Nonparametric adaptive estimation for pure jump Lévy processes. Fixed sample step data.

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Introduction:

Use of Lévy processes for modelling purposes:

very popular in many areas

especially in the field of finance

Eberlein and Keller (1995)
Barndorff-Nielsen and Shephard (2001)
Cont and Tankov (2004)
Bertoin (1996)
Sato (1999).

Distribution of a Lévy process: specified by its characteristic triple

(drift, Gaussian component and Lévy measure.)

Rather than by the distribution of its independent increments (intractable) $\Rightarrow$

standard parametric approach by likelihood methods difficult.

$\Rightarrow$ **nonparametric methods.**

Lévy measure interesting to estimate because specifies the jumps behavior.

Nonparametric estimation of the Lévy measure

Recent contributions:

Basawa and Brockwell (1982): non decreasing Lévy processes and observations of jumps with size larger than some positive ε, or discrete observations with fixed sampling interval.

Nonparametric estimators of a distribution function linked with the Lévy measure.

Figueroa-López and Houdré (2006): a continuous-time observation of a general Lévy process and study penalized projection estimators of the Lévy density.

Neumann and Reiss (2009).

Our aim. Nonparametric estimation of the Lévy measure for real-valued Lévy processes of **pure jump type**, *i.e.*

without drift and Gaussian component.

Assumption: the Lévy measure admits a density $n(x)$ on $\mathbb{R}$.

Notations: (L_t) the Lévy process. Observed random variables (i.i.d.):

$$(Z_k^{\Delta} = L_{k\Delta} - L_{(k-1)\Delta}, k = 1, \ldots, n)$$

Process discretely observed with sampling interval Δ.

Link between $n(x)$ and Z_k^{Δ}'s?

Characteristic function of $Z_1^\Delta = L_\Delta$:

$$\psi_\Delta(u) = \mathbb{E}(\exp iuZ_1^\Delta) = \exp\left(\Delta \int_{\mathbb{R}} (e^{iux} - 1)n(x)dx\right) \tag{1}$$

By derivating:

$$\psi'_\Delta(u) = i\mathbb{E}(Z_1^\Delta \exp iuZ_1^\Delta) = \left(i\Delta \int_{\mathbb{R}} e^{iux} xn(x)dx\right) \psi_\Delta(u).$$

Assume $\int_{\mathbb{R}} |x| n(x) dx < \infty$.

Denote $\mathbf{g(x) = xn(x)}$:

$$\mathbf{g^*(u) = \int e^{iux} g(x) dx = -i \frac{\psi'_{\Delta}(u)}{\Delta \psi_{\Delta}(u)}}. \qquad (2)$$

Nonparametric estimation strategy using

empirical estimators of the characteristic functions

and Fourier inversion.

See also Watteel-Kulperger (2003) and Neumann-Reiss (2009).

$\Rightarrow$ **Estimate** $g^*(u)$ **by using empirical counterparts of** $\psi_{\Delta}(u)$ **and** $\psi'_{\Delta}(u) = i\mathbb{E}(Z_1 e^{iuZ_1})$ **only.**

$\Rightarrow$ Problem of estimating g = **deconvolution**-type problem.

i.e. estimation of the density of X with observations

$$Z_i = X_i + \varepsilon_i$$

with ε_i centered i.i.d. noise. with density f_ε.

f_Z density of Z, g density of X, $u^*(x) = \int e^{itx} u(t)dt$,

$$f_Z^* = g^* f_\varepsilon^* \Rightarrow g^* = f_Z^* / f_\varepsilon^*.$$

f_Z^* estimated, f_ε^* known.

But: Problem of deconvolution from (4) is **not standard**

Both the numerator and the denominator are estimated

$\Rightarrow$ Deconvolution in presence of **unknown error density**.

$+$ Have to be estimated from the same data.

Moreover estimator of $1/\psi_\Delta(u)$ (like $1/f_\varepsilon^*(x)$) is not a simple empirical counterpart.

Truncated version analogous to the one used in Neumann (1997) and Neumann and Reiss (2009).

