Continuous-time random walks, fractional calculus and stochastic integrals

A model for high-frequency financial time series

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Outline

1. Motivation
   - Condensed matter
   - Finance

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   - Uncoupled continuous-time random walk (CTRW)
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   - Definition of the stochastic integral
   - Monte Carlo simulation

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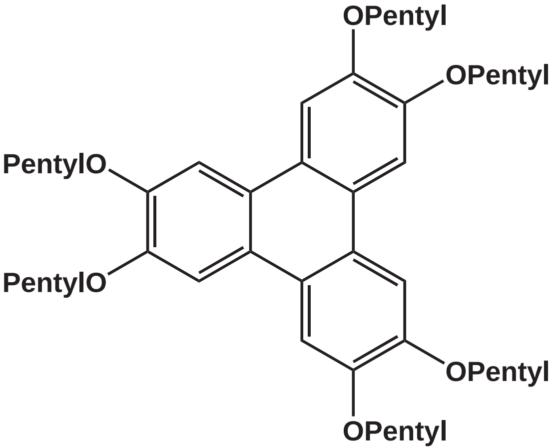
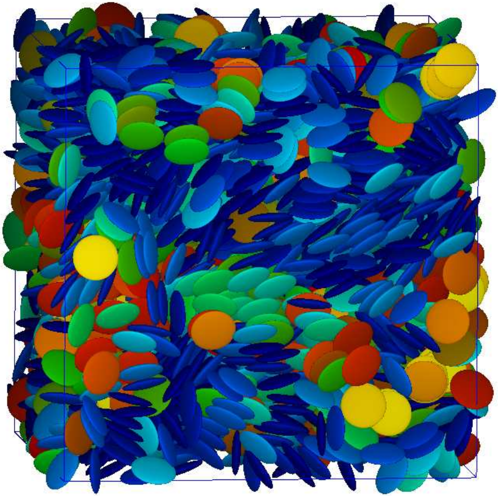
   - Monte Carlo simulation
4. Outlook
   - Statistical inference
   - Autoregressive processes (GARCH-ACD)

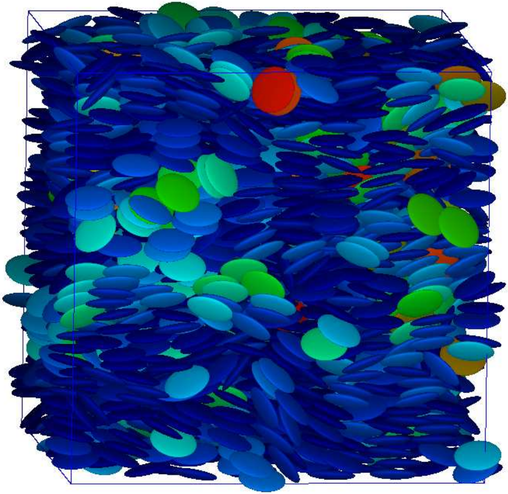
Motivation 1: Diffusion in complex liquids, e.g. liquid crystals (theory of soft condensed matter)



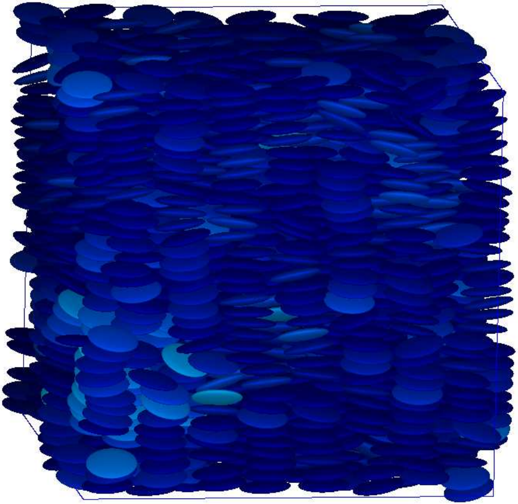
Hexakis(pentyloxy)triphenylene, a platelike molecule.



Isotropic phase $P^* = 200$ and $T^* = 13$.



Nematic phase at $P^* = 200$ and $T^* = 12$.



Columnar phase at $P^* = 200$ and $T^* = 11$.

