Random walk in dynamic random environment: a perturbative approach

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Setting

1. Dynamic random environment: spin system, i.e., Feller process $\{\eta_t : t \geq 0\}$ on $\Omega = \{0,1\}^{\mathbb{Z}}$ with generator

$$L_{IPS} f(\eta) = \sum_x c(x,\eta)\,(f(\eta^x) - f(\eta))$$

2. Random walk $\{X_t : t \geq 0\}$: jumping to right, left with rates $c^{\pm}(\eta_t)$ where

$$c^{\pm}(\eta) = \frac{\alpha+\beta}{2} \pm \frac{\alpha-\beta}{2}(2\eta(0)-1)$$

with $\alpha > \beta$

Hypotheses on the IPS

1. **Translation invariance**

$$c(x,\eta) = c(0,\tau_x\eta)$$

   with $\tau_x\eta(y) = \eta(y+x)$

2. **High noise: "$M < \epsilon$".**
   Exponentially fast convergence to the unique stationary measure μ, uniformly in the starting configuration.

$$\rho := \int \eta(0)\mu(d\eta)$$

Questions

1. Law of large numbers: there exists a constant v such that, almost surely,

$$\lim_{t\to\infty} \frac{X_t}{t} = v$$

2. Central limit theorem

$$\lim_{\lambda\to\infty} \frac{X_{\lambda t} - v\lambda t}{\sqrt{\lambda}} = W_{\sigma t}$$

for some $\sigma > 0$

3. Large deviations: what can be said about

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E} e^{\lambda X_t} =: F(\lambda)$$

High noise

Define for $f \in \mathcal{C}(\Omega)$

$$\|f\|_\infty = \sup_{\eta \in \Omega} |f(\eta)|$$

$$\delta_x f = \sup_\eta (f(\eta^x) - f(\eta))$$

$$|||f||| = \sum_x \delta_x f$$

There exists $c > 0$ (in Liggett's notation:($c = \epsilon - M$)) such that we have the estimates

$$|||S_{IPS}(t)f||| \leq e^{-ct}|||f|||$$

$$\|S_{IPS}(t)f - \int f d\mu\|_\infty \leq Ce^{-ct}|||f|||$$

Environment process

The environment as seen from the position of the walker determines the behavior of the walker:

$$\xi_t := \eta_t(X_t + \cdot)$$

This is a Markov process with generator

$$L = L_0 + L_*$$

$$L_0 f(\eta) = \frac{1}{2}(\alpha + \beta)\left(f(\tau_1\eta) + f(\tau_{-1}\eta) - 2f(\eta)\right) + L_{IPS} f(\eta)$$

$$L_* f(\eta) = \frac{\alpha - \beta}{2}(2\eta(0) - 1)\left(f(\tau_1\eta) - f(\tau_{-1}\eta)\right)$$

The unperturbed part of the generator

The part L_0 is easy: it is the generator of the process $\tau_{Y_t}\eta_t$ where Y_t is a continuous random walk jumping symmetrically between neighbors at rate $\alpha+\beta$, **independent of the environment** η_t. Since translations commute with the generator L_{IPS}, we have the same estimates on $\|\|_\infty$ and $|||\cdot|||$ as for L_{IPS}, i.e., putting $S_0(t) = e^{tL_0}$, we have

$$\|S_0(t)f - \int f d\mu\|_\infty < Ce^{-ct}|||f|||,\ |||S_0(t)f||| \leq e^{-ct}|||f|||$$

Adding the perturbation

Now we assume that L_* is "small":

$$(\alpha - \beta) < \frac{c}{2}$$

This means $\|L_*\| < c$, i.e., the operator norm of the perturbation is smaller than the "spectral gap" of the unperturbed part L_0. The result of the expansion is that the perturbed semigroup $S(t) = e^{t(L_0 + L_*)}$ satisfies the same estimates as $S_0(t)$ but with a smaller "gap" $c' = c - 2(\alpha - \beta)$.

The expansion

$$
\begin{aligned}
S(t)f &= e^{t(L_0+L_*)}f \\
&= e^{tL_0}f + \int_0^t S_0(t-s)L_*S(s)f\ ds \\
&= e^{tL_0}f + \int_0^t S_0(t-s)L_*S_0(s)f\ ds \\
&\quad + \int_0^t \int_0^s S_0(t-s)L_*S_0(s-r)L_*S(r)f\ dsdr \\
&= \ldots = \sum_{n=1}^{\infty} g_n(t,f)
\end{aligned}
$$

with

$$g_1(t,f) = S_0(t)f$$

$$g_{n+1}(t,f) = \int_0^t S_0(t-s)L_* g_n(s,f)$$

Using

$$\left|\left|\left|\int_0^t S_0(t-s)L_* g_n(s,f)\right|\right|\right| \leq \int_0^t e^{-c(t-s)} |||L_* g_n(s,f)||| ds$$

we get, by induction

$$|||g_n(t,f)||| \leq e^{-ct}\frac{(2(\alpha-\beta)t)^{n-1}}{(n-1)!}|||f|||$$

and from that a uniform bound in $t > 0$:

$$\begin{aligned}
\|g_{n+1}(t,f)\|_\infty &\leq \left\| \int_0^t S_0(t-s) L_* g_n(s,f) \right\|_\infty \\
&\leq \int_0^t \|L_* g_n(s,f)\|_\infty ds \\
&\leq 2(\alpha - \beta) \int_0^t |||g_n(s,f)||| ds \\
&\leq (\alpha - \beta)|||f||| \int_0^t e^{-cs} \frac{(2(\alpha-\beta)s)^{n-1}}{(n-1)!} ds \\
&\leq |||f||| \left(\frac{(2(\alpha-\beta)}{c} \right)^n
\end{aligned}$$

