Estimating Jump Tail Dependence in Lévy Copula Models

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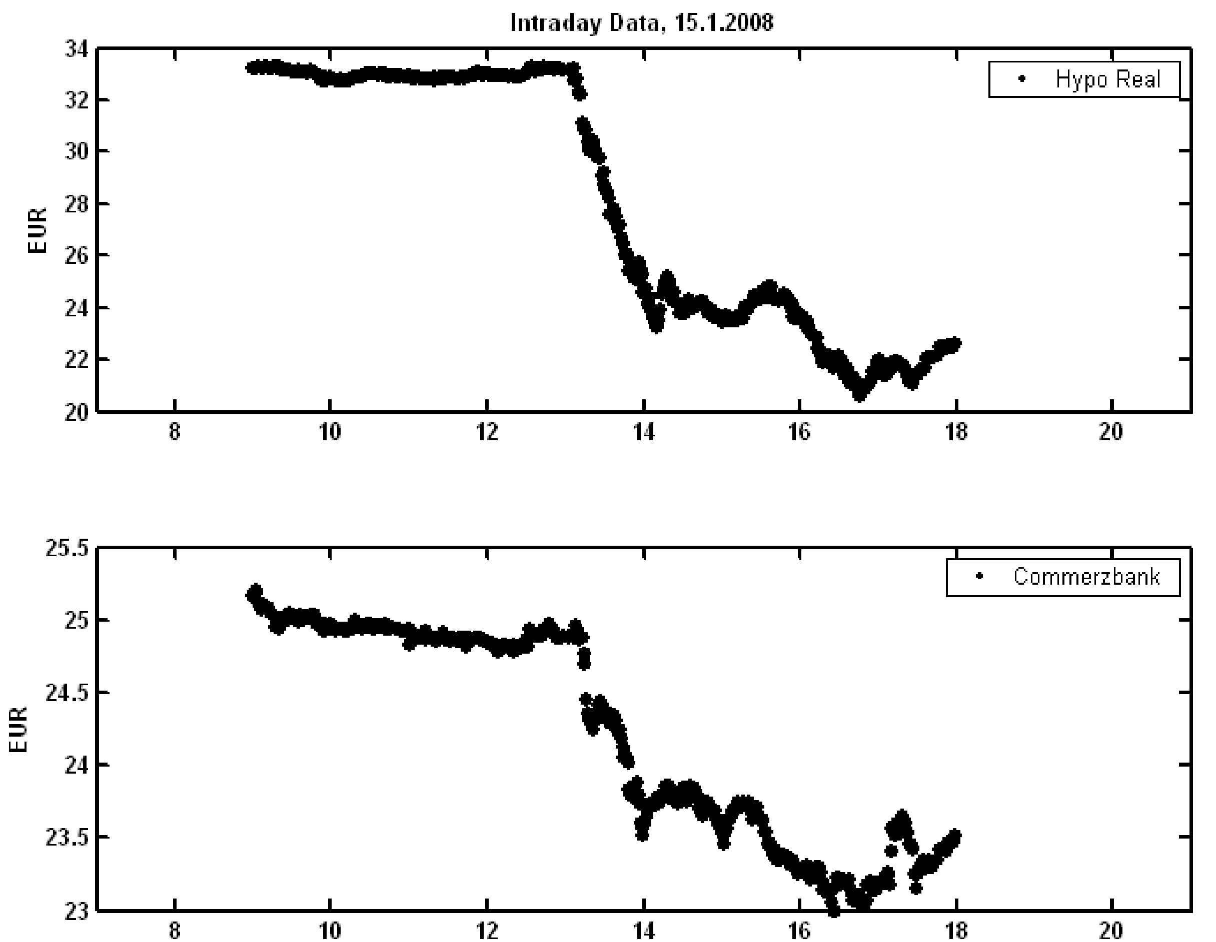
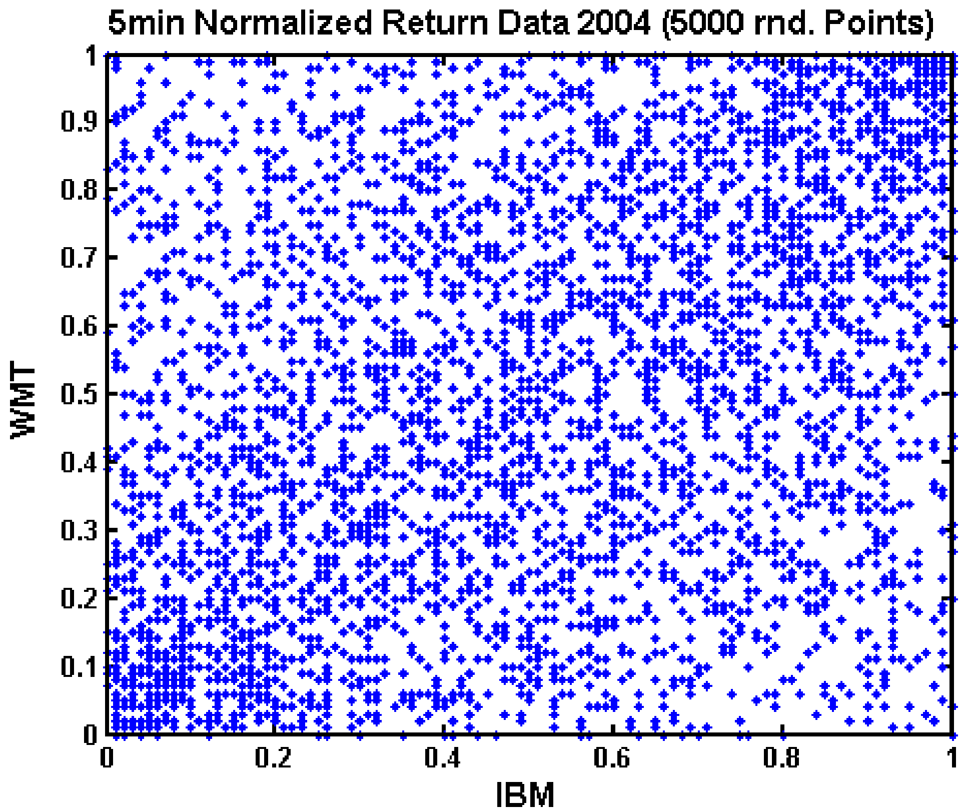
Motivation

