# Completeness and hedging in a Lévy bond market.

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Statistical Inference for Lévy Processes with Applications to Finance, July 15-17, Eindhoven

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# Outline

1. The model
   - The forward rates

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We assume that only the bank account is fixed exogenously and under the historical probability $P$. There is not any other primary asset in the market. So, any equivalent measure can serve to price derivatives, and a zero coupon bond is a derivative where the underlying is the bank account.
Then we have a bank account process $B$ that evolves as

$$B_t = \exp\{\int_0^t r(s)ds\}$$

where $r(t)$, the so-called short-rate interest. We consider a dynamics of the form

$$r(t) = \mu(t) + \int_0^t \gamma(s,t)dL_s, \tag{1}$$

where $L$ is a non-homogeneous Lévy process, or a process with independent increments and absolutely continuous characteristics $(\sigma_s^2, \upsilon_s, a_s)$:

$$\begin{aligned} & L_t \\ = \; & \int_0^t a_s ds + \int_0^t \sigma_s dW_s + \int_0^t \int_{\mathbb{R}} x \mathbf{1}_{\{|x| \leq 1\}} (J(dx, ds) - \upsilon_s(dx) ds) \\ & + \sum_{s \leq t} \Delta L_s \mathbf{1}_{\{|\Delta L_s| > 1\}} \\ = \; & \int_0^t a_s ds + \int_0^t \sigma_s dW_s + \int_0^t \int_{|x| \leq 1} x \mathrm{d}\widetilde{M}_s^P \\ & + \sum_{s \leq t} \Delta L_s \mathbf{1}_{\{|\Delta L_s| > 1\}} \end{aligned}$$

The coefficients $\mu(t)$ and $\gamma(s,t)$ are assumed to be deterministic and càdlàg. We also assume that the filtration generated by $L$ and $r$ is the same.

Fix $T > 0$ and consider a $T$-zero-coupon bond. This is a contract that guarantees the holder 1 monetary unit at time $T$. Write $P(t, T)$, $0 \leq t \leq T$, for the price of this contract. We know that $P(T, T) = 1$, take $Q$ equivalent to $P$ (the historical probability) and structure preserving (with respect to $L$) and *define*

$$\tilde{P}(t, T) = \mathbb{E}_Q(\exp\{-\int_0^T r(s)ds\}|\mathcal{F}_t)$$

this *discounted* price, whatever $Q$, equivalent to $P$, we choose, does not produce arbitrage.

From here we have that

$$\begin{aligned}
&\tilde{P}(t,T)\\
=\ & \mathbb{E}_Q\left[\exp\left\{-\int_0^T r(s)ds\right\} \mid \mathcal{F}_t\right]\\
=\ & \exp\left\{-\int_0^T \mu(s)ds\right\} \times \mathbb{E}_Q\left[\exp\left\{-\int_0^T\left(\int_u^T \gamma(u,s)ds\right)dL_u\right\} \mid \mathcal{F}_t\right]\\
=\ & \exp\left\{-\int_0^T \mu(s)ds\right\} \times \mathbb{E}_Q\left[\exp\left\{\int_0^T \Gamma(T)_u dL_u\right\} \mid \mathcal{F}_t\right]
\end{aligned}$$

where

$$\Gamma(T)_t = -\int_t^T \gamma(t,s)ds.$$

Under $Q$, $\int_0^{\cdot} \Gamma(T)_u dL_u$ is still a process with independent increments so

$$
\begin{aligned}
& \mathbb{E}_Q \left[ \exp\left\{ \int_0^T \Gamma(T)_u dL_u \right\} \mid \mathcal{F}_t \right] \\
= \; & \exp\left\{ \int_0^t \Gamma(T)_u dL_u \right\} \mathbb{E}_Q \left[ \exp\left\{ \int_t^T \Gamma(T)_u dL_u \right\} \right],
\end{aligned}
$$

Therefore

$$\tilde{P}(t,T) \propto \exp\left\{\int_0^t \Gamma(T)_u dL_u\right\},$$

and we can write

$$\tilde{P}(t,T) = P(0,T)\frac{\exp\left\{\int_0^t \Gamma(T)_u dL_u\right\}}{\mathbb{E}_Q\left[\exp\left\{\int_0^t \Gamma(T)_u dL_u\right\}\right]}. \tag{2}$$

