

# Estimation of the Degree of Jump Activity for a Lévy Process

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# Problem Formulation

- An underlying Lévy process  $(X_t)_{t \geq 0}$
- Observations at equally spaced discrete times:

$$X_0, X_\Delta, \dots, X_{n\Delta},$$

where  $\Delta > 0$  is fixed.

## Problem

*Determine the “degree of activity” of the jumps, when the number of observations  $n \rightarrow \infty$ :*

- *finitely many jumps ?*
- *or, if infinitely many, how “infinite” is it ?*

# Exponential Lévy Models

$$S_t = S_0 \exp(rt + X_t),$$

where  $X_t$  is a Lévy process, i.e.

- $X_u - X_t$  is independent of  $\mathcal{F}_t$  for  $u > t$ ,
- $X_u - X_t$  is distributed as  $X_{u-t}$  for  $u > t$ ,
- for any  $\varepsilon > 0$ ,

$$\lim_{h \rightarrow 0} \mathbb{P}(|X_{t+h} - X_t| \geq \varepsilon) = 0.$$

# Lévy-Khintchine Formula

It holds for  $\psi(u) := t^{-1} \log(\phi_t(u))$ , where  $\phi_t(u) = \mathbb{E} \exp(iuX_t)$

$$\psi(u) = -\frac{\sigma^2}{2}u^2 + i\gamma u + \int_{-\infty}^{\infty} (e^{iux} - 1 - iux\mathbf{1}_{\{|x|\leq 1\}})\nu(dx),$$

$$\nu(\{0\}) = 0, \quad \int_{\mathbb{R}} (|x|^2 \wedge 1)\nu(dx) < \infty,$$

$T := (\sigma, \gamma, \nu)$  – Lévy triple of  $X : X \sim \mathcal{L}(\sigma, \gamma, \nu)$ .

# Jump Activity Index of a Lévy Process

Let  $X_t$  be a Lévy process with the Lévy measure  $\nu$ . The value

$$\alpha = \inf \left\{ r \geq 0 : \int_{|x| \leq 1} |x|^r \nu(dx) < \infty \right\}$$

is called the **degree of jump activity** or the **fractional order** of the Lévy process  $X_t$

- if  $X_t$  is a stable process, then  $\alpha$  is the stability index of  $X_t$ ,
- if  $X_t$  has finite activity, i.e.  $\nu[-\varepsilon, \varepsilon] < \infty$ , then  $\alpha = 0$ ,
- if  $X_t$  has infinite activity and  $\nu((-\infty, -\varepsilon] \cup [\varepsilon, \infty)) \asymp \pi(\varepsilon)|\varepsilon|^{-\alpha}$ .

# Lévy Processes with Asymptotically Stable Behavior

Lévy processes with **asymptotically  $\alpha$ -stable behavior** (AS)

$$\psi(u) = i\gamma u - \sigma^2 u^2/2 + \vartheta(u), \quad \vartheta(u) = -\eta|u|^\alpha \tau(u), \quad u \in \mathbb{R},$$

where

- $0 < \alpha < 2$  – index of AS,  $\operatorname{Re} \eta > 0$ ,

- $\operatorname{Re} \tau(u) > 0$ ,  $|u| > 0$  and  $\tau(u) \rightarrow 1$ ,  $|u| \rightarrow \infty$ .

# Lévy Processes with Asymptotically Stable Behavior

Let  $\nu(x)$  be the Lévy density of AS Lévy process with

$$\tau(u) = 1 + D_{\pm} u^{-\kappa} + o(|u|^{-\kappa}), \quad u \rightarrow \pm\infty.$$

Then

$$\int_{|x| < \varepsilon} x^2 \nu(x) dx = c \varepsilon^{2-\alpha} \theta(\varepsilon),$$

where  $c > 0$  is a constant depending on  $\alpha$  and the function  $\theta(\varepsilon)$  satisfies

$$|\theta(\varepsilon) - 1| \lesssim |\varepsilon|^{\kappa}, \quad \varepsilon \rightarrow 0.$$

# Examples

Normal Inverse Gaussian (Barndorff-Nielsen)( $\alpha = 1$ )

$$\vartheta(u) = c \left[ (\theta^2 - (\beta + iu)^2)^{1/2} - (\theta^2 - \beta^2)^{1/2} \right], \quad \theta > |\beta| > 0$$

Hyperbolic (Eberlein)( $\alpha = 1$ )

$$\phi(u) = \frac{\theta}{K_1(\theta\delta)} \frac{K_1(\delta\sqrt{\theta^2 + u^2})}{\sqrt{\theta^2 + u^2}}, \quad \theta, \delta > 0$$