Determination of the diffusion constant

Diffusion equation (Fick's 2nd law)

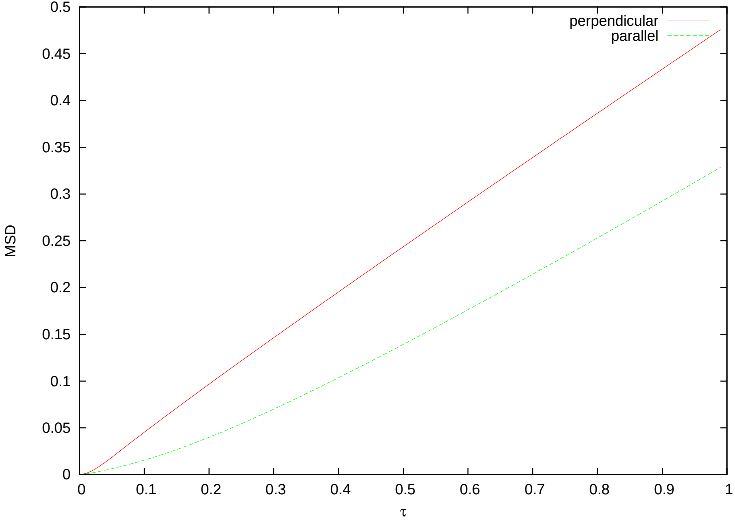
$$\frac{\partial \rho(\mathbf{r},t)}{\partial t} = D \frac{\partial^2 \rho(\mathbf{r},t)}{\partial \mathbf{r}^2}$$

Slope of mean square displacement vs. time (Einstein relation)

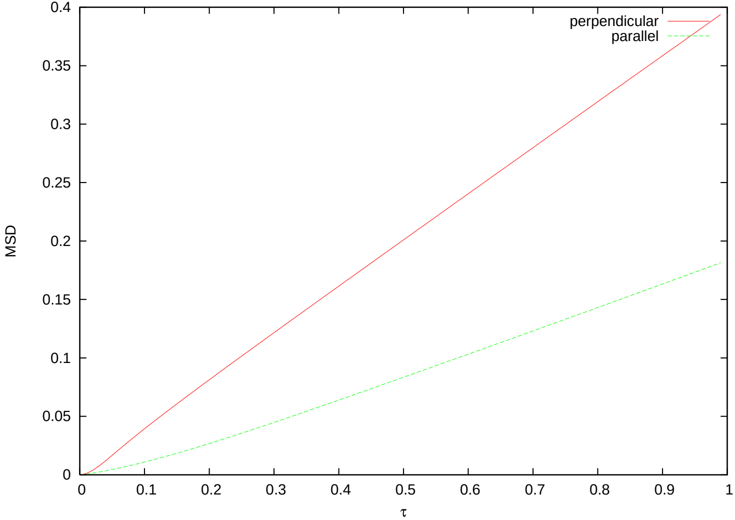
$$D = \lim_{\tau\to\infty} \frac{1}{6N\tau} \sum_{i=1}^{N} \langle |\mathbf{r}_i(t+\tau) - \mathbf{r}_i(t)|^2 \rangle_t$$

Integral of the velocity autocorrelation (Green-Kubo relation)

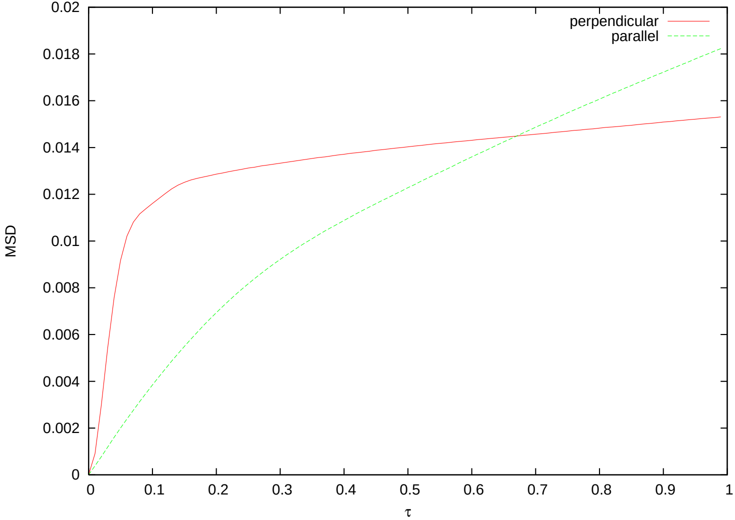
$$D = \frac{1}{3N} \sum_{i=1}^{N} \int_0^\infty \langle \mathbf{v}_i(t+\tau) \cdot \mathbf{v}_i(t) \rangle_t \, \mathrm{d}\tau$$