Jumps in Financial Data



Motivation

Dependence of Extreme Events



Tail Dependence

Property of Probabilistic Copula

Tail dependence coefficient

$$\lambda_L = \lim_{u \to 0+} P(X < F_X^{-1}(u) | Y < F_Y^{-1}(u)) = \lim_{u \to 0+} \frac{C(u,u)}{u}$$

measure of dependence of extreme events
(Embrechts et al. (2003), Malevergne/Sornette (2004))

Question: Can we define a similar measure, describing the dependence of extreme jump events in Lévy processes?

- we define the Jump Tail Dependence coefficient
- we discuss the estimation of this Lévy Copula property

Agenda

Jump Tail Dependence in Lévy Copulas

Lévy Copula Models

Distribution of Large Jumps

Jump Tail Dependence

How to Estimate?

Estimating Jump Tail Dependence

Lévy Copula Models

Dependence in Multivariate Lévy Processes

Lévy Measure v(A): numbers of exp. jumps with size in A.

Tail Integral U(x): numbers of exp. jumps larger/smaller x.

The Lévy measure is uniquely determined by its tail integral and all marginal tail integrals U^I.

Lévy Copula: function F such that

$$U^I((x_i)_{i\in I}) = F^I((U_i(x_i))_{i\in I}) \qquad I \subset \{1,\dots,d\}$$

F is not normalized, no interpretation as distribution function.

Example: Clayton Lévy Copula

$$F(u,v) = (|u|^{-\theta} + |v|^{-\theta})^{-1/\theta}(\eta 1_{\{uv\geq 0\}} - (1-\eta)1_{\{uv<0\}})$$

Lévy Copula Models II

Series Representation

The series

$$X_{\tau,t} = \sum_{-\tau \leq \Gamma_i \leq \tau} U^{(-1)}(\Gamma_i) 1_{V_i \leq t} - tA(\tau)$$

converges for adequate A for $\tau \to \infty$o a Lévy process with triplet (0,v,0); Rosinsky (2001), Tankov (2005).

In the multivariate case:

- use marginal tail integrals
- model dependence between the Γ_i by Lévy Copulas

Distribution of Large Jumps

Dependence in Multivariate Lévy Processes

Probability that largest jump J_1 is larger than $J^* = U^{(-1)}(u^*)$

Series representation: Probability of observing jumps larger than J* is probability of having $\Gamma_i < u^*$.