Tempered stable (Carr & Madan)

$$\vartheta(u) = c\Gamma(-\alpha)[\lambda_+^\alpha - (\lambda_+ + iu)^\alpha + (-\lambda_-)^\alpha - (-\lambda_- - iu)^\alpha]$$



# Asset Dynamics

We assume that the asset price  $S_t$  follows an exponential Lévy model under both historical measure  $\mathbb{P}$  and the risk neutral measure  $\mathbb{Q}$ :

$$S_t = \begin{cases} Se^{X_t}, & \text{under } \mathbb{P} \\ Se^{rt+Y_t}, & \text{under } \mathbb{Q}. \end{cases}$$

$\mathbb{P} \sim \mathbb{Q}$  implies

$$\int_{\mathbb{R}} (\sqrt{d\nu^{\mathbb{Q}}/d\nu^{\mathbb{P}}} - 1)^2 \nu^{\mathbb{P}}(dx) < \infty.$$

## Corollary

*If  $X_t$  and  $Y_t$  are two AS Lévy processes and  $X_t$  has index  $\alpha$ , then  $Y_t$  has the same index  $\alpha$ .*

# Main Idea

Assume first that  $\phi(u) = \exp(\psi(u))$  is known exactly. Define

$$\mathcal{L}(u) := \log(|\phi(u)|^2) = -\sigma^2 u^2 + 2 \operatorname{Re}[\vartheta(u)]$$

and

$$\mathcal{L}_\xi(u) := \xi^2 \mathcal{L}(u) - \mathcal{L}(\xi u) = \log \left( |\phi(u)|^{2\xi^2} / |\phi(\xi u)|^2 \right) =: \log(\varrho_\xi(u))$$

for some  $\xi > 1$ . Then it holds

$$\mathcal{L}_\xi(u) = -2\eta|u|^\alpha \left( \xi^2 \operatorname{Re}[\tau(u)] - \xi^\alpha \operatorname{Re}[\tau(\xi u)] \right) = -c_\xi(\alpha)|u|^\alpha \tau_\xi(u),$$

where  $c_\xi(\alpha) = 2\eta(\xi^2 - \xi^\alpha)^{-1}$  and  $\tau_\xi(u) \rightarrow 1$ .

# Truncation

Let  $\tilde{\phi}(u)$  be an estimate of  $\phi(u)$ . Define

$$\mathcal{Y}_\xi(u) := \log(-\log(\varrho_\xi(u))) = \log(c_\xi) + \alpha \log(u) + \log(\operatorname{Re} \tau_\xi(u)),$$

$$\tilde{\mathcal{Y}}_\xi(u) := \log(-\log(T_{\omega_-, \omega_+}[\tilde{\varrho}_\xi](u))), \quad u > 0,$$

where

$$\tilde{\varrho}_\xi(u) = |\tilde{\phi}(u)|^{2\xi^2} / |\tilde{\phi}(\xi u)|^2$$

and  $T_{\omega_-, \omega_+}$  is truncation operator with truncation levels  
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# Spectral Cut-Off Estimation

Let  $w^1(u)$  be a function supported on  $[\epsilon, 1]$  for some  $\epsilon > 0$  that satisfies

$$\int_0^1 w^1(u) \log(u) du = 1, \quad \int_0^1 w^1(u) du = 0$$

Set  $w^U(u) = U^{-1} w^1(uU^{-1})$  and define

$$\tilde{\alpha}_{\xi, U} = \int_0^{\infty} w^U(u) \tilde{\mathcal{Y}}_{\xi}(u) du.$$

# Spectral Cut-Off Estimation

If  $\tilde{\mathcal{Y}}_\xi = \mathcal{Y}_\xi$

$$\tilde{\alpha}_{\xi,U} = \log(c_\xi) \underbrace{\int_0^\infty w^U(u) du}_0 + \alpha \underbrace{\int_0^\infty w^U(u) \log(u) du}_1 + R_U,$$

where

$$R_U = \int_0^\infty w^U(u) \log(\operatorname{Re} \tau_\xi(u)) du - \text{“modelling bias”}.$$

# Minimax Model Class

Let us consider the class of AS Lévy models  $\mathcal{A}(\bar{\sigma}, \bar{\alpha}, \eta_-, \eta_+, \varkappa)$  with

$$\psi(u) = i\mu u - \sigma^2 u^2 / 2 + \vartheta(u), \quad \vartheta(u) = -\eta |u|^\alpha \tau(u), \quad u \in \mathbb{R},$$

where  $0 < \alpha \leq \bar{\alpha} < 2$ ,  $0 \leq \sigma \leq \bar{\sigma}$ ,

$$0 < \eta_- \leq |\eta| \leq \eta_+ < \infty,$$

$$|1 - \tau(u)| \lesssim \frac{1}{|u|^\varkappa}, \quad |u| \rightarrow \infty.$$