Technical assumptions up to now:

(H1) $\int_{\mathbb{R}} |x| n(x) dx < \infty$.

(H2(p)) For p integer, $\int_{\mathbb{R}} |x|^{p-1} |g(x)| dx < \infty$.

(H3) The function g belongs to $\mathbb{L}_2(\mathbb{R})$.

Our estimation procedure is based on the i.i.d. r.v.

$$\mathbf{Z_k^{\Delta} = L_{k\Delta} - L_{(k-1)\Delta}, k = 1, \ldots, n,} \tag{3}$$

with common characteristic function $\psi_\Delta(u)$.

Key formula

$$\mathbf{g^*(u) = \int e^{iux} g(x) dx = -i \frac{\psi'_{\Delta}(u)}{\Delta \psi_{\Delta}(u)}.} \tag{4}$$

Moments of Z_1^Δ linked with g:

Proposition 1 *Let $p \geq 1$ integer. Under (H2)(p), $\mathbb{E}(|Z_1^\Delta|^p) < \infty$. Moreover, setting, for $k = 1, \ldots p$, $M_k = \int_{\mathbb{R}} x^{k-1} g(x) dx$, we have*

$$\mathbb{E}(Z_1^\Delta) = \Delta M_1, \qquad \mathbb{E}[(Z_1^\Delta)^2] = \Delta M_2 + \Delta^2 M_1,$$

and more generally,

$$\mathbb{E}[(\mathbf{Z_1^\Delta})^{\mathbf{l}}] = \boldsymbol{\Delta}\ \mathbf{M_l} + \mathbf{o}(\boldsymbol{\Delta}) \textit{ for all } \mathbf{l} = \mathbf{1}, \ldots, \mathbf{p}.$$

Control of ψ_Δ.

(H4) $\forall \mathbf{x} \in \mathbb{R}, \mathbf{c}_\psi(\mathbf{1}+\mathbf{x}^2)^{-\boldsymbol{\Delta}\beta/2} \leq |\psi_{\boldsymbol{\Delta}}(\mathbf{x})| \leq \mathbf{C}_\psi(\mathbf{1}+\mathbf{x}^2)^{-\boldsymbol{\Delta}\beta/2}$,

for some given constants c_ψ, C_ψ and $\beta \geq 0$.

Also considered in Neumann and Reiss (2009).

For the **adaptive version** of our estimator, we need additional assumptions for g:

(H5) There exists some positive a such that $\int |g^*(x)|^2(1+x^2)^a dx < +\infty$,

and

(H6) $\int x^2 g^2(x) dx < +\infty$.

Independent assumptions for ψ_Δ and g: **there may be no relation at all between these two functions**.

Examples.

Compound Poisson processes.

$$L_t = \sum_{i=1}^{N_t} Y_i, \quad Y_i \text{ i.i.d. with density } f$$

(Y_i) independent of N_t, $N_t \sim \mathcal{P}\text{oisson}(c)$.

$\mathbb{P}(L_\Delta = 0) = e^{-c\Delta}$

$$\mathbf{n}(\mathbf{x}) = \mathbf{cf}(\mathbf{x}).$$

$$\mathbf{e}^{-\mathbf{2c}\boldsymbol{\Delta}} \leq |\psi_{\boldsymbol{\Delta}}(\mathbf{u})| \leq \mathbf{1}.$$

The Lévy Gamma process.

$$L_t \sim \Gamma(\beta t, \alpha)$$

$$\mathbf{n}(\mathbf{x}) = \beta \mathbf{x}^{-\mathbf{1}} \mathbf{e}^{-\alpha \mathbf{x}} \mathbf{1}(\mathbf{x} > \mathbf{0}).$$

$$\psi_{\boldsymbol{\Delta}}(\mathbf{u}) = \left(\frac{\alpha}{\alpha - \mathbf{iu}} \right)^{\beta \boldsymbol{\Delta}}.$$

Bilateral Gamma process. Küchler and Tappe (2008).
$L_t = L_t^{(1)} - L_t^{(2)}$, $L_t^{(1)}$ and $L_t^{(2)}$ independent and Lévy-Gamma.
Parameters $(\beta', \alpha'; \beta, \alpha)$.

