Correlation based calibration of Lévy interest rate models

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Statistical Inference for Lévy Processes wih Applications to Finance
EURANDOM, Eindhoven, The Netherlands July 15–17, 2009

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Basic interest rates

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$B(t,T)$: price at time $t \in [0,T]$ of a default-free zero coupon bond with maturity $T \in [0,T^*]$ $\quad (B(T,T) = 1)$

$f(t,T)$: instantaneous forward rate

$$B(t,T) = \exp\left(-\int_t^T f(t,u)\,\mathrm{d}u\right)$$

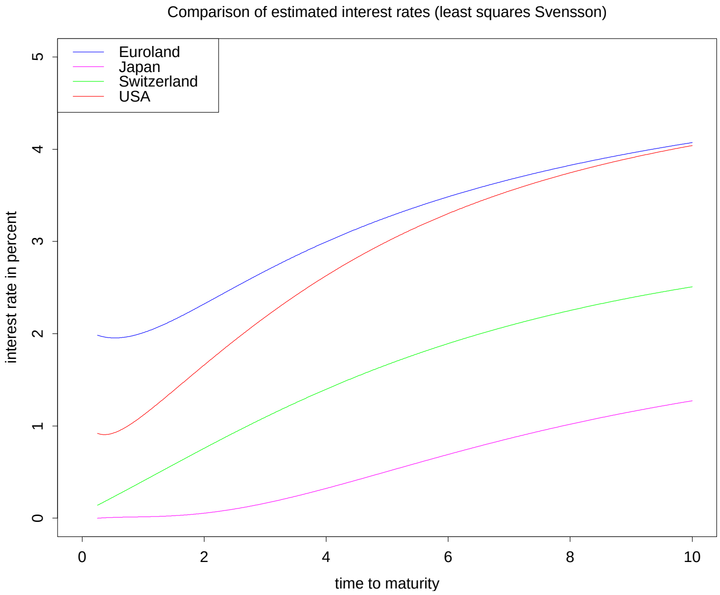
$L(t,T)$: default-free forward Libor rate for the interval T to $T+\delta$ as of time $t \le T$ $\quad$ (δ-forward Libor rate)

$$L(t,T) := \tfrac{1}{\delta}\left(\tfrac{B(t,T)}{B(t,T+\delta)} - 1\right)$$

$F_B(t,T,U)$: forward price process for the two maturities $T < U$

$$F_B(t,T,U) := \tfrac{B(t,T)}{B(t,U)}$$

$$\Longrightarrow \quad 1 + \delta L(t,T) = \frac{B(t,T)}{B(t,T+\delta)} = F_B(t,T,T+\delta)$$



Termstructure, February 17, 2004

The driving process

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$L = (L^1, \ldots, L^d)$ is a d-dimensional time-inhomogeneous Lévy process, i.e. L has independent increments and the law of L_t is given by the characteristic function

$$\mathbb{E}[\exp(\mathrm{i}\langle u, L_t\rangle)] = \exp \int_0^t \theta_s(\mathrm{i}u)\,\mathrm{d}s \qquad \text{with}$$

$$\theta_s(z) = \langle z, b_s\rangle + \frac{1}{2}\langle z, c_s z\rangle + \int_{\mathbb{R}^d} \left(e^{\langle z,x\rangle} - 1 - \langle z, x\rangle\right) F_s(\mathrm{d}x),$$

where $b_t \in \mathbb{R}^d$, c_t is a symmetric nonnegative-definite $d \times d$-matrix, and F_t is a Lévy measure.

Integrability: $$\int_0^{T^*} \left(|b_s| + ||c_s|| + \int_{\{|x|\le 1\}} |x|^2 F_s(\mathrm{d}x)\right) \mathrm{d}s < \infty$$

$$\int_0^{T^*} \int_{\{|x|>1\}} \exp\langle u, x\rangle F_s(\mathrm{d}x)\,\mathrm{d}s < \infty \quad \text{for } u \in [-(1+\varepsilon)M, (1+\varepsilon)M]^d$$

Description in terms of modern stochastic analysis

$L = (L_t)$ is a special semimartingale with canonical representation

$$L_t = \int_0^t b_s \, \mathrm{d}s + \int_0^t c_s^{1/2} \, \mathrm{d}W_s + \int_0^t \int_{\mathbb{R}^d} x(\mu^L - \nu)(\mathrm{d}s, \mathrm{d}x)$$

and characteristics

$$A_t = \int_0^t b_s \, \mathrm{d}s, \qquad C_t = \int_0^t c_s \, \mathrm{d}s, \qquad \nu(\mathrm{d}s, \mathrm{d}x) = F_s(\mathrm{d}x) \, \mathrm{d}s$$

$W = (W_t)$ is a standard d-dimensional Brownian motion ,

μ^L the random measure of jumps of L and ν is the compensator of μ^L

L is also called a process with independent increments and absolutely continuous characteristics (PIIAC).

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Generalized hyperbolic distributions

(O.E. Barndorff-Nielsen (1977))

Density: $$d_{GH}(x) = \alpha(\lambda,\alpha,\beta,\delta)\left(\delta^2+(x-\mu)^2\right)^{(\lambda-1/2)/2} \times K_{\lambda-1/2}\left(\alpha\sqrt{\delta^2+(x-\mu)^2}\right)\exp(\beta(x-\mu))$$

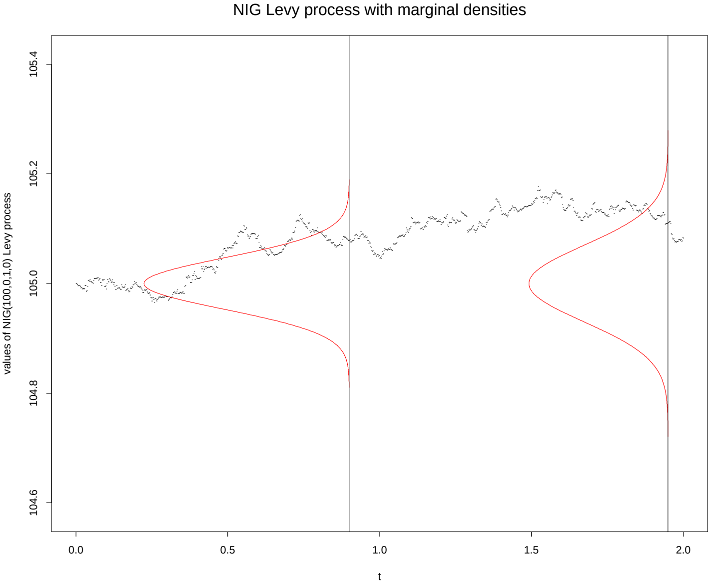
$$\alpha(\lambda,\alpha,\beta,\delta) = \frac{(\alpha^2-\beta^2)^{\lambda/2}}{\sqrt{2\pi}\alpha^{\lambda-1/2}\delta^{\lambda}K_\lambda\left(\delta\sqrt{\alpha^2-\beta^2}\right)}$$