The previous dynamics of $\tilde{P}(t,T)$ for all maturities, $T > t$, can be described in terms of the instantaneous forward rates, by using the definition

$$\tilde{P}(t,T) = \exp\{-\int_0^t r(s)ds - \int_t^T f(t,s)ds\}.$$

Then

$$\begin{aligned} f(t,T) &= -\partial_T \log \tilde{P}(t,T) \\ &= f(0,T) + A(T)_t + \int_0^t \gamma(s,T)dL_s \end{aligned}$$

where

$$A(T)_t = \frac{\partial_T \mathbb{E}_Q\left[\exp\left\{\int_0^t \Gamma(T)_u dL_u\right\}\right]}{\mathbb{E}_Q\left[\exp\left\{\int_0^t \Gamma(T)_u dL_u\right\}\right]}$$

In Eberlein, Jacod and Raible (2005), the authors assume that the forward rates are given and then study the arbitrage and completeness problems. In fact they start by assuming that (under $P$)

$$f(t,T) = f(0,T) + \int_0^t \alpha(s,T)\mathrm{d}s + \int_0^t \gamma(s,T)\mathrm{d}L_s,$$

and they look for martingale measures. They obtain that their existence implies some constrains for the process $\int_0^t \alpha(s,T)\mathrm{d}s$. This was already observed by Heath, Jarrow and Morton in 1992 in the case when $L$ is continuous. The constraint for the existence of "structure preserving" martingale measures is then

$$\int_0^t \alpha(s,T)\mathrm{d}s = A(T)_t.$$

for some $Q$ structure preserving.

We have considered that $Q$ is a given structure preserving equivalent measure, we construct the bond market with this $Q$ and deduce the form of the forward rates which are compatible with this prices. So the relevant problem is the completeness of the market. We take $Q$ and we check if this market is complete by trying to demonstrate that any (squared integrable) contingent claim can be replicated by a self-financing portfolio.

According with Eberlein, Jacod and Raible (2005) ([EJR05]) completeness is important by two reasons:

- Completeness amounts to the fact that any claim can be priced in a unique way: by taking the expectation of the discounted value of the claim with respect to the (unique) equivalent martingale measure. This means, in our context, that if we choose different, structure preserving equivalent measures, $Q$ we obtain different bond prices.
- Completeness is also related with hedging and is in fact "equivalent" to the property that any square integrable (discounted) contingent claim $Y$ can be written as

$$Y = E(Y) + \sum_{T_j \in J} \int_0^T H_s^j \mathrm{d}\tilde{P}(s, T_j), \qquad (3)$$

and this is equivalent to the martingale representation property

$$E(Y|\mathcal{F}_t) = E(Y) + \sum_{T_j \in J} \int_0^t H_s^j \mathrm{d}\tilde{P}(s, T_j).$$

Instead of looking if a system of bond prices implies a unique $Q$ we can use the martingale representation theorems for studying if and how we can replicate any contingent claim.
We have seen that

$$\tilde{P}(t,T) = P(0,T)\exp\{\bar{Z}_t\}$$

where $(\bar{Z}_t)_{0\le t<T}$ is a $Q$ non-homogeneous Lévy process (we shall omit the dependency on $T$) with characteristic triplet given say $(\bar{c}_t^2, \bar{\nu}_t, \bar{\gamma}_t)$. Note that

$$\begin{aligned} \bar{c}_t &= \Gamma(T)_t \sigma_t \\ \bar{\nu}_t(B) &= \upsilon_t\left(\frac{B}{\Gamma(T)_t}\right) \end{aligned}$$