The Γ_i are a sequence of standard Poisson variables.
We can therefore derive (u* small):

$$P(J_1 \geq J^*) = P(\Gamma_1 \leq u^*) = (1 - P(\Gamma_1 > u^*))$$
$$= 1 - \exp(-u^*) \approx u^*$$

Distribution of Large Jumps

Dependence in Bivariate Lévy Processes

With the help of the conditional probability

$$P(u_2 \leq u_2^* | u_1 = x) = F_x(u_2^*) = \frac{\partial}{\partial x} F(x, u_2^*)$$

we derive

$$\begin{aligned} P(J_2 \geq J_2^* | J_1 \geq J_1^*) &= P(J_2 \geq U^{(-1)}(u_2^*) | J_1 \geq U^{(-1)}(u_1^*)) \\ &= P(u_2 \leq u_2^* | u_1 \leq u_1^*) \\ &= \frac{\int_0^{u_1^*} P(u_2 \leq u_2^* | u_1 = x) P(u_1 = x) dx}{\int_0^{u_1^*} P(u_1 = x) dx} \\ &= \frac{\int_0^{u_1^*} \frac{\partial}{\partial x} F(x, u_2^*) P(u_1 = x) dx}{\int_0^{u_1^*} P(u_1 = x) dx} \end{aligned}$$

$$= \frac{\int_0^{u_1^*} \frac{\partial}{\partial x} F(x, u_2^*) P(u_1 = x) dx}{\int_0^{u_1^*} P(u_1 = x) dx}$$

$$= \dots$$

$$\approx \frac{F(u_1^*, u_2^*)}{u_1^*}$$

where the approximations are valid for small values of u_1^* und u_2^* ($o(u^{*2})$).

Derived probability is a function of the Lévy Copula only!

Jump Tail Dependence

Dependence of Extreme Jumps

With these results, we define:

$$\lambda_{PP} := \lim_{u^* \to 0+} P(u_1 \leq u^* | u_2 \leq u^*) = \lim_{u \to 0+} \frac{F(u,u)}{u}$$

the (pos.-pos.) jump tail dependence (JTD) coefficient
(in the same sense: neg.-neg., neg.-pos., pos.-neg.)

Jump tail dependence is

- a (bivariate) measure of jump dependence,
- a property of the Lévy copula only,
- a probability.

How to Estimate JTD?

Relationship to Tail Dependence

Use asymptotic relationship

$$F(u_1,\ldots,u_d) = \lim_{t\to 0}\frac{1}{t}C_t^{(\operatorname{sgn} u_1,\ldots,\operatorname{sgn} u_d)}(t|u_1|,\ldots,t|u_d|)\prod_{i=1}^{d}\operatorname{sgn} u_i$$

(Kallsen/Tankov (2006); Bäuerle et al. (2007)).

Thus, we can derive:

$$\begin{aligned}\lambda_{PP} = \lim_{u\to 0}\left[\frac{F(u,u)}{u}\right] &= \lim_{u\to 0}\left[\frac{1}{u}\lim_{t\to 0}\frac{1}{t}\overline{C_t}(tu,tu)\right]\\ &= \lim_{t\to 0}\left[\lim_{v\to 0}\frac{\overline{C_t}(v,v)}{v}\right]\\ &= \lim_{t\to 0}\lambda_U^{(C_t)}.\end{aligned}$$

Idea: Estimate JTD with estimators of TD on high frequencies.

Estimating Tail Dependence

Nonparametric Estimator

With $\overline{C}_n$ the empirical survival copula

$$\hat{\lambda}_U = \frac{n}{k}\overline{C}_n\left(\frac{k}{n}, \frac{k}{n}\right) = \frac{1}{k}\sum_{j=1}^{n} \mathbf{1}_{\{R_{n1}^{(j)} > n-k \text{ and } R_{n2}^{(j)} > n-k\}}$$

is an estimator for the upper tail dependence coefficient.

Schmidt/Stadtmüller (2006) show weak convergence, asymptotic normality and strong consistency.

Variance can be estimated by means of nonparametric bootstrap; Schmid/Schmidt (2007).