Special case: $\beta' = \beta$ and $\alpha' = \alpha$.

Variance-Gamma (Madan and Seneta, 1990).

$$L_t = W_{Z_t}, \quad (W) \text{ Brownian motion independent of } Z$$

and Z Lévy-Gamma.

- Bilateral Gamma. $n(x) = x^{-1} g(x)$

$$\mathbf{g}(\mathbf{x}) = \beta' \mathbf{e}^{-\alpha' \mathbf{x}} \mathbf{1}(\mathbf{x} > \mathbf{0}) - \beta \mathbf{e}^{-\alpha |\mathbf{x}|} \mathbf{1}(\mathbf{x} > \mathbf{0}).$$

$$\psi_{\mathbf{\Delta}}(\mathbf{u}) = \left(\frac{\alpha}{\alpha - \mathbf{iu}} \right)^{\beta \mathbf{\Delta}} \left(\frac{\alpha'}{\alpha' + \mathbf{iu}} \right)^{\beta' \mathbf{\Delta}}.$$

Notations

u^* the Fourier transform of the function u: $u^*(y) = \int e^{iyx} u(x) dx$,

$$\|u\|^2 = \int |u(x)|^2 dx,$$

$$< u, v >= \int u(x)\overline{v}(x) dx \text{ with } z\overline{z} = |z|^2.$$

For any integrable and square-integrable functions u, u_1, u_2,

$$(\mathbf{u}^*)^*(\mathbf{x}) = \mathbf{2}\pi\mathbf{u}(-\mathbf{x}) \text{ and } \langle \mathbf{u_1}, \mathbf{u_2} \rangle = (\mathbf{2}\pi)^{-\mathbf{1}} \langle \mathbf{u_1^*}, \mathbf{u_2^*} \rangle. \qquad (5)$$

Definition of the estimator.

$$g^*(x) = -i\frac{\psi'_\Delta(x)}{\Delta\psi_\Delta(x)} = \frac{\boldsymbol{\theta_\Delta(x)}}{\boldsymbol{\Delta\psi_\Delta(x)}}, \tag{6}$$

with

$$\psi_\Delta(x) = \mathbb{E}(e^{ixZ_1^\Delta}), \quad \theta_\Delta(x) = -i\psi'_\Delta(x) = \mathbb{E}(Z_1^\Delta e^{ixZ_1^\Delta}).$$

$$\boldsymbol{\hat{\psi}_\Delta(x) = \frac{1}{n}\sum_{k=1}^{n} e^{ixZ_k^\Delta}, \quad \hat{\theta}_\Delta(x) = \frac{1}{n}\sum_{k=1}^{n} Z_k^\Delta e^{ixZ_k^\Delta}}.$$

Although $|\psi_\Delta(x)| > 0$ for all x, this is not true for $\hat{\psi}_\Delta$.

As Neumann (1997) and Neumann and Reiss (2007), truncate $1/\hat{\psi}_\Delta$

$$\frac{1}{\tilde{\psi}_\Delta(x)} = \frac{\mathbf{1}}{\boldsymbol{\hat{\psi}_\Delta(x)}} \mathbb{I}_{|\hat{\psi}_\Delta(\mathbf{x})| > \kappa_\psi \mathbf{n}^{-1/2}}. \tag{7}$$

$$\widehat{g^*}(x) = \frac{\hat{\theta}_\Delta(x)}{\Delta\tilde{\psi}_\Delta(x)}.$$

Inverse Fourier transform with cutoff m:

$$\mathbf{\hat{g}_m(x) = \frac{1}{2\pi}\int_{-\pi m}^{\pi m} e^{-ixu}\frac{\hat{\theta}_\Delta(u)}{\Delta\tilde{\psi}_\Delta(u)}du.}$$

Because integrals on $\mathbb{R}$ may not be finite.