Then, by Itô's formula and the Lévy-Itô representation of $\bar{Z}$

$$
\begin{aligned}
& \mathrm{d}\tilde{P}(t,T) \\
= \;& \tilde{P}(t-,T)((\bar{\gamma}_t + \frac{\bar{c}_t^2}{2} + \int_{\mathbb{R}} \left(e^x - 1 - x\mathbf{1}_{\{|x|<1\}}\right) \bar{\nu}_t(dx))\mathrm{d}t + \bar{c}_t \mathrm{d}\bar{W}_t \\
& + \int_{\mathbb{R}} (e^x - 1)\, \mathrm{d}\widetilde{\bar{M}}_t) \\
= \;& \tilde{P}(t-,T)\left(\bar{c}_t \mathrm{d}\bar{W}_t + \int_{\mathbb{R}} (e^x - 1)\, \mathrm{d}\widetilde{\bar{M}}_t\right) \\
= \;& \tilde{P}(t-,T)\left(\Gamma(T)_t \sigma_t \mathrm{d}\bar{W}_t + \int_{\mathbb{R}} \left(e^{\Gamma(T)_t x} - 1\right) \mathrm{d}\widetilde{M}_t^Q\right) \\
= \;& \tilde{P}(t-,T)\left(\Gamma(T)_t \sigma_t \mathrm{d}W_t + \int_{\mathbb{R}} \left(e^{\Gamma(T)_t x} - 1\right) \mathrm{d}\widetilde{M}_t^P\right) \\
& + \tilde{P}(t-,T)\left(\Gamma(T)_t \sigma_t G_t + \int_{\mathbb{R}} \left(e^{\Gamma(T)_t x} - 1\right) H_t(x) \mathrm{d}\upsilon_t\right) \mathrm{d}t
\end{aligned}
$$

Given $Q$ we have a system of bond prices determined by the mappings, fixed $t$ and $\omega$ :

$$\mathcal{K}_t^{\tilde{P}} : (G, H(x)) \to \tilde{P}(t-, T)\left(\Gamma(T)_t \sigma_t G + \int_{\mathbb{R}} \left(e^{\Gamma(T)_t x} - 1\right) H(x) \mathrm{d}\upsilon_t\right).$$

Uniqueness of the martingale measure is equivalent to the injectivity of these mappings.
In [EJR05] authors consider a bond market in the interval $I := [0, \bar{T}]$ and with zero-coupon bonds with maturities $T_j \in J$ where $J$ is a dense set in $I$, and they prove the uniqueness of $Q$. That is there is not another structure preserving and equivalent measure that gives the same bond prices. Equivalently this means that we should be able to replicate (in an $L^2$-sense) any $Q$-square integrable contingent claim.

In fact, incompleteness, in $L^2(Q)$ sense, implies that there is $X \geq 0$, bounded such that cannot be replicated and consequently, by taking

$$Z = \frac{X - \mathrm{proj}_{\mathcal{R}} X}{||X||_{L^2(Q)}},$$

where $\mathcal{R}$ is the closure of the subspace of r.v. of the form (3), we have that

$$\mathrm{d}P^* := (1 + Z)\mathrm{d}Q$$

provides another martingale measure.

Instead of looking if a system of bond prices implies a unique $Q$ we can use the martingale representation theorems for studying if and how we can replicate any contingent claim.
In Björk, Kabanov and Runggaldier (1997) ([BKR97]) the authors consider measure value self-financing portfolios. A portfolio is a pair $\{g_t, h_t(\mathrm{d}T)\}$, where

- $g$ is a predictable process (units of the risk-free asset)
- $h_t(\omega, \cdot)$ is signed finite on $\mathcal{B}((t, +\infty))$.
- $h_t(\cdot, A)$ is a predictable process, for every $A \in \mathcal{B}((t, +\infty))$.

$h_t(\mathrm{d}T)$ is interpreted as the "number" of bonds with maturities in the interval $[T, T + \mathrm{d}T]$. The value of this portfolio at time $t$ is given by

$$V_t = g_t e^{\int_0^t r_s \mathrm{d}s} + \int_t^\infty P(t, T) h_t(\mathrm{d}T),$$

and the self-financing condition is given by

$$
\begin{aligned}
\mathrm{d}\tilde{V}_t &= \int_t^\infty \mathrm{d}\tilde{P}(t,T)h_t(\mathrm{d}T) \\
&= \int_t^\infty \tilde{P}(t-,T)\left(\Gamma(T)_t\sigma_t\mathrm{d}\bar{W}_t + \int_{\mathbb{R}}\left(e^{\Gamma(T)_t x}-1\right)\mathrm{d}\widetilde{M}_t^Q\right)h_t(\mathrm{d}T).
\end{aligned}
$$