K_λ modified Bessel function of the third kind with index λ
λ Class parameter, α Shape, β Skewness,
μ Location, δ Scale parameter (Volatility)

Generalized hyperbolic distributions are infinitely divisible

$\Longrightarrow$ d_{GH} generates a Lévy process $(L_t)_{t\geq 0}$ s.t. $\mathcal{L}(L_1) \sim d_{GH}$

Simulation of a GH Lévy motion



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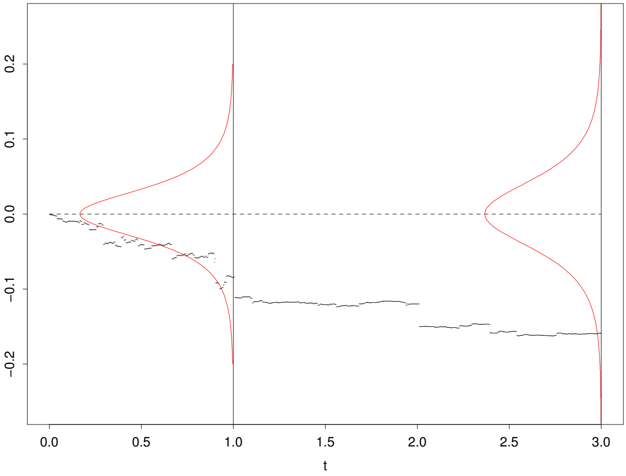
Simulation of a Lévy process

NIG(10,0,0.100,0) on [0,1]
NIG(10,0,0.025,0) on [1,3]



Dynamics of the forward rates

(Eb–Raible (1999), Eb–Özkan (2003),
Eb–Jacod–Raible (2005), Eb–Kluge (2006)

$$\mathrm{d}f(t,T) = \alpha(t,T)\,\mathrm{d}t - \sigma(t,T)\,\mathrm{d}L_t \qquad (0 \le t \le T \le T^*)$$

$\alpha(t,T)$ and $\sigma(t,T)$ satisfy measurability and boundedness conditions and $\alpha(s,T) = \sigma(s,T) = 0$ for $s > T$

Define $\quad A(s,T) = \int_{s\wedge T}^{T} \alpha(s,u)\,\mathrm{d}u \quad$ and $\quad \Sigma(s,T) = \int_{s\wedge T}^{T} \sigma(s,u)\,\mathrm{d}u$

Assume $0 \le \Sigma^i(s,T) \le M \qquad (1 \le i \le d)$

For most purposes we can consider deterministic α and σ

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Implications

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Savings account and default-free zero coupon bond prices are given by

$$B_t = \frac{1}{B(0,t)} \exp\left(\int_0^t A(s,T)\,\mathrm{d}s - \int_0^t \Sigma(s,t)\,\mathrm{d}L_s \right) \quad \text{and}$$

$$B(t,T) = B(0,T) B_t \exp\left(-\int_0^t A(s,T)\,\mathrm{d}s + \int_0^t \Sigma(s,T)\,\mathrm{d}L_s \right).$$

If we choose $A(s,T) = \theta_s(\Sigma(s,T))$, then bond prices, discounted by the savings account, are martingales.

In case $d = 1$, the martingale measure is unique (see Eberlein, Jacod, and Raible (2004)).

Key tool

$L = (L^1, \dots, L^d)$ d-dimensional time-inhomogeneous Lévy process

$$\mathbb{E}[\exp(i\langle u, L_t\rangle)] = \exp \int_0^t \theta_s(iu)\,\mathrm{d}s \qquad \text{where}$$

$$\theta_s(z) = \langle z, b_s\rangle + \frac{1}{2}\langle z, c_s z\rangle + \int_{\mathbb{R}^d} \left(e^{\langle z,x\rangle} - 1 - \langle z, x\rangle\right) F_s(\mathrm{d}x)$$

in case L is a (time-homogeneous) Lévy process, $\theta_s = \theta$ is the cumulant (log-moment generating function) of L_1.

Proposition Eberlein, Raible (1999)

Suppose $f : \mathbb{R}_+ \to \mathbb{C}^d$ is a continuous function such that $|\mathcal{R}(f^i(x))| \leq M$ for all $i \in \{1, \dots, d\}$ and $x \in \mathbb{R}_+$, then

$$\mathbb{E}\left[\exp\left(\int_0^t f(s)dL_s\right)\right] = \exp\left(\int_0^t \theta_s(f(s))ds\right)$$

Take $f(s) = \sum(s, T)$ for some $T \in [0, T^*]$

Correlation of zero coupon bond prices

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Theorem

Let $0 \leq t \leq T_1 \leq T_2 \leq T^*$*. The correlation of* $B(t, T_1)$ *and* $B(t, T_2)$ *is*

$$\mathrm{Corr}(B(t,T_1), B(t,T_2)) = \frac{g_1(t,T_1,T_2) - g_2(t,T_1,T_2)}{\sqrt{h(t,T_1)}\sqrt{h(t,T_2)}},$$

where

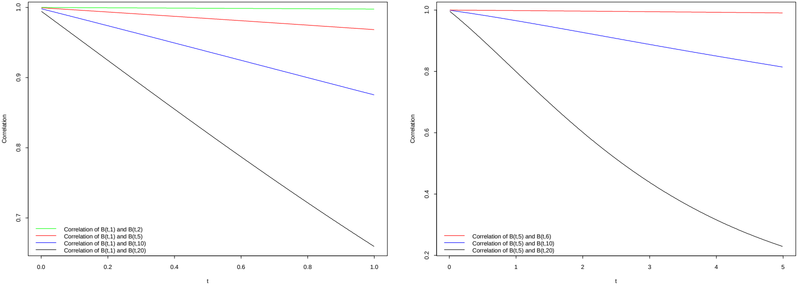
$$g_1(t,T_1,T_2) := \exp\left(\int_0^t \theta_s\Big(\Sigma(s,t,T_1) + \Sigma(s,t,T_2)\Big)\, ds\right),$$

$$g_2(t,T_1,T_2) := \exp\left(\int_0^t \Big(\theta_s(\Sigma(s,t,T_1)) + \theta_s(\Sigma(s,t,T_2))\Big)\, ds\right)$$

and

$$h(t,T) := \exp\left(\int_0^t \theta_s(2\Sigma(s,t,T))\, ds\right) - \exp\left(\int_0^t 2\theta_s(\Sigma(s,t,T))\, ds\right).$$