$$g_m(x) = \frac{1}{2\pi}\int_{-\pi m}^{\pi m} e^{-ixu}\frac{\theta_\Delta(u)}{\Delta\psi_\Delta(u)}du.$$

Risk bound for fixed m

$$\begin{aligned} \|g - \hat{g}_m\|^2 &= \|g - g_m\|^2 + \|g_m - \hat{g}_m\|^2 \text{ (Pythagoras)} \\ &\leq \|g - g_m\|^2 + 2\|g_m - \mathbb{E}(\hat{g}_m)\|^2 + 2\|\mathbb{E}(\hat{g}_m) - \hat{g}_m\|^2. \end{aligned}$$

We define:

$$\mathbf{\Phi}_\psi(\mathbf{m}) = \int_{-\pi \mathbf{m}}^{\pi \mathbf{m}} \frac{\mathbf{dx}}{|\psi_{\mathbf{\Delta}}(\mathbf{x})|^{\mathbf{2}}}, \tag{8}$$

(analogy with deconvolution setting)

Proposition 2 *Under* (H1)-(H2)(4)-(H3), *for all m:*

$$\mathbb{E}(\|\mathbf{g} - \hat{\mathbf{g}}_\mathbf{m}\|^\mathbf{2}) \leq \|\mathbf{g} - \mathbf{g}_\mathbf{m}\|^\mathbf{2} + \mathbf{K} \frac{\mathbb{E}^{\mathbf{1/2}}[(\mathbf{Z}_\mathbf{1}^{\mathbf{\Delta}})^\mathbf{4}]\mathbf{\Phi}_\psi(\mathbf{m})}{\mathbf{n\Delta^2}}.$$

where K is a constant.

Discussion about the rates

$$\|g - g_m\|^2 = \int_{|x| \geq \pi m} |g^*(x)|^2 dx.$$

Suppose that g belongs to the Sobolev class

$$\mathcal{S}(a, L) = \{f, \int |f^*(x)|^2 (x^2 + 1)^a dx \leq L\}.$$

Then, the bias term satisfies

$$\|g - g_m\|^2 = \mathbf{O(m^{-2a})}.$$

Under (H4), the bound of the variance term satisfies

$$\frac{\int_{-\pi m}^{\pi m} dx / |\psi_\Delta(x)|^2}{n\Delta} = \mathbf{O}\left(\frac{\mathbf{m}^{\mathbf{2}\beta\mathbf{\Delta}+\mathbf{1}}}{\mathbf{n\Delta}}\right).$$

The optimal choice for m is $O((n\Delta)^{1/(2\beta\Delta+2a+1)})$ and the resulting rate for the risk is

$$(\mathbf{n\Delta})^{-\mathbf{2a}/(\mathbf{2}\beta\mathbf{\Delta}+\mathbf{2a}+\mathbf{1})}.$$

Sampling interval Δ explicitly appears in the exponent of the rate.

$\Rightarrow$ for positive β, the **rate is worse for large** Δ than for small Δ.

Thus we can state the following corollary of Proposition 2:

Corollary 1 *Under assumptions* (H1)-(H2(4))-(H3)-(H5), *then*

$\mathbb{E}(\|\hat{g}_m - g\|^2) = O(n\Delta)^{-2a/(2\beta\Delta+2a+1)}$ *when* $m = O((n\Delta)^{1/(2\beta\Delta+2a+1)})$.

Projection formulation.

$$\varphi(\mathbf{x}) = \frac{\sin(\pi \mathbf{x})}{\pi \mathbf{x}} \quad \text{and} \quad \varphi_{\mathbf{m},\mathbf{j}}(\mathbf{x}) = \sqrt{\mathbf{m}}\varphi(\mathbf{m}\mathbf{x} - \mathbf{j}).$$

$$\varphi^*_{\mathbf{m},\mathbf{j}}(\mathbf{x}) = \frac{\mathbf{e}^{\mathbf{i}\mathbf{x}\mathbf{j}/\mathbf{m}}}{\sqrt{\mathbf{m}}} \mathbb{I}_{[-\pi\mathbf{m},\pi\mathbf{m}]}(\mathbf{x}). \tag{9}$$

$$S_m \quad = \quad \text{Span}\{\varphi_{\mathbf{m},\mathbf{j}},\ \mathbf{j} \in \mathbb{Z}\} = \{h \in \mathbb{L}^2(\mathbb{R}), \text{supp}(h^*) \subset [-m\pi, m\pi]\}.$$