Suppose now that we want to replicate $X \in L^2(Q)$, then $\tilde{V}_T = \tilde{X}$ and

$$M_t := \mathbb{E}_Q(\tilde{X}|\mathcal{F}_t) = \mathbb{E}_Q(\tilde{V}_T|\mathcal{F}_t) = \tilde{V}_t.$$

It is well known that $M_t$, as every $L^2(Q)$-martingale, has an integral representation of the form

$$M_t = \int_0^t \gamma_s \mathrm{d}\bar{W}_s + \int_0^t \int_{\mathbb{R}} \varphi(x,s)\mathrm{d}\widetilde{M}_s^Q,$$

then, the replicating self-financing portfolio should satisfy

$$\begin{aligned} \gamma_t &= \int_t^\infty \tilde{P}(t-,T)\Gamma(T)_t\sigma_t h_t(\mathrm{d}T) \\ \varphi(x,t) &= \int_t^\infty \tilde{P}(t-,T)\left(e^{\Gamma(T)_t x} - 1\right) h_t(\mathrm{d}T). \end{aligned}$$

This system, fixed $t$ and $\omega$, corresponds to the mappings

$$\mathcal{K}_t^{\tilde{P}_*} : h(dT) \rightarrow$$
$$\left(\int_t^\infty \tilde{P}(t-,T)\Gamma(T)_t\sigma_t h(\mathrm{d}T), \int_t^\infty \tilde{P}(t-,T)\left(e^{\Gamma(T)_t x}-1\right)h(\mathrm{d}T)\right)$$

$\mathcal{K}_t^{\tilde{P}*}$ and $\mathcal{K}_t^{\tilde{P}}$ are dual and compact operators then uniqueness of the martingale measure (that is injectivity of $\mathcal{K}_t^{\tilde{P}}$) implies that this equation can been solved only in a dense set of $\mathbb{R} \times L^2(\upsilon_t(\mathrm{d}x))$ and this set is a proper subset of $\mathbb{R} \times L^2(\upsilon_t(\mathrm{d}x))$ if the support of $\upsilon_t$ is infinite. This means that a perfect (a.s) replication of an $L^2(Q)$ contingent claim is in general not possible.

We could consider Lévy measures $\upsilon_t(\mathrm{d}x)$ with a countable number of different jump sizes an portfolios with a countable number of different bonds then

$$
\begin{aligned}
\gamma_t &= \sum_{k\geq 1} \tilde{P}(t-, T_k)\Gamma(T_k)_t \sigma_t h_t(T_k) \\
\varphi(x,t) &= \sum_{k\geq 1} \tilde{P}(t-, T_k)\left(e^{\Gamma(T)_t x} - 1\right) h_t(T_k).
\end{aligned}
$$

but the situation does not change: we cannot replicate (a.s.) in general an $L^2(Q)$ contingent claim.

If we have only $n$ different jump sizes: $x_1, x_2, ..., x_n$ we will have a system

$$\begin{aligned} \gamma_t &= \int_t^\infty \tilde{P}(t-, T)\Gamma(T)_t \sigma_t h_t(\mathrm{d}T) \\ \varphi(x_i, t) &= \int_t^\infty \tilde{P}(t-, T)\left(e^{\Gamma(T)_t x_i} - 1\right) h_t(\mathrm{d}T), i = 1, ..., n, \end{aligned}$$

and it is reasonable to look for a measure $h_t(\mathrm{d}T)$ concentrated in $n+1$ points, $T_1(t), .., T_{n+1}(t)$,

$$\begin{aligned} \gamma_t &= \sigma_t \sum_{i=1}^{n+1} \Gamma(T_i(t))_t G_t^i \\ \varphi(x_j, t) &= \sum_{i=1}^{n+1} \left(e^{\Gamma(T_i(t))_t x_j} - 1\right) G_t^i, j = 1, ..., n, \end{aligned}$$

where $G_t^i$ is the discounted amount invested at $t$ in bonds with maturity $T_i(t)$.