Similar for $\mathrm{Corr}(B(t_1,T_1), B(t_2,T_2))$.



Correlation of zero coupon bond prices for $\alpha = 100$, $\beta = 0$, $\delta = 1$ and $a = 0.05$

Basic interest rates

$B(t,T)$: price at time $t \in [0, T]$ of a default-free zero coupon bond

$f(t,T)$: instantaneous forward rate

$$B(t,T) = \exp\left(-\int_t^T f(t,u)\,\mathrm{d}u\right)$$

$L(t,T)$: default-free forward Libor rate for the interval T to $T+\delta$

$$L(t,T) := \frac{1}{\delta}\left(\frac{B(t,T)}{B(t,T+\delta)} - 1\right)$$

$F_B(t,T,U)$: forward price process for the two maturities T and U

$$F_B(t,T,U) := \frac{B(t,T)}{B(t,U)}$$

$$\Longrightarrow \quad 1 + \delta L(t,T) = \frac{B(t,T)}{B(t,T+\delta)} = F_B(t,T,T+\delta)$$

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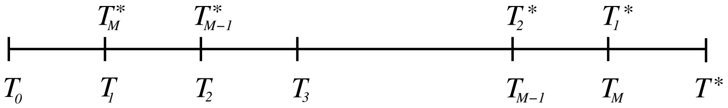
The Lévy Libor model

(Eb–Özkan (2005))

Tenor structure $T_0 < T_1 < \cdots < T_M < T_{M+1} = T^*$
with $T_{i+1} - T_i = \delta$, set $T_i^* = T^* - i\delta$ for $i = 1, \ldots, M$



Assumptions

(LR.1): For any maturity T_i there is a bounded deterministic function $\lambda(\cdot, T_i)$, which represents the volatility of the forward Libor rate process $L(\cdot, T_i)$.

(LR.2): We assume a strictly decreasing and strictly positive initial term structure $B(0, T)$ $(T \in]0, T^*])$. Consequently the initial term structure of forward Libor rates is given by

$$L(0, T) = \frac{1}{\delta} \left(\frac{B(0, T)}{B(0, T + \delta)} - 1 \right)$$

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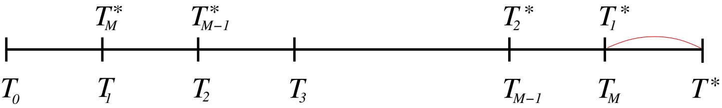
Backward Induction (1)

Given a stochastic basis $(\Omega, \mathcal{F}_{T^*}, \mathbb{P}_{T^*}, (\mathcal{F}_t)_{0 \leq t \leq T^*})$



We postulate that under $\mathbb{P}_{T^*}$

$$L(t, T_1^*) = L(0, T_1^*) \exp\left(\int_0^t \lambda(s, T_1^*) dL_s^{T^*} \right)$$

$$\text{where} \qquad L_t^{T^*} = \int_0^t b_s ds + \int_0^t c_s^{1/2} dW_s^{T^*} + \int_0^t \int_{\mathbb{R}} x(\mu^L - \nu^{T^*,L})(ds, dx)$$

is a non-homogeneous Lévy process with random measure of jumps μ^L and $\mathbb{P}_{T^*}$-compensator $\nu^{T^*,L}(ds, dx) = F_s(dx)ds$

Backward Induction (2)

In order to make $L(t, T_1^*)$ a $\mathbb{P}_{T^*}$-martingale, choose the drift characteristic (b_s) s.t.

$$\int_0^t \lambda(s, T_1^*) b_s ds = -\frac{1}{2}\int_0^t c_s \lambda^2(s, T_1^*) ds - \int_0^t \int_{\mathbb{R}} \left(e^{\lambda(s, T_1^*)x} - 1 - \lambda(s, T_1^*)x\right) \nu^{T^*, L}(ds, dx)$$

Transform $L(t, T_1^*)$ in a stochastic exponential

$$L(t, T_1^*) = L(0, T_1^*)\mathcal{E}(H(t, T_1^*))$$

where

$$H(t, T_1^*) = \int_0^t \lambda(s, T_1^*) c_s^{1/2} dW_s^{T^*} + \int_0^t \int_{\mathbb{R}} \left(e^{\lambda(s, T_1^*)x} - 1\right)(\mu^L - \nu^{T^*, L})(ds, dx)$$

Backward Induction (3)

Equivalently

$$dL(t, T_1^*) = L(t-, T_1^*)\Big(\lambda(t, T_1^*)c_t^{1/2}dW_t^{T^*} + \int_{\mathbb{R}} \left(e^{\lambda(t,T_1^*)x} - 1\right)(\mu^L - \nu^{T^*,L})(dt, dx)\Big)$$

with initial condition

$$L(0, T_1^*) = \frac{1}{\delta}\left(\frac{B(0, T_1^*)}{B(0, T^*)} - 1\right)$$

Backward Induction (4)