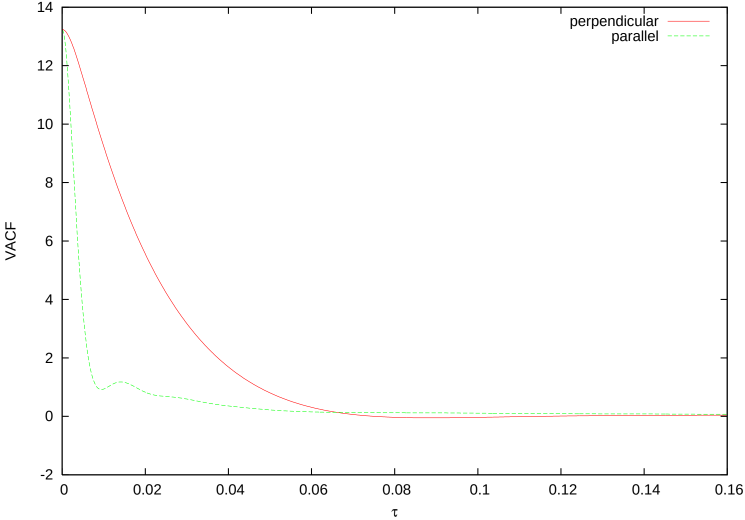
Mean square displ. in the isotropic phase at $P^* = 200, \ T^* = 13$.



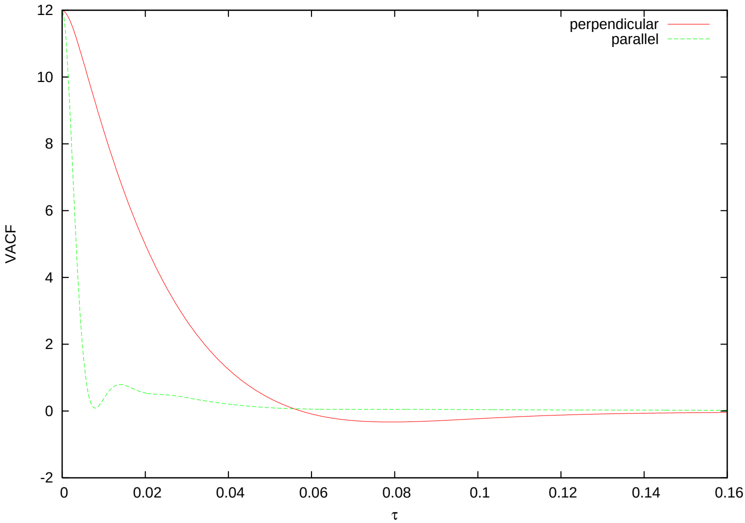
Mean square displ. in the nematic phase at $P^* = 200,\ T^* = 12$.



Mean square displ. in the columnar phase at $P^* = 150,\ T^* = 10$.



Velocity autocorr. in the isotropic phase at $P^* = 200,\ T^* = 13$.



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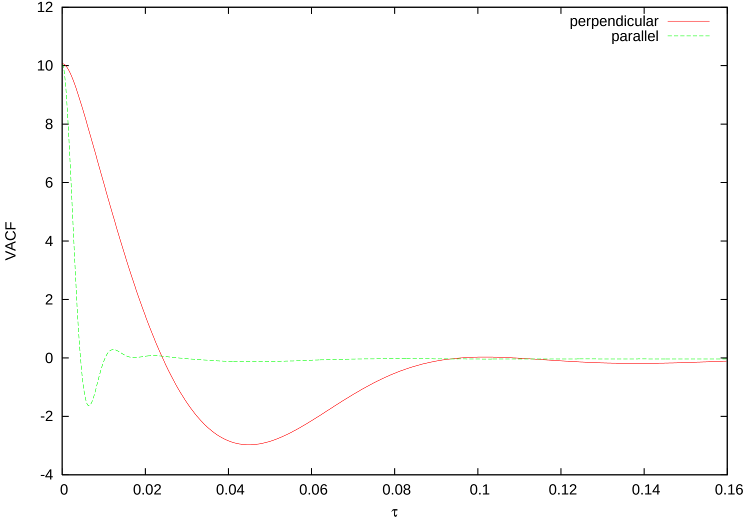