So

$$\sum_{i=1}^{n+1} A_{ji}(t) G_t^i = \phi_j, j = 1, ..., n+1,$$

where

$$\begin{aligned} A_{1i}(t) &= \Gamma(T_i(t))_t, \quad i = 1, ..., n+1; \phi_1 = \gamma_t/\sigma_t \\ A_{ji}(t) &= e^{\Gamma(T_i(t))_t x_{j-1}} - 1; \phi_j = \varphi(x_{j-1}, t) \quad j = 2, ..., n+1, i = 1, ..., n+1 \end{aligned}$$

And we will have a solution provided that $A$ is non singular. Let $\bar{T}$ be the horizon of the market, it can be shown that if $\Gamma(T)_t$ is analytic in $T$ and $t$ we can choose any maturity times bigger than $\bar{T}$ and independent of $t$.

Take $T_1 = T > \bar{T}$ and assume now that for any $k \geq 2$ there is $T_k(t)$, such that

$$k\Gamma(T)_t = \Gamma(T_k(t))_t,$$

then it is easy to see that

$$\det A = \Gamma(T)_t \Pi_{j=1}^n (e^{\Gamma(T)_t x_j} - 1)^2 \Pi_{1 \leq i < j \leq n}^n (e^{\Gamma(T)_t x_j} - e^{\Gamma(T)_t x_i}) \neq 0,$$

$\forall t \in [0, T]$.

If we consider affine term structure models, where $\Gamma(T)_t$ is given by

$$\Gamma(T)_t = a(T)\,b(t) + c(t)\,, \tag{4}$$

with $a(\cdot)$ strictly increasing and $C^1$ function and $b(t) \neq 0$, we have

$$T_k(t) = a^{-1}(k(a(T) - a(t)) + a(t)).$$

In particular this is true for the Vasiček model

$$\Gamma(T)_t = C(1 - e^{-a(T-t)})$$

and the Ho-Lee model

$$\Gamma(T)_t = C(T - t).$$

We also can write

$$\frac{d\tilde{P}(t,T)}{\tilde{P}(t-,T)} = dZ_t,$$

and where $(Z_t)_{0\leq t<T}$ is also a non-homogeneous Lévy process with characteristic triplet given by $\left(c_t^2, \nu_t, \gamma_t\right)$, where

$$\begin{aligned} \gamma_t &= -\int_{|x|>1} \left(e^{\Gamma(T)_t x} - 1\right) \upsilon_t\left(\mathrm{d}x\right), \\ c_t^2 &= \Gamma(T)_t^2 \sigma_t^2, \\ \nu_t\left(G\right) &= \int_{\mathbb{R}} \mathbf{1}_G \left(e^{\Gamma(T)_t x} - 1\right) \upsilon_t\left(dx\right), \qquad G \in \mathcal{B}\left(\mathbb{R}\right). \end{aligned}$$

Note that the Lévy measures $\nu_t$ have support in $(-1, +\infty)$.

Set

$$Z_t^{(i)} = \sum_{0<s\leq t} (\Delta Z_s)^i, \qquad i \geq 2,$$

where $\Delta Z_s = Z_s - Z_{s-}$. Define the $Q$-martingales

$$H_t^{(i)} = Z_t^{(i)} - \mathbb{E}_Q(Z_t^{(i)}), \qquad i = 1, 2, \ldots,$$

with $Z_t^{(1)} = Z_t$.

Assume that $\{\nu_t\}_{t\in[0,\bar{T}]}$ satisfies, for some $\varepsilon > 0$ and $\lambda > 0$,

$$\sup_{t\in[0,\bar{T}]} \int_{(-\varepsilon,\varepsilon)^c} \exp(\lambda|x|)\nu_t(\mathrm{d}x) < \infty. \tag{5}$$

Then it is known that any $Q$-square-integrable martingale $M_t$ can be expressed as

$$M_t = M_0 + \sum_{n=1}^{\infty} \int_0^t \beta_s^n \mathrm{d}\bar{H}_s^{(n)}$$

where $\bar{H}^{(n)}$ are the orthogonal version of the $H^{(n)}$ defined previously and the $\beta^i$ are predictable processes.

Then we can complete our market (with finite horizon $\bar{T} < T$), $M$, with a series of additional assets, $Y^{(i)} = \{Y_t^{(i)}, 0 \leq t \leq \bar{T}\}$, based on the above mentioned processes:

$$Y_t^{(i)} = e^{\int_0^t r_s \mathrm{d}s} H_t^{(i)}, \qquad i \geq 2, \quad 0 \leq t \leq \bar{T}$$

the so-called "power-jump" assets.
The market model, $M_Q$, obtained by enlarging the market $M$ with the power-jump assets is complete, in the sense that any square-integrable contingent claim $X \in L^2(Q)$ can be replicated by a self-financing portfolio.