$\{\varphi_{m,j}\}_{j\in\mathbb{Z}}$ orthonormal basis

$(S_m)_{m\in\mathcal{M}_n}$ the collection of linear spaces,

$$\mathcal{M}_{\mathbf{n}} = \{\mathbf{1}, \ldots, \mathbf{m}_{\mathbf{n}}\}$$

and $m_n \leq n$ is the maximal admissible value of m, subject to constraints to be given later.

Consider g_m orthogonal projection of g on S_m

$$g_m = \sum_{j\in\mathbb{Z}} a_{m,j}(g)\varphi_{m,j} \text{ with } a_{m,j}(g) = \int_{\mathbb{R}} \varphi_{m,j}(x)g(x)dx = \langle \varphi_{m,j}, g\rangle.$$

and

$$\langle \varphi_{\mathbf{m,j}}, \mathbf{g}\rangle = \frac{\mathbf{1}}{\mathbf{2}\pi}\langle \varphi^*_{\mathbf{m,j}}, \mathbf{g}^*\rangle = \frac{\mathbf{1}}{\mathbf{2}\pi}\langle \varphi^*_{\mathbf{m,j}}, \frac{\theta_{\mathbf{\Delta}}}{\mathbf{\Delta}\psi_{\mathbf{\Delta}}}\rangle.$$

The estimator can be defined by:

$$\hat{\mathbf{g}}_{\mathbf{m}} = \sum_{\mathbf{j}\in\mathbb{Z}} \hat{\mathbf{a}}_{\mathbf{m,j}}\varphi_{\mathbf{m,j}}, \text{ with } \hat{\mathbf{a}}_{\mathbf{m,j}} = \frac{\mathbf{1}}{\mathbf{2}\pi\mathbf{n}\mathbf{\Delta}}\sum_{\mathbf{k=1}}^{\mathbf{n}} \mathbf{Z}^{\mathbf{\Delta}}_{\mathbf{k}} \int \mathbf{e}^{\mathbf{ixZ}^{\mathbf{\Delta}}_{\mathbf{k}}} \frac{\varphi^*_{\mathbf{m,j}}(-\mathbf{x})}{\tilde{\psi}_{\mathbf{\Delta}}(\mathbf{x})}\mathbf{dx},$$

$$\text{or} \quad \hat{a}_{m,j} = \frac{1}{2\pi\Delta}\int \hat{\theta}_{\Delta}(x)\frac{\varphi^*_{m,j}(-x)}{\tilde{\psi}_{\Delta}(x)}dx.$$

Let $t \in S_m$ of the collection $(S_m)_{m\in\mathcal{M}_n}$, and define

$$\begin{aligned} \gamma_{\mathbf{n}}(\mathbf{t}) &= \|t\|^2 - \frac{1}{\pi\Delta}\frac{1}{n}\sum_{k=1}^{n} Z_k^\Delta \int e^{ixZ_k^\Delta}\frac{t^*(-x)}{\tilde{\psi}_\Delta(x)}dx, \\ &= \|\mathbf{t}\|^2 - \frac{\mathbf{1}}{\pi\boldsymbol{\Delta}}\int \mathbf{t}^*(\mathbf{x})\frac{\hat{\theta}_{\boldsymbol{\Delta}}(\mathbf{x})}{\tilde{\psi}_{\boldsymbol{\Delta}}(\mathbf{x})}\mathbf{dx}. \end{aligned} \tag{10}$$

Consider $\gamma_n(t)$ as an approximation of the theoretical contrast

$$\gamma_n^{th}(t) = \|t\|^2 - \frac{1}{\pi\Delta}\int \hat{\theta}_\Delta(x)\frac{t^*(-x)}{\psi_\Delta(x)}dx,$$

$\mathbb{E}(\gamma_{\mathbf{n}}^{\mathbf{th}}(\mathbf{t})) = \|\mathbf{t}\|^2 - \mathbf{2}\langle \mathbf{g}, \mathbf{t}\rangle = \|\mathbf{t}-\mathbf{g}\|^2 - \|\mathbf{g}\|^2$ minimal for $t = g$.