Recall $F_B(t, T_1^*, T^*) = 1 + \delta L(t, T_1^*)$, therefore,

$$
\begin{aligned}
dF_B(t, T_1^*, T^*) &= \delta dL(t, T_1^*) \\
&= F_B(t-, T_1^*, T^*)\Bigg(\underbrace{\frac{\delta L(t-, T_1^*)}{1 + \delta L(t-, T_1^*)} \lambda(t, T_1^*)}_{= \alpha(t, T_1^*, T^*)} c_t^{1/2} dW_t^{T^*} \\
&\quad + \int_{\mathbb{R}} \underbrace{\frac{\delta L(t-, T_1^*)}{1 + \delta L(t-, T_1^*)} \left(e^{\lambda(t, T_1^*)x} - 1 \right)}_{= \beta(t, x, T_1^*, T^*) - 1} (\mu^L - \nu^{T^*, L})(dt, dx) \Bigg)
\end{aligned}
$$

Define the forward martingale measure associated with T_1^*

$$
\frac{d\mathbb{P}_{T_1^*}}{d\mathbb{P}_{T^*}} = \mathcal{E}_{T_1^*}(M^1) \qquad \text{where}
$$

$$
\begin{aligned}
M_t^1 &= \int_0^t \alpha(s, T_1^*, T^*) c_s^{1/2} dW_s^{T^*} \\
&\quad + \int_0^t \int_{\mathbb{R}} (\beta(s, x, T_1^*, T^*) - 1)\, (\mu^L - \nu^{T^*, L})(ds, dx)
\end{aligned}
$$

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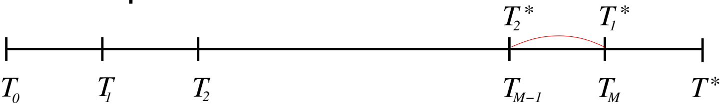
Backward Induction (5)

Then $W_t^{T_1^*} = W_t^{T^*} - \int_0^t \alpha(s, T_1^*, T^*) c_s^{1/2} ds$

is the forward Brownian motion for date T_1^* and

$\nu^{T_1^*, L}(dt, dx) = \beta(t, x, T_1^*, T^*) \nu^{T^*, L}(dt, dx)$ is the $\mathbb{P}_{T_1^*}$-compensator for μ^L.

Second step



We postulate that under $\mathbb{P}_{T_1^*}$

$$L(t, T_2^*) = L(0, T_2^*) \exp\left(\int_0^t \lambda(s, T_2^*) dL_s^{T_1^*} \right) \text{ where}$$

$$L_t^{T_1^*} = \int_0^t b_s^{T_1^*} ds + \int_0^t c_s^{1/2} dW_s^{T_1^*} + \int_0^t \int_{\mathbb{R}} x(\mu^L - \nu^{T_1^*, L})(ds, dx)$$

Backward Induction (6)

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Second measure change

$$\frac{d\mathbb{P}_{T_2^*}}{d\mathbb{P}_{T_1^*}} = \mathcal{E}_{T_2^*}(M^2)$$

where

$$M_t^2 = \int_0^t \alpha(s, T_2^*, T_1^*) c_s^{1/2} dW_s^{T_1^*} + \int_0^t \int_{\mathbb{R}} \left(\beta(s, x, T_2^*, T_1^*) - 1\right)(\mu^L - \nu^{T_1^*, L})(ds, dx)$$

This way we get for each time point T_j^* in the tenor structure a Libor rate process which is under the forward martingale measure $\mathbb{P}_{T_{j-1}^*}$ of the form

$$L(t, T_j^*) = L(0, T_j^*) \exp\left(\int_0^t \lambda(s, T_j^*) dL_s^{T_{j-1}^*}\right)$$

Backward Induction (7)

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Simpler formula for the forward martingale measure

$$\frac{\mathrm{d}\mathbb{P}_{T_i}}{\mathrm{d}\mathbb{P}_{T_{i+1}}} = \frac{1 + \delta L(T_i, T_i)}{1 + \delta L(0, T_i)}$$

Under the forward martingale measure $\mathbb{P}_{T_{j+1}}$ the difference of the driving processes can be given in the form

$$L_t^{T_{i+1}} - L_t^{T_{j+1}} = \int_0^t \mathrm{d}_s^{i,j} \, \mathrm{d}s$$

Correlations of the Libor rates

In order to stay within the class of time-inhomogeneous Lévy processes

$$\frac{\delta L(s-, T_i)}{1 + \delta L(s-, T_i)} \longrightarrow \frac{\delta L(0, T_i)}{1 + \delta L(0, T_i)}$$

Set

$$\widetilde{\theta}_s^{T_{i+1}}(z) := \frac{1}{2} c_s z^2 + \int_{\mathbb{R}} (e^{zx} - 1 - zx) F_s^{T_{i+1}}(\mathrm{d}x)$$

Correlations of the Libor rates (2)

Theorem

Let $i, j, k \in \{1, \ldots, M+1\}$ and $0 \le t \le \min\{T_i, T_j\}$. Then under the forward martingale measure $\mathbb{P}_{T_{k+1}}$ (and under the approximation), the correlation of the LIBOR rates $L(t, T_i)$ and $L(t, T_j)$ is

$$\mathrm{Corr}_{\mathbb{P}_{T_{k+1}}}(L(t,T_i), L(t,T_j)) = \frac{g_1(t,i,j,k) - g_2(t,i,j,k)}{\sqrt{h(t,i,k))}\sqrt{h(t,j,k)}},$$

where

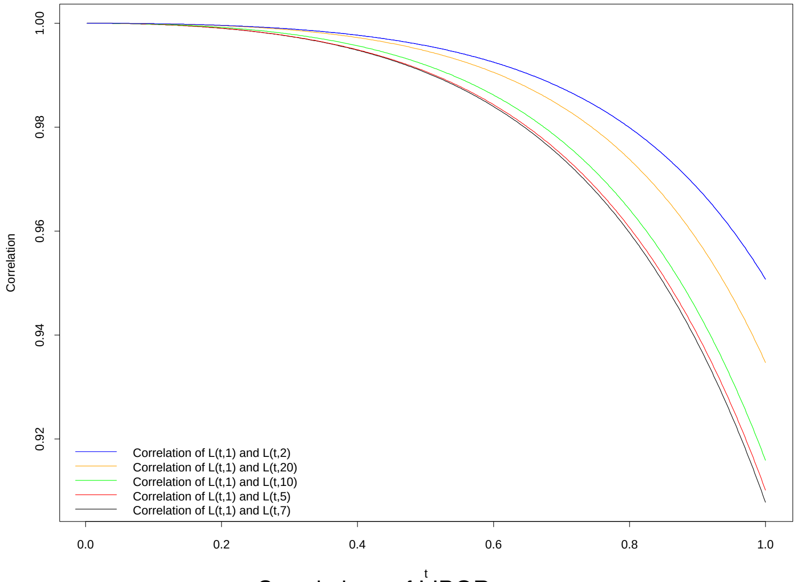
$$g_1(t,i,j,k)) := \exp\left(\int_0^t \tilde{\theta}_s^{T_{k+1}}\Big(\lambda(s,T_i) + \lambda(s,T_j)\Big)\, ds\right),$$