Let $X$ be a square-integrable (with respect to $Q$) contingent claim. Consider the squared-integrable martingale $M_t := \mathbb{E}_Q(e^{-\int_0^{\bar{T}} r_s \mathrm{d}s} X | \mathcal{F}_t)$. We can write

$$
\begin{aligned}
\mathrm{d}M_t &= \sum_{n=1}^{\infty} \beta_t^n \mathrm{d}\bar{H}_t^{(n)} \\
&= \beta_t^1 \frac{\mathrm{d}\tilde{P}_t}{\tilde{P}_{t-}} + \sum_{n=2}^{\infty} \beta_t^n \mathrm{d}\bar{\tilde{Y}}_t^{(n)},
\end{aligned}
$$

where $P_t := P(t, T)$.

Then if we take a self-financing portfolio: $((\phi_t^i)_{i\geq 1})_{0\leq t\leq T}$, where $\phi^1$ denotes the number of units of the stock, and $(\phi^i)_{i\geq 2}$ the number of power-jump assets of different order, we will have that the discounted value of this portfolio evolves as

$$\mathrm{d}\tilde{V}_t = \phi_t^1 \mathrm{d}\tilde{P}_t + \sum_{n=2}^{\infty} \phi_t^n \mathrm{d}\bar{\tilde{Y}}_t^{(n)}.$$

So, by taking $\phi_t^1 = \frac{\beta_t^1}{\tilde{P}_{t-}}$ and $\phi_t^i = \beta_t^i$ we obtain the replicating portfolio.

In certain cases we can obtain hedging formulas directly, by using the Itô formula. In fact, assume that the discounted price of a contingent claim at time $t$ can be written as $\tilde{F}(s, \tilde{P}_s)$, $\tilde{F}$ smooth, then by the Itô formula

$$
\begin{aligned}
& \mathrm{d}\tilde{F}(s, \tilde{P}_s) \\
= \; & \frac{\partial \tilde{F}}{\partial \tilde{P}_s} \mathrm{d}\tilde{P}_s \\
& + \int_{-\infty}^{+\infty} \left( \tilde{F}(s, \tilde{P}_{s-}(1+y)) - \tilde{F}(s, \tilde{P}_{s-}) - y \tilde{P}_{s-} \frac{\partial \tilde{F}}{\partial \tilde{P}_s} \right) \mathrm{d}\tilde{M}_s,
\end{aligned}
$$

where $\mathrm{d}\tilde{M}_s$ is the compensated random measure associated with $Z$.

Then if we assume now that $\tilde{F}(s, \tilde{P}_{s-}(1+y))$ is analytical in $y$ we have

$$
\begin{aligned}
& \mathrm{d}\tilde{F}(s, \tilde{P}_s) \\
= \;& \frac{\partial \tilde{F}}{\partial \tilde{P}_s}\mathrm{d}\tilde{P}_s + \int_{-\infty}^{+\infty} \sum_{k\geq 2} \frac{1}{k!} \frac{\partial^k \tilde{F}}{\partial y^k}_{|y=0} y^k \mathrm{d}\tilde{M}_s \\
= \;& \frac{\partial \tilde{F}}{\partial \tilde{P}_s}\mathrm{d}\tilde{P}_s + \sum_{k\geq 2} \frac{1}{k!} \frac{\partial^k \tilde{F}}{\partial y^k}_{|y=0} \mathrm{d}\,\tilde{Y}_s^{(k)}
\end{aligned}
$$

For instance, consider derivatives with payoff $e^{\int_0^{\bar{T}} r_s \mathrm{d}s} \tilde{P}_{\bar{T}}^k$, $k \geq 2$. Then its discounted price will be given by

$$\tilde{F}^{(k)}(t, \tilde{P}_t) = \mathbb{E}_Q(\tilde{P}_{\bar{T}}^k | \mathcal{F}_t) = \tilde{P}_t^k \mathbb{E}_Q \left( \left( \frac{\tilde{P}_{\bar{T}}}{\tilde{P}_t} \right)^k | \mathcal{F}_t \right)$$

$$= \tilde{P}_t^k \mathbb{E}_Q \left( \left( \frac{\tilde{P}_{\bar{T}}}{\tilde{P}_t} \right)^k \right) = \varphi^{(k)}(t, \bar{T}, T) \tilde{P}_t^k .$$