We have also

$$\hat{\mathbf{g}}_{\mathbf{m}} = \mathrm{Argmin}_{\mathbf{t}\in\mathbf{S_m}}\gamma_{\mathbf{n}}(\mathbf{t}). \tag{11}$$

Study of the adaptive estimator

We have to select an adequate value of m.

$$\mathrm{pen}(m) = \kappa(1 + \mathbb{E}[(Z_1^\Delta)^2]/\Delta)\frac{\Phi_\psi(m)}{n\Delta}. \tag{12}$$

We set

$$\hat{\mathbf{m}} = \arg\min_{\mathbf{m}\in\mathcal{M}_\mathbf{n}} \left\{\gamma_\mathbf{n}(\hat{\mathbf{g}}_\mathbf{m}) + \mathrm{pen}(\mathbf{m})\right\},$$

and study first the "risk" of $\hat{g}_{\hat{m}}$.

And $\mathcal{M}_n = \{1, \dots, n\}$ with m_n such that $\mathrm{pen}(m_n) \leq C$, where C is a given constant.

Result:

Theorem 1 *Assume that assumptions* (H1)-(H2)(8)-(H3)-(H6) *hold. Then*

$$\mathbb{E}(\|\hat{\mathbf{g}}_{\hat{\mathbf{m}}} - \mathbf{g}\|^{\mathbf{2}}) \leq \mathbf{C} \inf_{\mathbf{m} \in \mathcal{M}_{\mathbf{n}}} \left(\|\mathbf{g} - \mathbf{g}_{\mathbf{m}}\|^{\mathbf{2}} + \mathrm{pen}(\mathbf{m})\right) + \mathbf{K} \frac{\ln^{\mathbf{2}}(\mathbf{n})}{\mathbf{n}\boldsymbol{\Delta}},$$

where K is a constant.

Automatic squared bias $\|g - g_m\|^2$ / variance **compromise** as pen(m) has the order of the variance.

Theoretical estimator because $\Phi_\psi(m)$ **unknown.**

$\Rightarrow$ To get an estimator, we replace the theoretical penalty by:

$$\widehat{\text{pen}}(\mathbf{m}) = \kappa' \left(\mathbf{1} + \frac{\mathbf{1}}{\mathbf{n}\boldsymbol{\Delta}^{\mathbf{2}}} \sum_{\mathbf{i}=\mathbf{1}}^{\mathbf{n}} (\mathbf{Z}_{\mathbf{i}}^{\boldsymbol{\Delta}})^{\mathbf{2}} \right) \frac{\int_{-\pi\mathbf{m}}^{\pi\mathbf{m}} \mathbf{dx}/|\tilde{\psi}_{\boldsymbol{\Delta}}(\mathbf{x})|^{\mathbf{2}}}{\mathbf{n}}.$$

Assumption on the collection of models $\mathcal{M}_n = \{1, \ldots, m_n\}$, $m_n \leq n$:

(H7) $\quad \exists \varepsilon, \mathbf{0} < \varepsilon < \mathbf{1}, \;\; \mathbf{m}_{\mathbf{n}}^{\mathbf{2}\beta\boldsymbol{\Delta}} \leq \mathbf{C}\mathbf{n}^{\mathbf{1}-\varepsilon}$,

where C is a fixed constant and β is defined by (H4).

For instance, **Assumption (H7) is fulfilled if**:

1. $\text{pen}(m_n) \leq C$. In such a case, we have $m_n \leq C(n\Delta)^{1/(2\beta\Delta+1)}$.
2. Δ is small enough to ensure $2\beta\Delta < 1$. Take $\mathcal{M}_n = \{1, \ldots, n\}$.