Scenario Analysis

Simulation of Lévy Copula Portfolios

Two portfolios of Lévy processes:

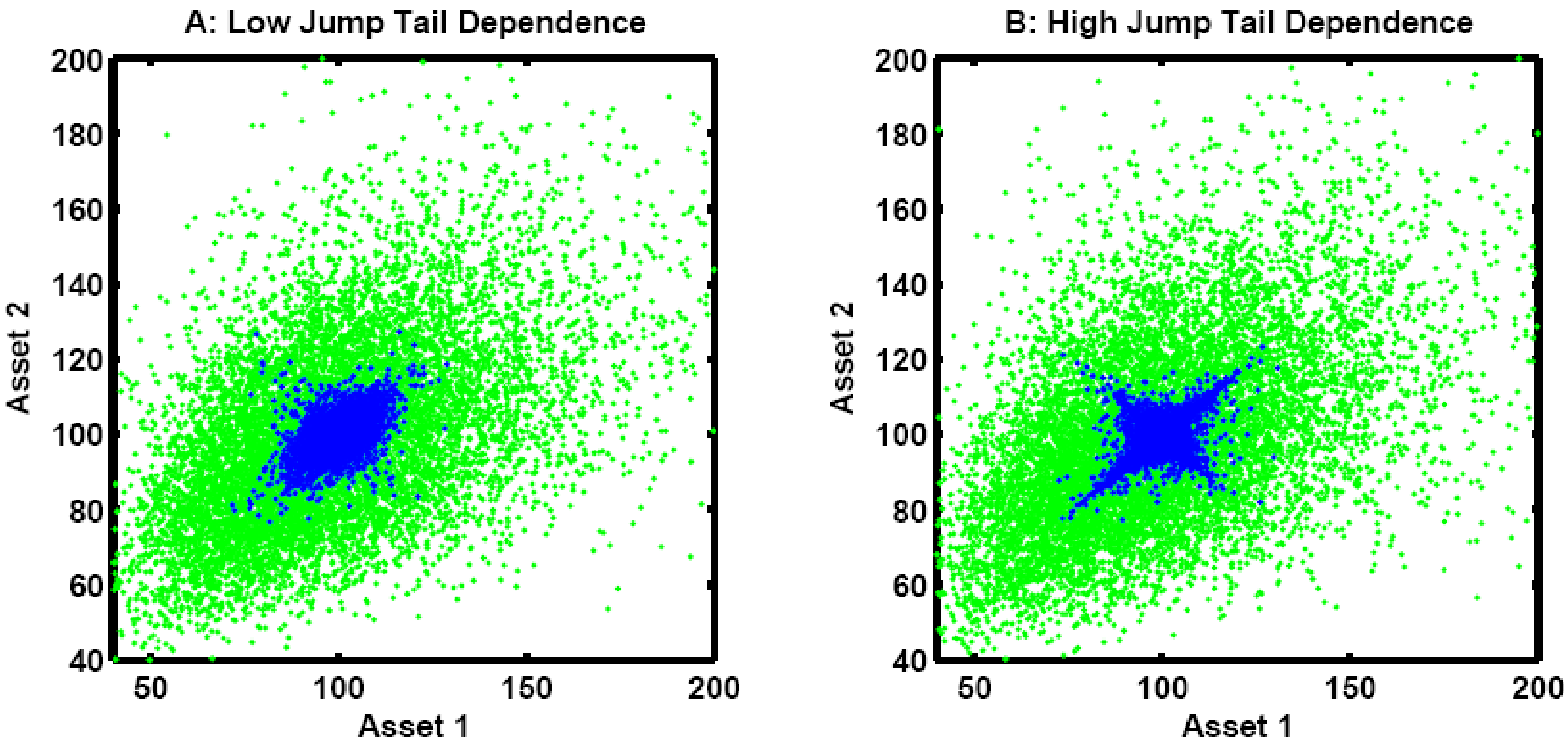
$$100\exp(r_1 t + X_t^{(1)}) \qquad\qquad 100\exp(r_2 t + X_t^{(2)})$$

- Variance-Gamma Lévy process; Madan/Senata (1990)
- Dependence modelled by Clayton Lévy Copula $F_{\theta\eta}$
- Yearly Volatility of 30% (1) and 25% (2)
- Two Scenarios, high JTD and low JTD