Then this derivative can be replicated by using the power-jump assets

$$\mathrm{d}\tilde{F}^{(k)}(t,\tilde{P}_t)=\frac{k\tilde{F}^{(k)}(t,\tilde{P}_{t-})}{\tilde{P}_{t-}}\mathrm{d}\tilde{P}_t+\sum_{i=2}^{k}\tilde{F}^{(k)}(t,\tilde{P}_{t-})\binom{k}{i}\mathrm{d}\tilde{Y}_t^{(i)}.$$

Define $\tilde{F}^{(1)}(t, \tilde{P}_t) = \tilde{P}_t$, and

$$\mathrm{d}\tilde{Y}_t^{(1)} = \frac{\mathrm{d}\tilde{P}_t}{\tilde{P}_{t-}}.$$

Then we can write

$$\mathrm{d}\tilde{F}^{(k)}(t, \tilde{P}_t) = \sum_{i=1}^{k} \tilde{F}^{(k)}(t, P_{t-}) \binom{k}{i} \mathrm{d}\tilde{Y}_t^{(i)}$$

and

$$\mathrm{d}\tilde{Y}_t^{(k)} = \sum_{i=1}^{k} \binom{k}{i} (-1)^{k-i} \frac{1}{\tilde{F}^{(i)}(t, \tilde{P}_{t-})} \mathrm{d}\tilde{F}^{(i)}(t, \tilde{P}_{t-}).$$

The market model, $M_Q$, obtained by enlarging the market $M$ with the derivatives $F^{(k)}, k \geq 2$ is complete, in the sense that any square-integrable contingent claim $X \in L^2(Q)$ can be replicated by a self-financing portfolio.

In our market with horizon $\bar{T}$ we have considered only the bond with maturity $T$, power-jump assets or derivatives with power payoffs, but obviously we can consider bonds with different maturity time to complete the market. Then the question is if these bonds are sufficient to complete the market. We know that, to complete the market, it is sufficient to produce discounted payoffs with values $\tilde{P}(\bar{T}, T)^k$, $k \geq 2$ at time $\bar{T}$.

Let us consider again affine term structure models such that $\Gamma(T)_t$ is given by

$$\Gamma(T)_t = a(T)\,b(t) + c(t)\,, \tag{6}$$

where $a(\cdot)$ is a strictly increasing function and $b(\cdot) \neq 0$.

## Proposition

*If the term structure model satisfies (6), then the discounted price function of the contingent claim with payoff* $X^{(k)}$ *is given by*

$$\widetilde{F}^{(k)}\left(t, \widetilde{P}(t, T)\right) = \psi^{(k)}\left(t, \overline{T}, T\right) \widetilde{P}(t, T_k(t)) B_t^{1-k},$$

*where* $\psi^{(k)}$ *is the deterministic function*

$$\psi^{(k)}\left(t, \overline{T}, T\right) = \varphi^{(k)}\left(t, \overline{T}, T\right) \left(\frac{P(0, T)}{P(0, t)}\right)^k \frac{1}{P(0, T_k)}$$

*and*

$$T_k(t) = a^{-1}\left(k(a(T) - a(t)) + a(t)\right).$$

Using the formula deduced for $\widetilde{P}(t,T)$ in (2), we have that

$$P(t,T)=\frac{P(0,T)}{P(0,t)}\exp\{(a(T)-a(t))\int_0^t b(u)\mathrm{d}L_u\},$$

So,

$$P(t,T)^k=\left(\frac{P(0,T)}{P(0,t)}\right)^k\exp\{k\,(a(T)-a(t))\int_0^t b(u)\mathrm{d}L_u\}$$

and taking $T_k(t)=a^{-1}(k\,(a(T)-a(t))+a(t))$ we have that

$$\begin{aligned}P(t,T)^k &= \left(\frac{P(0,T)}{P(0,t)}\right)^k\exp\{(a(T_k)-a(t))\int_0^t b(u)\mathrm{d}L_u\}\\ &= \left(\frac{P(0,T)}{P(0,t)}\right)^k\frac{P(t,T_k)}{P(0,T_k)}.\end{aligned}$$