We shall write

$$(\sigma, \alpha, \eta, \tau) \in \mathcal{A}(\bar{\sigma}, \bar{\alpha}, \eta_-, \eta_+, \varkappa).$$

## Estimation of $\phi$

Assume that the values of the log-price process  $X_t = \log(S_t)$  on the equidistant time grid  $\pi = \{t_0, t_1, \dots, t_n\}$  are observed.

- Estimate the characteristic function  $\phi(u)$  by its empirical counterpart

$$\tilde{\phi}(u) = \frac{1}{n} \sum_{j=1}^n e^{iu(X_{t_j} - X_{t_{j-1}})}.$$



# Upper Bounds

For  $U = \bar{U}$  with

$$\bar{U} = \left[ \frac{1}{2\bar{\sigma}} \log(n \log^{-\beta}(n)) \right]^{1/2}, \quad \beta > 0$$

it holds

$$\sup_{(\sigma, \alpha, \eta, \varkappa) \in \mathcal{A}(\bar{\sigma}, \bar{\alpha}, \eta_-, \eta_+, \varkappa)} \mathbb{E} |\tilde{\alpha}_{\xi, \bar{U}} - \alpha|^2 \lesssim \mathcal{R}_n,$$

where

$$\mathcal{R}_n = c_{\xi}^{-1}(\alpha) \left[ \frac{1}{2\bar{\sigma}} \log(n) \right]^{-\varkappa}.$$

# Lower Bounds

For any fixed  $\varkappa > 0$

$$\liminf_{n \rightarrow \infty} \inf_{\tilde{\alpha}} \sup_{(\sigma, \alpha, \eta, \tau) \in \mathcal{A}(\bar{\sigma}, \bar{\alpha}, \eta_-, \eta_+, \varkappa)} \delta_n^{-2} \mathbb{E}(|\tilde{\alpha} - \alpha|^2) = O(1),$$

where

$$\delta_n = \left[ \frac{1}{2\bar{\sigma}} \log(n) \right]^{-\varkappa/2},$$

and the infimum is taken over all estimators  $\tilde{\alpha}$  of  $\alpha$ .

# Adaptive Procedure

Fix a sequence of cut-off parameters  $U_1 > U_2 > \dots > U_K$  and define

$$\tilde{\alpha}_k = \int_0^\infty w^{U_k}(u) \tilde{\mathcal{Y}}_\xi(u) du, \quad k = 1, \dots, K.$$

For the sequence of estimates  $\tilde{\alpha}_k$  consider the sequence of nested hypothesis  $H_k : \alpha_1 = \dots = \alpha_k = \alpha$  with

$$\alpha_k = \int_0^\infty w^{U_k}(u) \mathcal{Y}_\xi(u) du, \quad k = 1, \dots, K.$$

# Adaptive Procedure

Compute sequentially

$$\hat{\alpha}_k = \gamma_k \tilde{\alpha}_k + (1 - \gamma_k) \hat{\alpha}_{k-1},$$

where  $\hat{\alpha}_1 = \tilde{\alpha}_1$  and the mixing parameter  $\gamma_k$  is defined as

$$\gamma_k := \mathcal{K}(T_k / \mathcal{V}_k), \quad T_k := (\tilde{\alpha}_k - \hat{\alpha}_{k-1})^2 / \nu_k$$

- $\nu_k$  is the variance of  $\tilde{\alpha}_k$
- $\mathcal{K}$  is a kernel supported on  $[0, 1]$
- $\{\mathcal{V}_k\}$  is a set of critical values

## Choice of Critical Values

The critical values  $\nu_1, \dots, \nu_{K-1}$  are selected to provide the prescribed performance under the simplest (null) hypothesis

$$\alpha_1 = \dots = \alpha_K = \alpha.$$

Under the null hypothesis, every estimate  $\tilde{\alpha}_k$  fulfills asymptotically

$$\sqrt{n}(\tilde{\alpha}_k - \alpha) \sim \mathcal{N}(\mathbf{0}, \mathbf{v}_k), \quad n \rightarrow \infty.$$

and therefore

$$E_0 | \mathbf{v}_{k,n}^{-1} (\tilde{\alpha}_k - \alpha) |^r \approx C_r,$$

where  $\mathbf{v}_{k,n} = n^{-1} \mathbf{v}_k$ ,  $C_r = E|\xi|^{2r}$  and  $\xi$  is standard normal.