$$g_2(t,i,j,k)) := \exp\left(\int_0^t \left(\tilde{\theta}_s^{T_{k+1}}(\lambda(s,T_i)) + \tilde{\theta}_s^{T_{k+1}}(\lambda(s,T_j))\right)\, ds\right)$$

and for $l \in \{i, j\}$ we set

$$h(t,l,k) := \exp\left(\int_0^t \tilde{\theta}_s^{T_{k+1}}(2\lambda(s,T_l))\, ds\right) - \exp\left(2\int_0^t \tilde{\theta}_s^{T_{k+1}}(\lambda(s,T_l))\, ds\right).$$

Suitable volatility structure $\lambda(t, T_i) = (T_i - t)\exp(-b(T_i - t)) + c$



Correlations of LIBOR rates

for $\alpha = 100$, $\beta = 0$, $\delta = 0.01$, $b = 0.5$ and $c = 0.1$

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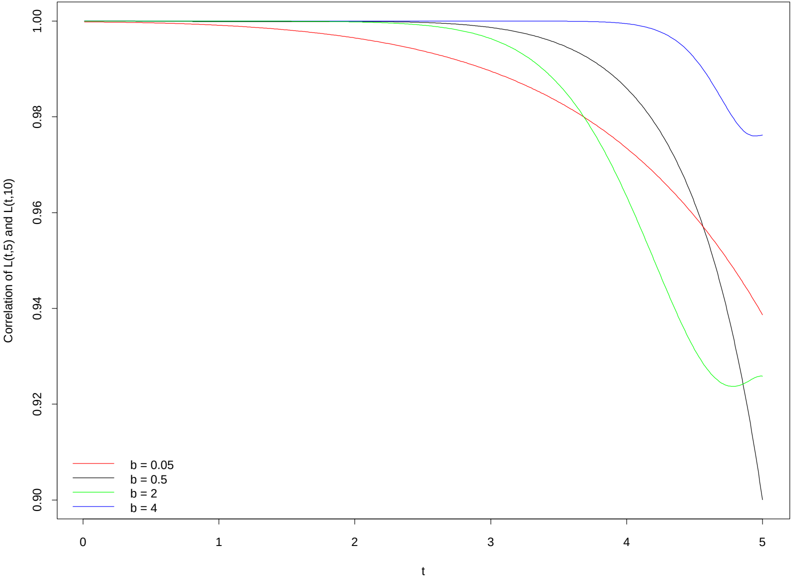
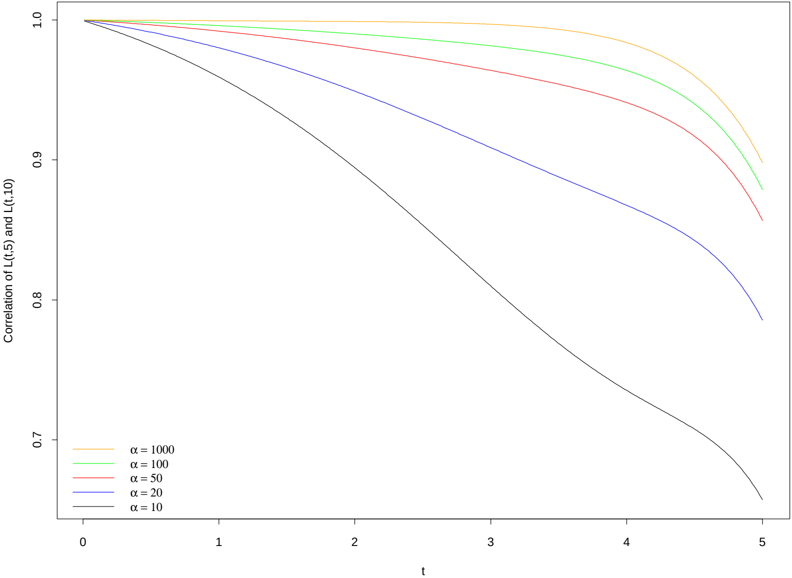


Correlation of $L(t, 5)$ and $L(t, 10)$ for $\alpha = 100$, $\beta = 0$, $\delta = 0.01$ and $c = 0.1$



Correlation of $L(t,5)$ and $L(t,10)$ for $\beta = 0$, $\delta = 10$, $b = 0.5$ and $c = 0.1$

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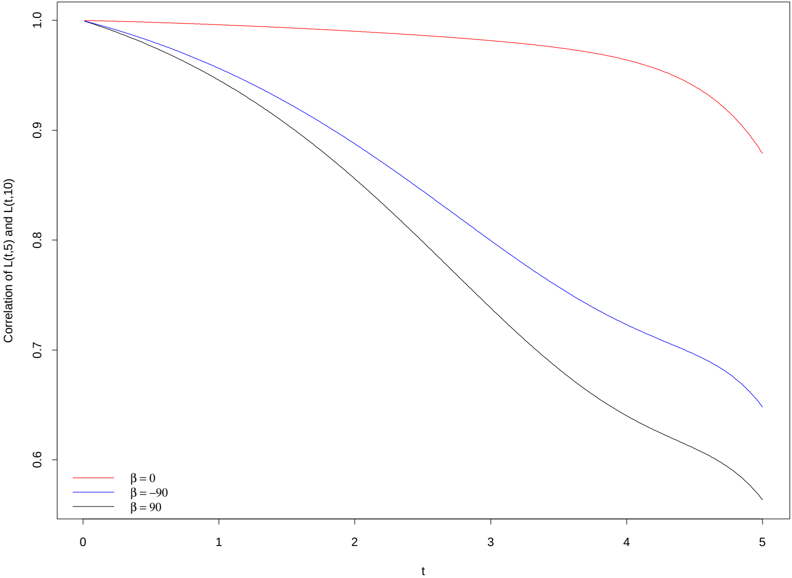