Finally

$$\widetilde{P}(t,T)^k=\frac{P(t,T)^k}{B_t^k}=\left(\frac{P(0,T)}{P(0,t)}\right)^k\frac{\widetilde{P}(t,T_k)}{P(0,T_k)}B_t^{1-k}.$$

The idea is to use the fact that

$$\widetilde{F}^{(k)}\left(t, \widetilde{P}(t,T)\right) = \psi^{(k)}\left(t, \overline{T}, T\right) \widetilde{P}(t, T_k(t)) B_t^{1-k}$$

is a martingale and from this, we have that

$$\mathrm{d}\widetilde{F}^{(k)}\left(t, \widetilde{P}(t,T)\right) = ...\mathrm{d}t + \psi^{(k)}\left(t, \overline{T}, T\right) B_t^{1-k} \mathrm{d}\widetilde{P}(t, T_k(t))$$

and

$$\frac{\mathrm{d}\widetilde{F}_s^{(k)}}{\widetilde{F}_s^{(k)}} = \frac{\mathrm{d}\widetilde{P}(s, T_k(s))}{\widetilde{P}(s, T_k(s))},$$

We have finally

$$\mathrm{d}\tilde{Y}_t^{(k)} = \sum_{i=1}^{k} \binom{k}{i} (-1)^{k-i} \frac{d\widetilde{P}(s, T_i(s))}{\widetilde{P}(s, T_i(s))}.$$

which gives us a way of obtaining the replicating self-financing portfolio and the corresponding hedging formula.

In our market with horizon $\overline{T}$ we have considered first only the bond with maturity $T > \overline{T}$ and we want to complete the market with bonds with maturity times $T^* > \bar{T}$. Then the question is if these bonds are sufficient to complete the market. First we see that for any measurable function $h$, can be approximate by a linear combination of $P(t, T^*)$ with $T^* > \bar{T}$. In fact if

$$\mathbb{E}_Q[h(P(t,T))P(t,T^*)] = 0, \text{for all} \quad T^* > \overline{T},$$

we have that $h(P(t, T)) = 0$ a.s. since

$$\mathbb{E}_Q[h(P(t,T))P(t,T^*)] = \mathbb{E}_Q[g(\int_0^t b(s)\mathrm{d}L_s)\exp\{\lambda\int_0^t b(s)\mathrm{d}L_s] = 0,$$

for all $\lambda = a(T^*) - a(t)$, then, in particular,

$$P(t,T)^k = \sum_{j\in J}\lambda_j^k(t)P(t,T_j^{k*}(t))$$

and

$$\begin{aligned}\widetilde{F}^{(k)}\left(t,\widetilde{P}(t,T)\right) &= \varphi^{(k)}\left(t,\overline{T},T\right)\left(\widetilde{P}(t,T)\right)^k = \varphi^{(k)}\left(t,\overline{T},T\right)\frac{P(t,T)^k}{B_t^k} \\ &= \varphi^{(k)}\left(t,\overline{T},T\right)B_t^{1-k}\sum_{j\in J}\lambda_j^k(t)\tilde{P}(t,T_j^{k*}(t)).\end{aligned}$$

Consequently the result follows as in the previous case but with $\sum_{j\in J}\lambda_j^k(t)\tilde{P}(t,T_j^{k*}(t))$ instead of $\left(\frac{P(0,T)}{P(0,t)}\right)^k\frac{\tilde{P}(t,T_k(t))}{P(0,T_k(t))}$.

Note that if in the expression

$$\mathrm{d}M_t = \gamma_t \mathrm{d}\bar{W}_t + \int_{\mathbb{R}} \varphi(x,s)\mathrm{d}\widetilde{M}_s^Q,$$

$\varphi(\cdot, t)$ is analytic. for fixed $t$, then we can write a.s. that

$$\mathrm{d}M_t = \sum_{k\geq 1} a_k \mathrm{d}\tilde{Y}_s^{(k)},$$

and the corresponding contingent claim will be able replicated in a perfect (a.s.) sense. This was already observed, by using different arguments, in [BKR97].

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