	θ	η	$\lambda_{PP} = \lambda_{NN}$	$\lambda_{NP} = \lambda_{PN}$	$\hat{\rho}$	$\widehat{\text{stdev}}(\hat{\rho})$
Scenario A	0.900	0.841	0.389	0.074	0.501	0.010
Scenario B	10	0.750	0.700	0.233	0.502	0.011

Scenario Analysis

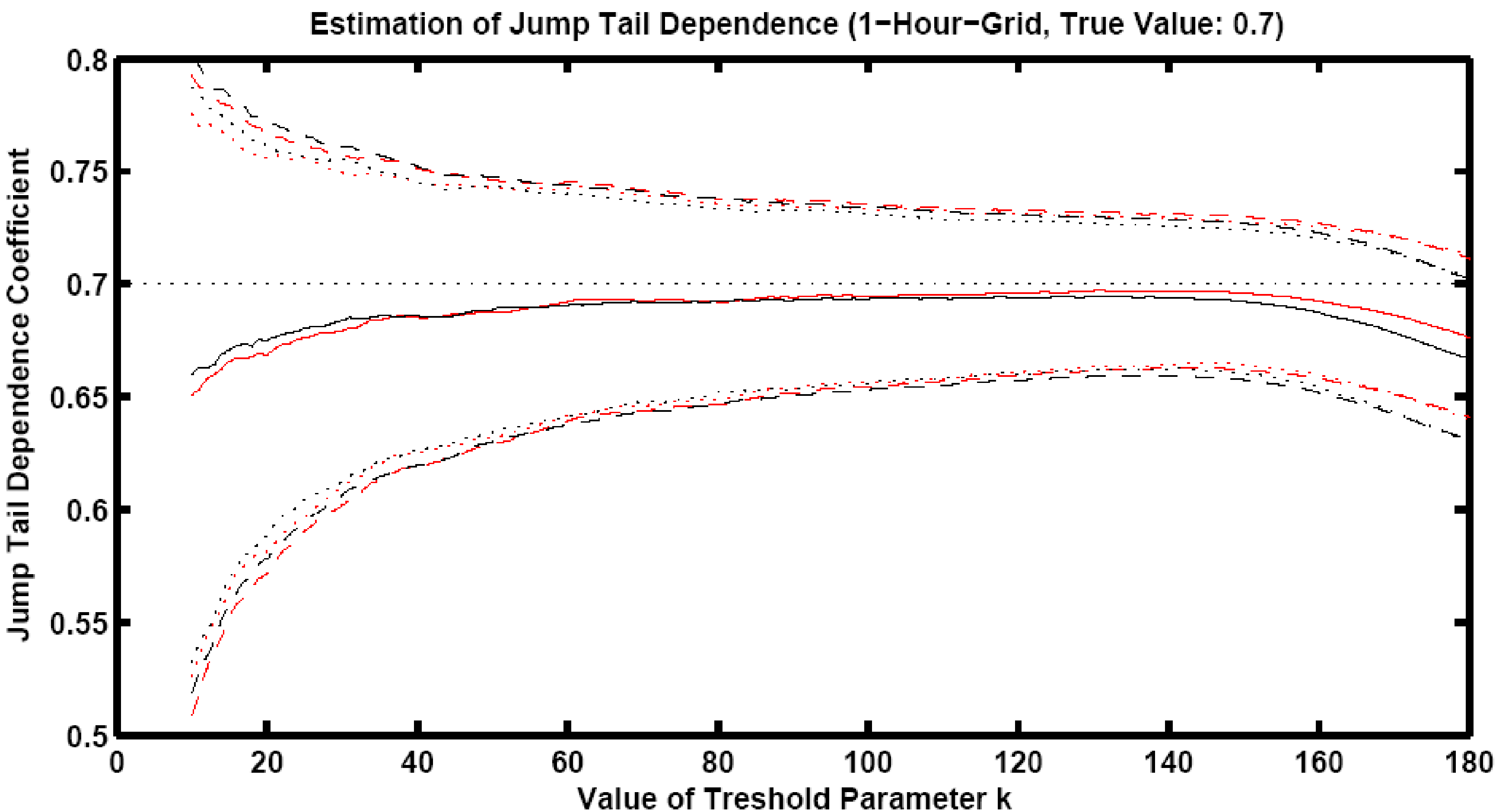
Effect of JTD on Portfolio Distribution



one week horizon, **one year horizon**

Scenario Analysis

Estimating Jump Tail Dependence