Finally, we can estimate

$$h_n(t,f) = g_n(t,f) - < g_n(t,f) >_\mu$$

$$\begin{aligned}
& \|h_{n+1}(t,f)\|_\infty && (1)\\
= & \left\| \int_0^t (S_0(t-s)(L_* g_n(s,f)) - < L_* g_n(s,f) >_\mu) \; ds \right| && (2)\\
\leq & \; C \int_0^t e^{-c(t-s)} |||L_* g_n(s,f)||| \; ds && (3)\\
= & \; C \int_0^t e^{-c(t-s)} |||L_* h_n(s,f)||| \; ds && (4)\\
\leq & \; C 2(\alpha - \beta) \int_0^t e^{-c(t-s)} |||h_n(s,f)||| && (5)
\end{aligned}$$

The fruits of the expansion

1. Uniform convergence for all finite $t > 0$:

$$S(t)f = \sum_{n=0}^{\infty} g_n(t, f)$$

2. Exponentially rapid convergence:

$$\|S(t)f - < S(t)f >_\mu \|_\infty < Ce^{-(c-2(\alpha-\beta))t} |||f|||$$

3. Uniform convergence to the limit $t \to \infty$:

$$\lim_{t\to\infty} S(t)f = \sum_{n=0}^{\infty} \lim_{t\to\infty} < g_n(t, f) >_\mu$$

4. Unique stationary distribution μ_e of the environment process with

$$< f >_{\mu_e} = \sum_{n=1}^{\infty} < \Psi_n >_\mu$$

with recursion of Ψ_n:

$$\Psi_1 = f$$

$$\Psi_{n+1} = L_* L_0^{-1}(\Psi_n - < \Psi_n >_\mu)$$

To see e.g. Ψ_2:

$$\lim_{t\to\infty} \left\langle \int_0^t S_0(t-s) L_* g_1(s,f) \right\rangle_\mu$$

$$\lim_{t\to\infty} \left\langle \int_0^t L_* g_1(s,f) \right\rangle_\mu$$

$$\lim_{t\to\infty} \left\langle \int_0^t L_* S_0(s) f \right\rangle_\mu$$

$$\lim_{t\to\infty} \left\langle \int_0^t L_*(S_0(s)f - < f >_\mu) \right\rangle_\mu$$

$$\left\langle L_* \int_0^\infty S_0(s)(f - < f >_\mu) \right\rangle_\mu = < L_* L_0^{-1}(f - < f >_\mu) >_\mu$$

Law of large numbers

$$\frac{X_t}{t} \longrightarrow v$$

$$v = (2\tilde{\rho} - 1)(\alpha - \beta)$$

with

$$\tilde{\rho} = < \phi_0 >_{\mu_e}$$

with $\phi_0(\eta) = \eta(0)$. Using the expansion gives

$$v = (2\rho - 1)(\alpha - \beta) + \sum_{n=2}^{\infty} c_n(\alpha - \beta)^n$$

So , for $\rho > 1/2$, $\alpha > \beta$, and $\alpha - \beta$ small enough, the speed is strictly positive.

About c_n

$$c_2 = 2 \int < \phi_0 S_0(t)(\phi_1 - \phi_{-1}) > dt$$

In the reversible case, i.e., when $S^*_{IPS}(t) = S_{IPS}(t)$ then

$$< \phi_0 S_0(t)(\phi_1 - \phi_{-1}) >=< (S_0(t)\phi_0)\phi_1 > - < \phi_1(S_0(t)\phi_0) >= 0$$

So, in that case,

$$v = (2\rho - 1)(\alpha - \beta) + O((\alpha - \beta)^3)$$

If the dynamics is independent spin-flip with rates
$c(x,\eta) = \gamma(1-\eta(x)) + \delta\eta(x)$

$$c_3 = \frac{4}{U^2}\rho(1-\rho)(2\rho-1)f(U,V)$$

with $U = \alpha+\beta$, $V = \gamma+\delta$

$$f(U,V) = \frac{2U+V}{\sqrt{V^2+2UV}} - \frac{2U+2V}{\sqrt{V^2+UV}} + 1 < 0$$

"slowing down" w.r.t. "averaged process"

Central limit theorem

$$X_t - vt = \int_0^t F(\xi_s)ds + M_t$$

the quadratic variation of the martingale is independent of the environment η_t (because the rates add up to $\alpha + \beta$) Since for the environment process we have

$$\|S(t)F - < F >_{\mu_e} \|_\infty < C|||F|||e^{-t(c-2(\alpha-\beta))}$$

we have that for all F with $< F >_{\mu_e} = 0$,

$$F = LG$$

hence,

$$X_t - vt = M_t + \int_0^t LG(\xi_s)ds = M_t + M'_t + \epsilon_t$$

where $\epsilon_t = -G(\xi_t) + G(\xi_0)$ is negligeable when divided by $\sqrt{t}$,

Large deviations

From the estimates

$$\|S(t)F- < F >_{\mu_e} \|_\infty < C|||F|||e^{-t(c-2(\alpha-\beta))}$$

and

$$|||S(t)F||| \leq |||F|||e^{-t(c-2(\alpha-\beta))}$$

one infers Gaussian concentration bounds for additive functionals of the environment process, i.e.,

$$\mathbb{P}_{\mu_e}\left(\left|\int_0^t F(\xi_s) - \int F d\mu_e\right| > \delta\right) \leq \exp\left(-\frac{C'\delta^2}{|||f|||^2}\right)$$

which implies that the large-deviation rate function (for X_t/t) cannot contain flat pieces