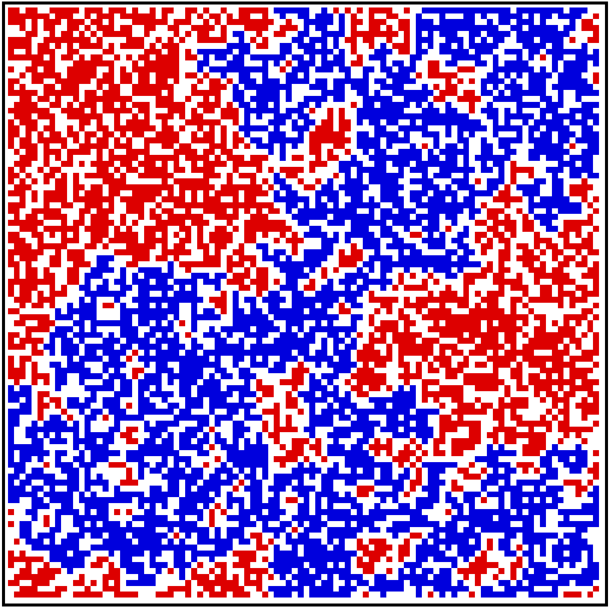
Arm Exponents for high-d percolation

Gady Kozma (speaker) and Asaf Nachmias

Dynamic Random Environments, Eindhoven, 2009

Critical $2d$ percolation



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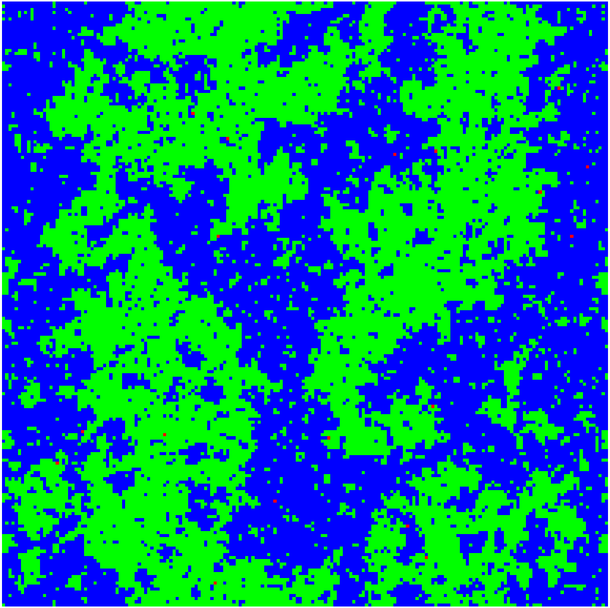

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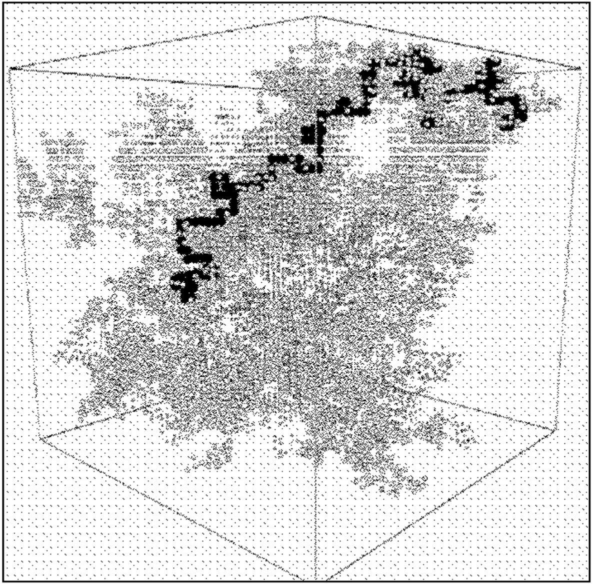