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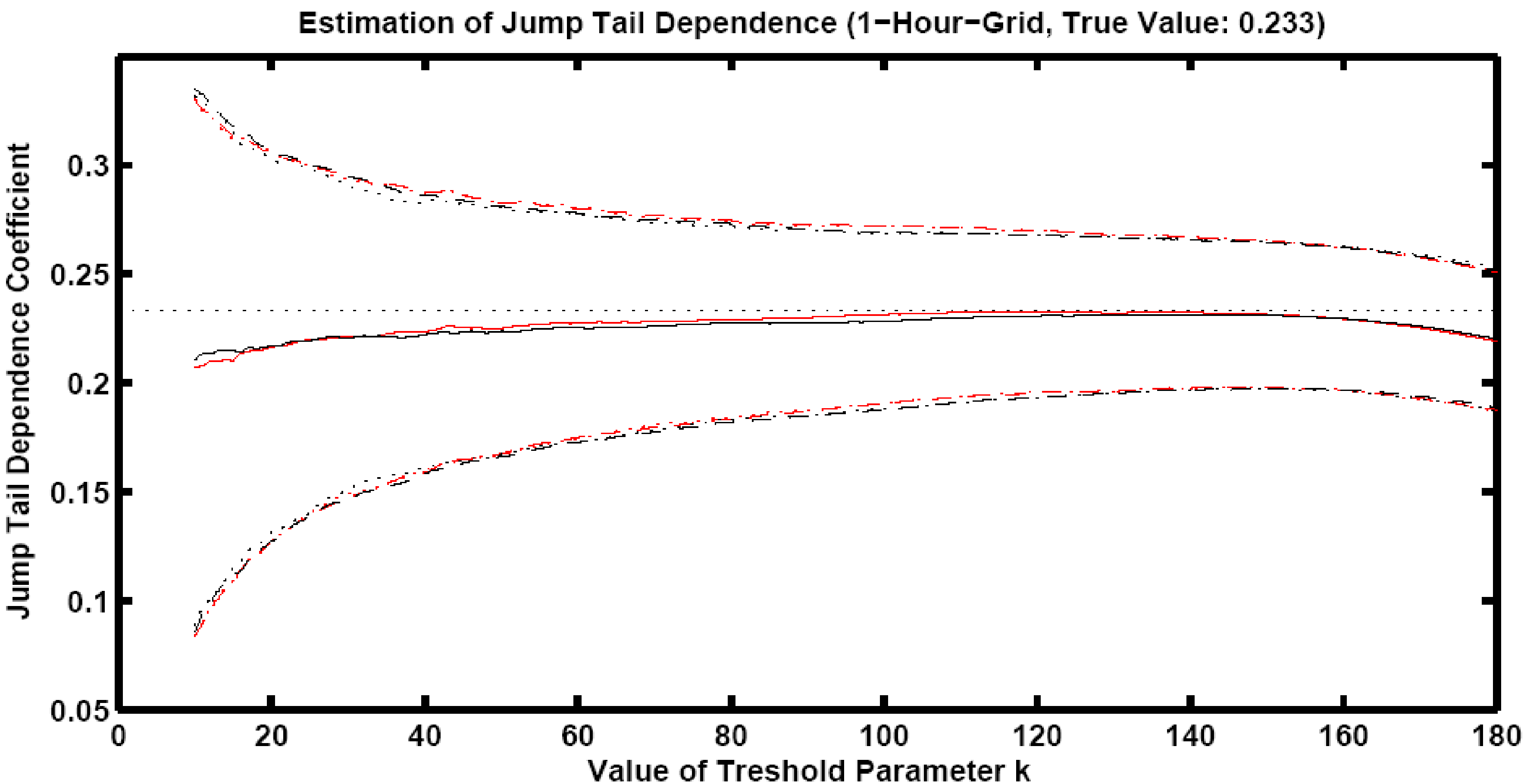
Scenario Analysis

Estimating Jump Tail Dependence



Summary

Estimating Jump Tail Dependence

- Jump Tail Dependence
  - probability of large jump (2) given large jump (1)
  - property of the Lévy copula only
- Nonparametric estimation on high frequencies
  - weak convergence
  - asymptotic normality and strong consistency
- Bootstrap applicable