Correlation of $L(t, 5)$ and $L(t, 10)$ for $\alpha = 100$, $\delta = 10$, $b = 0.5$ and $c = 0.1$

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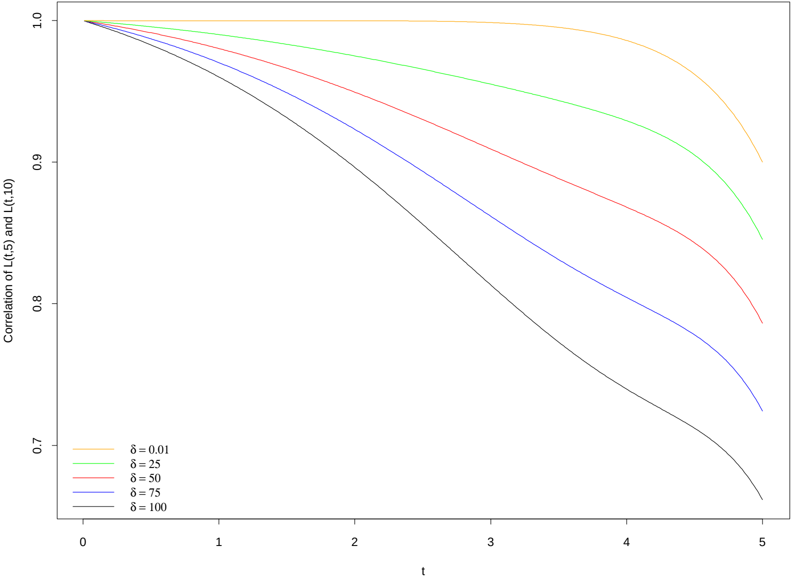
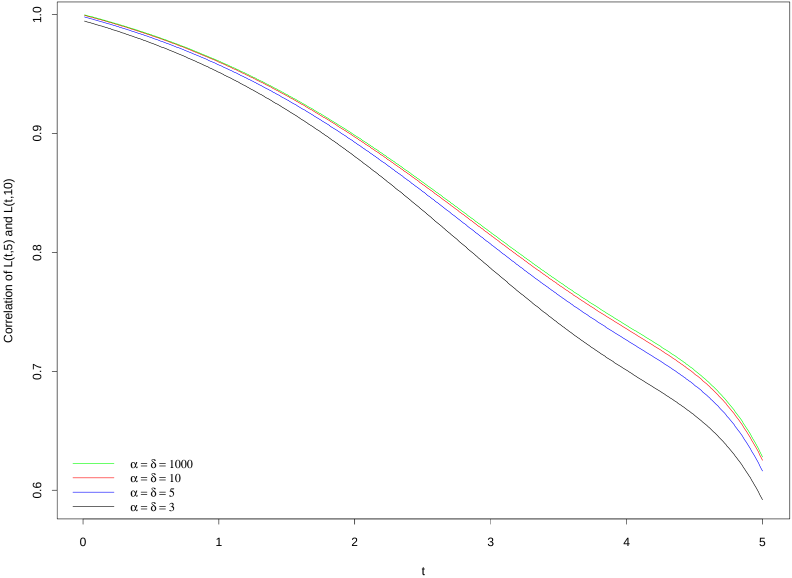


Correlation of $L(t,5)$ and $L(t,10)$ for $\alpha = 100$, $\beta = 0$, $b = 0.5$ and $c = 0.1$



Correlation of $L(t, 5)$ and $L(t, 10)$ for $\beta = 100$, $b = 0.5$ and $c = 0.1$

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Forward process model (1)

Postulate

$$1 + \delta L(t, T_1^*) = (1 + \delta L(0, T_1^*)) \exp\left(\int_0^t \lambda(s, T_1^*) dL_s^{T^*} \right)$$

equivalently

$$F_B(t, T_1^*, T^*) = F_B(0, T_1^*, T^*) \exp\left(\int_0^t \lambda(s, T_1^*) dL_s^{T^*} \right)$$

In differential form

$$\begin{aligned} \mathrm{d}F_B(t, T_1^*, T^*) \; = \; & F_B(t-, T_1^*, T^*)\Big(\lambda(t, T_1^*) c_t^{1/2} dW_t^{T^*} \\ & + \int_{\mathbb{R}} \left(\mathrm{e}^{\lambda(t, T_1^*)x} - 1\right)(\mu^L - \nu^{T^*,L})(dt, dx)\Big) \end{aligned}$$

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Forward process model (2)

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Define the forward martingale measure associated with T_1^*

$$\frac{d\mathbb{P}_{T_1^*}}{d\mathbb{P}_{T^*}} = \mathcal{E}_{T_1^*}(\widetilde{M}^1)$$

where

$$\widetilde{M}_t^1 = \int_0^t \lambda(s, T_1^*) c_s^{1/2} dW_s^{T^*} + \int_0^t \int_{\mathbb{R}} \left(\mathrm{e}^{\lambda(s,T_1^*)x} - 1\right)(\mu^L - \nu^{T^*,L})(ds, dx).$$

Forward process model (3)

Then $W_t^{T_1^*} = W_t^{T^*} - \int_0^t \lambda(s, T_1^*) c_s^{1/2} ds$ is the forward Brownian motion for date T_1^* and

$\nu^{T_1^*, L}(dt, dx) = \exp(\lambda(t, T_1^*)x)\, \nu^{T^*, L}(dt, dx)$ is the $\mathbb{P}_{T_1^*}$-compensator of μ^L.