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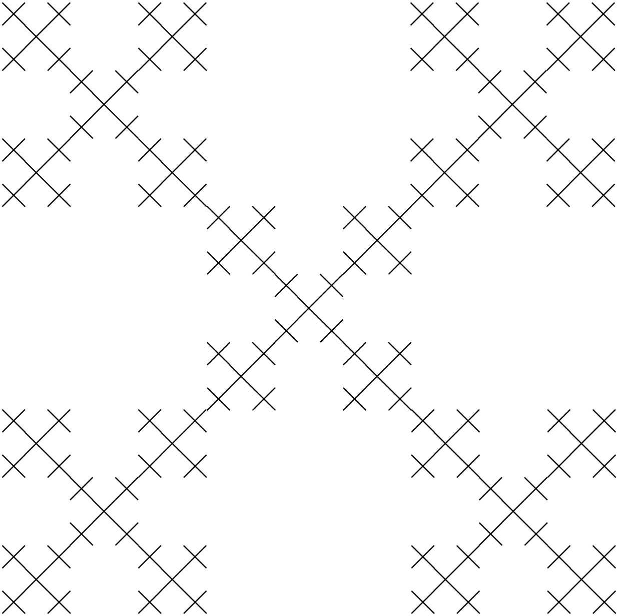

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Onset of high-dimensionality is usually related to the finiteness of some diagram. For percolation this is the triangle diagram.

Definition

Let G be an transitive infinite graph and let $0 < p < 1$. Define

$$\nabla = \sum_{x,y} \mathbb{P}(0 \leftrightarrow x)\mathbb{P}(x \leftrightarrow y)\mathbb{P}(0 \leftrightarrow y)$$

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and a simple calculation shows that the triangle condition holds whenever $p < \frac{1}{\sqrt{k-1}}$. In particular it holds for $p_c = \frac{1}{k-1}$. (the triangle condition cannot hold for $p > p_c$ for $\mathbb{Z}^d$, or in general for any amenable graph)

Corollaries

Assume our graph satisfies the triangle condition at p_c

- $\mathbb{E}_p(|\mathscr{C}(0)|) \approx (p_c - p)^{-1}$. "$\gamma = 1$". Aizenman & Newman (1984).

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Mean-field vs. $2d$

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- The case $d > 6$ was proved when the lattice is sufficiently spread out or when d is sufficiently large. Hara & Slade (1990), Hara, van der Hofstad & Slade (2003). Most results go via $\nabla < \infty$.

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- When $3 \leq d \leq 5$ there is no good conjecture for the values of these exponentials.
- ρ_{gr} is not known in $d = 2$, probably because the graph distance is not a conformal invariance

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- η defined by $\mathbb{P}(0 \leftrightarrow x) \approx |x|^{2-d+\eta}$.
- ρ defined by $\mathbb{P}(0 \leftrightarrow \partial B(0,r)) \approx r^{-1/\rho}$.

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- Aizenman (1997): the number of crossing clusters of a box of side-length r is $\approx r^{d-6}$.
- Barsky & Aizenman may be simplified considerably by using $\eta = 0$, $d > 6$ instead of $\nabla < \infty$.

Our result

Let G be a lattice in $\mathbb{R}^d$ satisfying

1. $d > 6$,
2. $\mathbb{P}(0 \leftrightarrow x) \approx |x|^{2-d}$,
3. The lattice is invariant to coordinate permutations and reflections.

Then

$$\mathbb{P}(0 \leftrightarrow \partial B(0,r)) \approx r^{-2}$$

In short: $\eta = 0$ implies $\rho = 1/2$.

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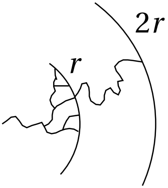

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- A careful choice of what is a “bad vertex”.

A lemma

Lemma 1. Let G be a lattice in $\mathbb{R}^d$ for *any* d, invariant to coordinate permutations and reflections. Let Q_r be a box of size-length r centered at 0. Let $z \in \partial Q_r$. Then

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- Nevertheless, any subexponential estimate (and even an exponential estimate with sufficiently low constant) would have been enough for the application.
- This lemma is the only place in the proof the symmetries of the lattice are needed.