## Choice of Critical Values

The critical values  $\nu_1, \dots, \nu_{K-1}$  are selected in such a way that

$$E_0 | \nu_{k,n}^{-1} (\hat{\alpha}_k - \tilde{\alpha}_k)^2 |^r \leq \gamma C_r, \quad k = 2, \dots, K,$$

where  $\gamma$  is the preselected confidence level of the procedure.

## Numerical Example

Consider the GH Lévy model with parameters  $(\kappa, \beta, \delta, \lambda)$  and c.f.

$$\Phi_{GH}(u) = e^{i\mu u - \sigma^2 u^2 / 2} \frac{\left(\sqrt{\kappa^2 + \beta^2}\right)^\lambda}{\left(\sqrt{\kappa^2 - (\beta + iu)^2}\right)^\lambda} \frac{K_\lambda\left(\delta\sqrt{\kappa^2 - (\beta + iu)^2}\right)}{K_\lambda\left(\delta\sqrt{\kappa^2 + \beta^2}\right)}.$$

Lévy-Khintchine representation

$$\Phi_{GH}(u) = \exp\left(ibu - \sigma^2 u^2 / 2 + \int_{-\infty}^{\infty} (e^{iux} - 1 - iux)g(x) dx\right),$$

where  $g(x) = \frac{\delta}{\pi}x^{-2} + \frac{\lambda + \frac{1}{2}}{2}|x|^{-1} + \frac{\delta\beta}{\pi}x^{-1} + o(|x|^{-1})$ ,  $x \rightarrow 0$ .

# Numerical Example

- Simulate 500 samples of  $(X_0, X_\Delta, \dots, X_{n\Delta})$  from the GH model with  $\lambda = 1$ ,  $\beta = 0$ ,  $\kappa = 1$  and  $\delta = 4$ .
- Construct

$$\tilde{\phi}(u) = \frac{1}{n} \sum_{k=1}^n e^{iu(X_{k\Delta} - X_{(k-1)\Delta})}.$$

- Compute

$$\tilde{\mathcal{Y}}_\xi(u) := \log(-\log(T_{\omega_-, \omega_+}[\tilde{\varrho}_\xi](u))),$$

where  $\tilde{\varrho}_\xi(u) = |\tilde{\phi}(u)|^{2\xi^2} / |\tilde{\phi}(\xi u)|^2$  and  $\xi = 2$ .



# Numerical Example

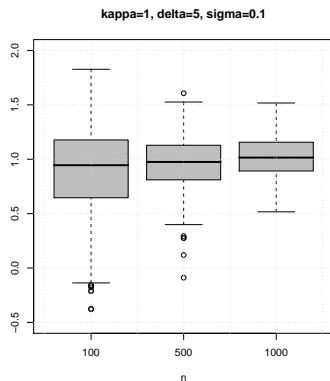
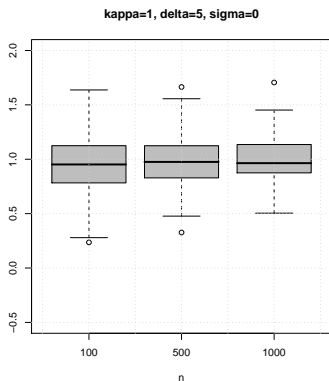
- Solve the minimization problem

$$(l_0^U, l_1^U) = \operatorname{argmin}_{l_0, l_1} \int_0^U w^U(u) (\tilde{\mathcal{Y}}_\xi(u) - l_1 \log(u) - l_0)^2 du,$$

where  $w^U(u) = U^{-1} w^1(U^{-1}u)$  and  $w^1(u) = u \mathbf{1}_{\{0 \leq u \leq 1\}}$ .

- Let  $U_k = 100(1.25)^{-(k-1)}$ ,  $k = 1, \dots, 30$  and define  $\tilde{\alpha}_k = l_k^U$ .
- Construct aggregated estimates  $\hat{\alpha}_1, \dots, \hat{\alpha}_{30}$  via SA with  $\mathcal{K}(x) = (1-x) \mathbf{1}_{\{0 \leq x \leq 1\}}$ ,  $r = 1$  and  $\gamma = 0.5$ .





# Numerical Example



Box plots of the estimates  $\hat{\alpha}$  (left) and  $\hat{\alpha}_\xi$  (right) for different sample sizes.

# Future Work

- Extension to the affine models
- Extension to time the changed Lévy models
- Investigate the case  $\Delta \rightarrow 0$

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