Continuing this way we get for each time point T_j^* in the tenor structure a Libor rate process under $\mathbb{P}_{T_{j-1}^*}$ in the form

$$1 + \delta L(t, T_j^*) = (1 + \delta L(0, T_j^*)) \exp\left(\int_0^t \lambda(s, T_j^*) dL_s^{T_{j-1}^*} \right).$$

with successive compensators

$$\nu^{T_j^*, L}(dt, dx) = \exp\left(\sum_{i=1}^{j} \lambda(t, T_i^*) x \right) F_t(dx) dt.$$

Consequence of this alternative approach: negative Libor rates can occur

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Extensions of the basic Lévy market model

Lévy market model
(Eb–Özkan (2005))

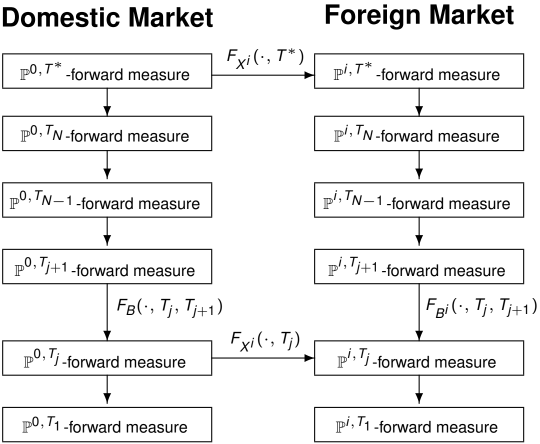
→ Multi-currency setting
(Eb–Koval (2006))

→ Credit risk model
(Eb–Kluge–Schönbucher (2006))

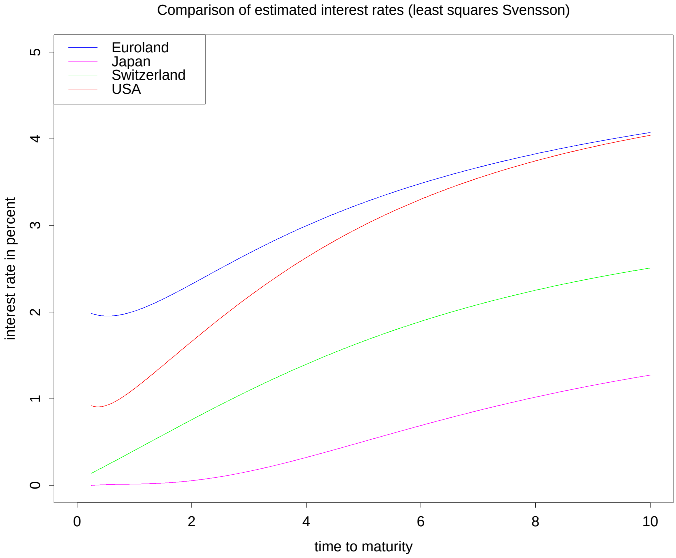
→ Swap rate model
(Eb–Liinev (2006))

→ Duality principle
(Eb–Kluge–Papapantoleon (2006))

Cross-currency Lévy market model



Relationship between domestic and foreign fixed income markets in a discrete-tenor framework.



Termstructure, February 17, 2004

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The foreign forward exchange rate for date T^* (1)

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Assumption

(FXR.1): *For every market $i \in \{0, \dots, m\}$ there are strictly decreasing and strictly positive zero-coupon bond prices $B^i(0, T_j)(j = 0, \dots, N+1)$ and for every foreign economy $i \in \{1, \dots, m\}$ there are spot exchange rates $X^i(0)$ given.*

Consequently the initial foreign forward exchange rate corresponding to T^* is

$$F_{X^i}(0, T^*) = \frac{B^i(0, T^*) X^i(0)}{B^0(0, T^*)}$$

The foreign forward exchange rate for date T^* (2)

Assumption

(FXR.2): *For every foreign market $i \in \{1, \dots, m\}$ there is a continuous deterministic function $\xi^i(\cdot, T^*) : [0, T^*] \to \mathbb{R}_+^d$.*

For every coordinate $1 \leq k \leq d$ we assume

$$(\xi^i(s, T^*))_k \leq \overline{M} \quad (s \in [0, T^*], \; 1 \leq i \leq m)$$

where $\overline{M} < \dfrac{M}{N+2}$.

The foreign forward exchange rate for date T^* (3)

Assumption

(FXR.3): *For every $i \in \{1, \ldots, m\}$ the foreign forward exchange rate for date T^* is given by*

$$F_{X^i}(t, T^*) = F_{X^i}(0, T^*) \exp\left(\int_0^t \gamma^i(s, T^*)\,\mathrm{d}s + \int_0^t \xi^i(s, T^*)^\top \,\mathrm{d}L_s^{0,T^*} \right)$$

where

$$\begin{aligned} \gamma^i(s, T^*) &= -\xi^i(s, T^*)^\top b_s^{0,T^*} - \frac{1}{2}\left|\xi^i(s, T^*)^\top c_s\right|^2 \\ &\quad - \int_{\mathbb{R}^d} \left(e^{\xi^i(s,T^*)^\top x} - 1 - \xi^i(s, T^*)^\top x \right) \lambda_s^{0,T^*}(\mathrm{d}x) \end{aligned}$$

Equivalently

$$\begin{aligned} F_{X^i}(t, T^*) &= F_{X^i}(0, T^*) \mathcal{E}_t \Bigg(\int_0^{\cdot} \xi^i(s, T^*)^\top c_s \,\mathrm{d}W_s^{0,T^*} \\ &\quad + \int_0^{\cdot} \int_{\mathbb{R}^d} \left(\exp\left(\xi^i(s, T^*)^\top x\right) - 1 \right)(\mu - \nu_{0,T^*})(\mathrm{d}s, \mathrm{d}x) \Bigg) \end{aligned}$$

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The foreign forward exchange rate for date T^* (4)

Consequences: $F_{X^i}(\cdot, T^*)$ is a $\mathbb{P}^{0,T^*}$-martingale

$$E_{\mathbb{P}^{0,T^*}}\left[\frac{F_{X^i}(t,T^*)}{F_{X^i}(0,T^*)}\right] = 1$$

Define

$$\left.\frac{d\mathbb{P}^{i,T^*}}{d\mathbb{P}^{0,T^*}}\right|_{\mathcal{F}_t} = \frac{F_{X^i}(t,T^*)}{F_{X^i}(0,T^*)}$$

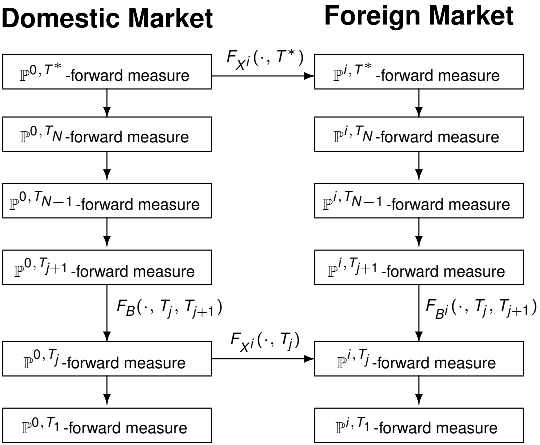
By Girsanov's theorem we get a $\mathbb{P}^{i,T^*}$-standard Brownian motion

$$W_t^{i,T^*} = W_t^{0,T^*} - \int_0^t c_s \xi^i(s,T^*)\,ds$$

and a $\mathbb{P}^{i,T^*}$-compensator

$$\nu_{i,T^*}(dt,dx) = \exp(\xi^i(t,T^*)^\top x)\nu_{0,T^*}(dt,dx)$$

Cross-currency Lévy market model



Relationship between domestic and foreign fixed income markets in a discrete-tenor framework.

