{\varepsilon \to 0} d((\check{Y}_t^{(\varepsilon,\delta)}), (\bar{Y}_t^{(\varepsilon)})) = 0 \text{ in } \mathbb{P}(\cdot|U) \text{ probability},$$

$$\lim_{\varepsilon \to 0}(\check{Y}_t^{(\varepsilon,\delta)}) = (Z_t^\delta),\ \lim_{\delta \to 0}(Z_t^\delta) = (Z_t) \text{ in } \mathbb{P}(\cdot|U) \text{ distribution}.$$

Conclusion of proof of lemma

For a.e. U,

$$\lim_{\delta\to 0}\limsup_{\varepsilon\to 0} d((\check{Y}_t^{(\varepsilon,\delta)}),(\bar{Y}_t^{(\varepsilon)})) = 0 \text{ in } \mathbb{P}(\cdot|U) \text{ probability,}$$

$$\lim_{\varepsilon\to 0}(\check{Y}_t^{(\varepsilon,\delta)}) = (Z_t^\delta),\ \lim_{\delta\to 0}(Z_t^\delta) = (Z_t) \text{ in } \mathbb{P}(\cdot|U) \text{ distribution.}$$

(Use

$$L(0,bn) - L(0,an) \to 0$$

in probability as $n\to\infty$, $b>a>0$.)

Conclusion of proof of lemma

For a.e. U,

$$\lim_{\delta\to 0}\limsup_{\varepsilon\to 0} d((\check{Y}_t^{(\varepsilon,\delta)}),(\bar{Y}_t^{(\varepsilon)}))=0 \text{ in } \mathbb{P}(\cdot|U) \text{ probability,}$$

$$\lim_{\varepsilon\to 0}(\check{Y}_t^{(\varepsilon,\delta)})=(Z_t^{\delta}),\ \lim_{\delta\to 0}(Z_t^{\delta})=(Z_t) \text{ in } \mathbb{P}(\cdot|U) \text{ distribution.}$$

(Use

$$L(0,bn)-L(0,an)\to 0$$

in probability as $n\to\infty$, $b>a>0$.)

For aging, also use

$$\lim_{\delta\to 0}\limsup_{\varepsilon\to 0}\mathbb{P}(\bar{Y}_t^{(\varepsilon)}\neq \check{Y}_t^{(\varepsilon,\delta)})=0.$$