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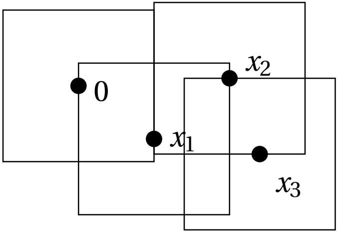

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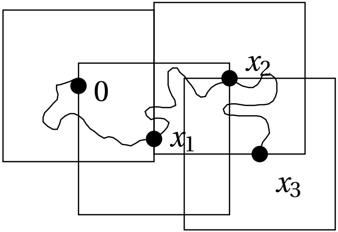

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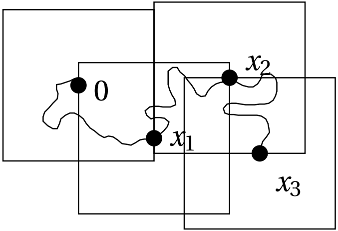
$$\sum_{x_1, \ldots, x_n} E(x_1, \ldots, x_n) \leq (1 - \epsilon)^n.$$



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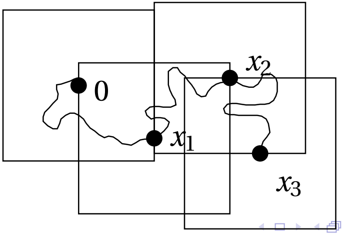

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Proof of lemma 1

Lemma 1. Let Q_r be a box of size-length r centered at 0. Let $z \in \partial Q_r$. Then

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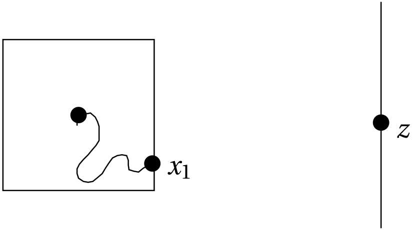

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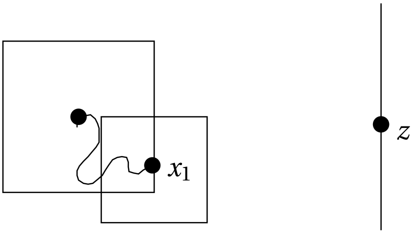

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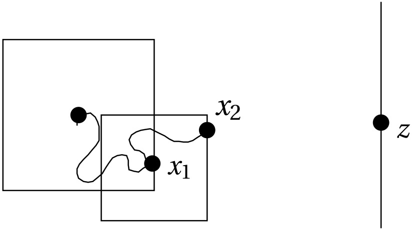

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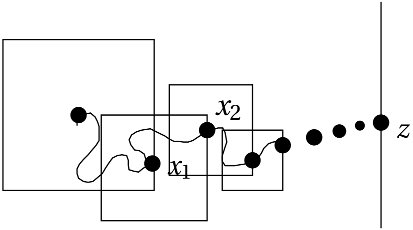

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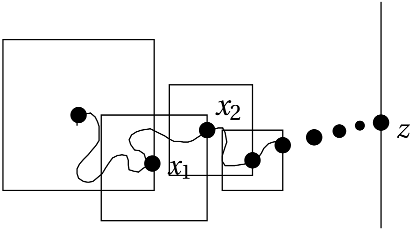

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$$\mathbb{P}(0 \xleftrightarrow{Q_r} z) \geq \prod_i \mathbb{P}(x_i \xleftrightarrow{Q_i} x_{i+1}) \geq \left(c r^{1-d}\right)^{C \log r}. \quad \square$$

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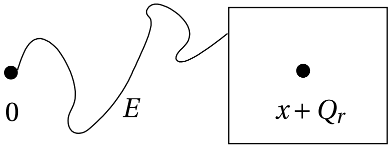
- Of course, $0 \leftrightarrow x$ implies in particular $0 \leftrightarrow x + \partial Q_r$. Denote this event by E.



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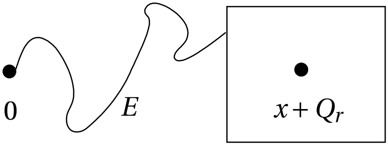

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The proof of lemma 3 is a straightforward exploration of $\mathscr{C}(0)$.

Conclusions

- Working with the graph distance is far easier than with the Euclidean distance (3 pages vs. 40).
- Regularity analysis can replace a direct use of the triangle condition in many cases.

Thank you