In the compound Poisson model, $\beta = 0$ and nothing is needed.

(H7) = problem because **depends on the unknown** β

But concrete implementation requires the knowledge of m_n.
Analogous **deconvolution with unknown error density**.

In that case we can prove:

Theorem 2 *Assume that assumptions* (H1)-(H2)(8)-(H3)-(H7) *hold and let* $\tilde{g} = \hat{g}_{\widehat{\widehat{m}}}$ *be the estimator defined with* $\widehat{\widehat{m}} = \arg\min_{m \in \mathcal{M}_n}(\gamma_n(\hat{g}_m) + \widehat{\mathrm{pen}}(m))$*. Then*

$$\mathbb{E}(\|\tilde{\mathbf{g}} - \mathbf{g}\|^2) \leq \mathbf{C} \inf_{\mathbf{m} \in \mathcal{M}_\mathbf{n}} \left(\|\mathbf{g} - \mathbf{g}_\mathbf{m}\|^2 + \mathrm{pen}(\mathbf{m})\right) + \mathbf{K}'_{\boldsymbol{\Delta}} \frac{\ln^2(\mathbf{n})}{\mathbf{n}}$$

where K'_Δ *is a constant depending on* Δ *(and on fixed quantities but not on* n*).*

If g belongs to the Sobolev ball $\mathcal{S}(a, L)$, and under (H4), the rate is automatically of order $O((n\Delta)^{-2a/(2\beta\Delta+2a+1)})$.

Proofs rely on control of empirical processes via Talagrand's type inequality and precise bounds on residual terms.

$$\gamma_n(t) - \gamma_n(s) = \|t-g\|^2 - \|s-g\|^2 - 2\nu_n^{(1)}(t-s) - 2\nu_n^{(2)}(t-s) - 2\sum_{i=1}^{4} R_n^{(i)}(t-s),$$

$$\nu_n^{(1)}(t) = \frac{1}{2\pi\Delta}\int t^*(-x)\frac{\hat{\theta}_\Delta^{(1)}(x) - \theta_\Delta^{(1)}(x)}{\psi_\Delta(x)}dx,$$

$$\nu_n^{(2)}(t) = \frac{1}{2\pi\Delta}\int t^*(-x)\frac{\theta_\Delta(x)}{[\psi_\Delta(x)]^2}(\psi_\Delta(x) - \hat{\psi}_\Delta(x))dx,$$

$$R_n^{(1)}(t) = \frac{1}{2\pi\Delta}\int t^*(-x)(\hat{\theta}_\Delta(x) - \theta_\Delta(x))\left(\frac{1}{\tilde{\psi}_\Delta(x)} - \frac{1}{\psi_\Delta(x)}\right)dx$$

$$R_n^{(2)}(t) = \frac{1}{2\pi\Delta}\int t^*(-x)\frac{\theta_\Delta(x)}{\psi_\Delta(x)}(\psi_\Delta(x) - \hat{\psi}_\Delta(x))\left(\frac{1}{\tilde{\psi}_\Delta(x)} - \frac{1}{\psi_\Delta(x)}\right)dx,$$

$$R_n^{(3)}(t) = \frac{1}{2\pi\Delta}\int t^*(-x)\frac{\hat{\theta}_\Delta^{(2)}(x) - \theta_\Delta^{(2)}(x)}{\psi_\Delta(x)}dx,$$

$$R_n^{(4)}(t) = -\frac{1}{2\pi\Delta}\int t^*(-x)\frac{\theta_\Delta(x)}{\psi_\Delta(x)}\mathbb{I}_{|\hat{\psi}_\Delta(x)|\le\kappa_\psi/\sqrt{n}}dx.$$

Further works:

Deconvolution setting with unknown error density:

a solution with a **random** set $\mathcal{M}_n$,

but two **independent** samples are available.

Maybe a way of generalisation.

But non pure jump processes : only for small sample step!