Correlations of the Libor rates (3)

Define

$$\tilde{\theta}_s^{i,T_{j+1}}(z) := \frac{1}{2}c_s z^2 + \int_{\mathbb{R}} \left(e^{zx} - 1 - zx\right) F_s^{i,T_{j+1}}(dx).$$

Then the correlation of $L^{i_1}(t, T_{j_1})$ and $L^{i_2}(t, T_{j_2})$ under $\mathbb{P}_{\ell,T_{k+1}}$ is

$$\mathrm{Corr}_{\mathbb{P}_{l,T_{k+1}}}(L^{i_1}(t, T_{j_1}), L^{i_2}(t, T_{j_2})) = \frac{g_1(t, i_1, i_2, j_1, j_2, l, k) - g_2(t, i_1, i_2, j_1, j_2, l, k)}{\sqrt{h(t, i_1, j_1, l, k)}\sqrt{h(t, i_2, j_2, l, k)}},$$

where

$$g_1(t, i_1, i_2, j_1, j_2, l, k) = \exp\left(\int_0^t \tilde{\theta}_s^{l,T_{k+1}}\left(\lambda^{i_1}(s, T_{j_1}) + \lambda^{i_2}(s, T_{j_2})\right) ds\right),$$

$$g_2(t, i_1, i_2, j_1, j_2, l, k) = \exp\left(\int_0^t \left(\tilde{\theta}_s^{l,T_{k+1}}(\lambda^{i_1}(s, T_{j_1})) + \tilde{\theta}_s^{l,T_{k+1}}(\lambda^{i_2}(s, T_{j_2}))\right) ds\right)$$

and for $p \in \{1, 2\}$ we define

$$h(t, i_p, j_p, l, k) := \exp\left(\int_0^t \tilde{\theta}_s^{l,T_{k+1}}(2\lambda^{i_p}(s, T_{j_p}))ds\right) - \exp\left(2\int_0^t \tilde{\theta}_s^{l,T_{k+1}}(\lambda^{i_p}(s, T_{j_p}))ds\right).$$

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Calibration of the Lévy forward rate model

First step: Estimate correlations between zero coupon bond prices

Second step: Use the correlations to estimate the parameters of the driving Lévy process

Data set: Yield curve of German government bonds
August 7, 1997 – April 9, 2008
$\rightarrow$ 2707 trading days

How to get independent samples for each price?

Derivation of samples

For t, T, Δ where $t < T$ and $\Delta \in [0, T^* - T]$ define

$$B^\Delta(t, T) := B(t + \Delta, T + \Delta)\frac{B(0, T)}{B(0, t)}\frac{B(\Delta, t + \Delta)}{B(\Delta, T + \Delta)},$$

then $B^\Delta(t, T)$ has the same distribution as $B(t, T)$. For $\Delta \geq t$, $B^\Delta(t, T)$ is independent of $B(t, T)$.

Choose $\Delta = t, 2t, 3t, \ldots \implies$ independent samples $B^\Delta(t, T)$

Furthermore, for all $\Delta \geq t$

$$\text{Corr}(B^\Delta(t, T_1), B^\Delta(t, T_2)) = \text{Corr}(B(t, T_1), B(t, T_2))$$

Estimation

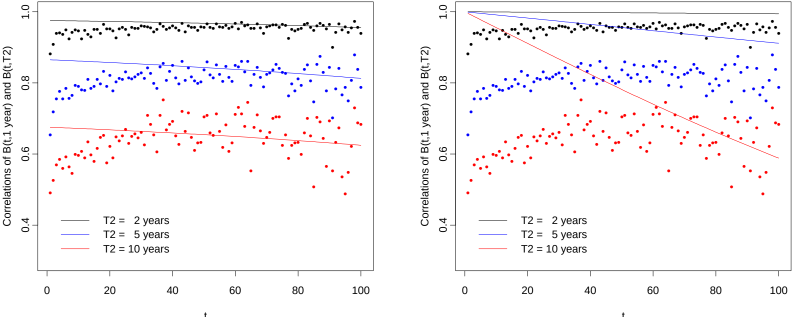
To estimate $\mathrm{Corr}(B(t,T_1), B(t,T_2))$ use the empirical correlation

$$\widehat{\mathrm{Corr}}(B(t,T_1), B(t,T_2)) = \frac{\sum_{i=0}^{n}(B^{it}(t,T_1) - \bar{B}(t,T_1))(B^{it}(t,T_2) - \bar{B}(t,T_2))}{\sqrt{\sum_{i=0}^{n}(B^{it}(t,T_1) - \bar{B}(t,T_1))^2}\sqrt{\sum_{i=0}^{n}(B^{it}(t,T_2) - \bar{B}(t,T_2))^2}},$$

where $n = \left[\frac{2707}{t}\right]$ and $\bar{B}(t,T_1)$, $\bar{B}(t,T_2)$ denote the arithmetic means.

To estimate the parameters, minimize

$$\sum_{t=1\text{ day}}^{100\text{ days}} \sum_{T_1=1\text{ year}}^{10\text{ years}} \sum_{T_2=1\text{ year}}^{10\text{ years}} \left(\widehat{\mathrm{Corr}}(B(t,T_1), B(t,T_2)) - \mathrm{Corr}(B(t,T_1), B(t,T_2))\right)^2.$$



Empirical correlations (points) and correlations calculated from the models (lines) for the calibrations with NIG Lévy processes (left) and with Brownian motions (right)

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