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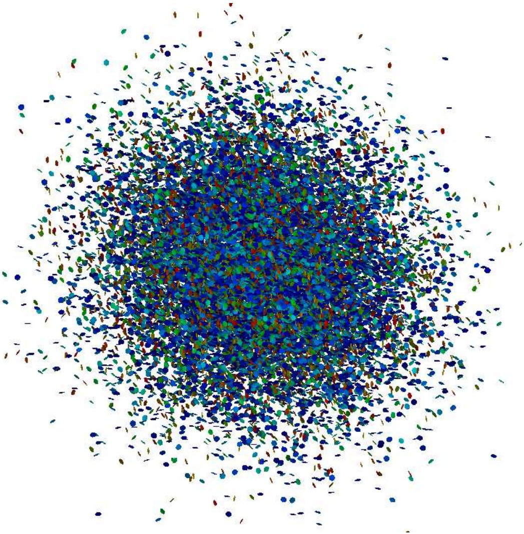
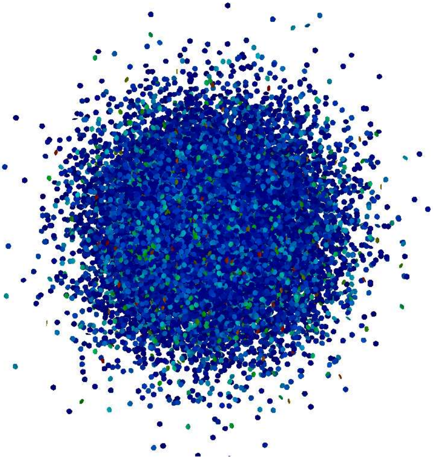
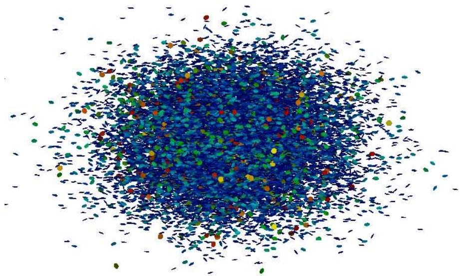


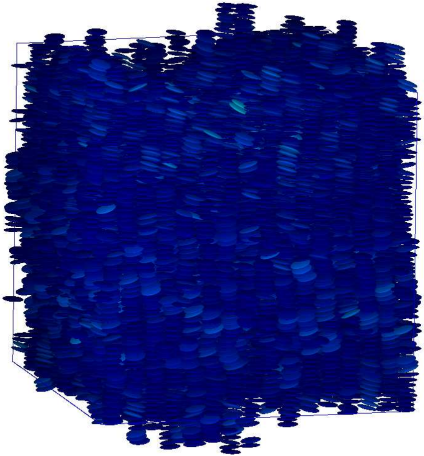
Diffusion in the isotropic phase at $P^* = 150$ and $T* = 11$.



Diffusion in the nematic phase at $P^* = 200$ and $T^* = 12$ (top).



Diffusion in the nematic phase at $P^* = 200$ and $T^* = 12$ (side).



Diffusion in the columnar phase at $P^* = 150$ and $T^* = 9$ (side).

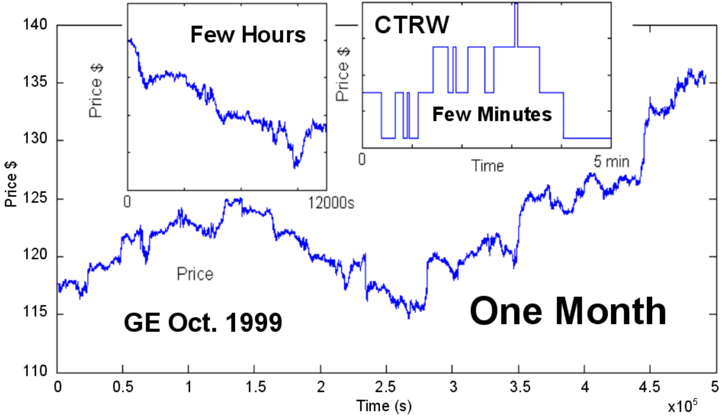
Motivation 2: Diffusion in finance

In modern finance theory, stock prices $S(t)$ are modelled customarily with geometric Brownian motion ($W(t)$ is the Wiener process):

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t).$$

This has many convenient mathematical properties, but is not very realistic, as has been pointed out already a long time ago: B. Mandelbrot, "The variation of certain speculative prices", *Journal of Business* **36**, 394–419 (1963).

Search for realistic high-frequency stock price processes beyond geometric Brownian motion



Continuous-time random walks

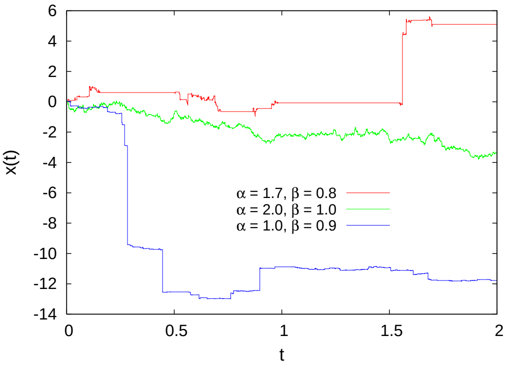
A CTRW is a pure jump process; it consists of a sequence of independent identically distributed (IID) random jumps (events) ξ_i separated by IID random waiting times τ_i, $i = 1, \ldots, n$, with $i, n \in \mathbb{N}$,

$$t_n = \sum_{i=1}^{n} \tau_i, \quad \tau_i = t_i - t_{i-1}, \quad \tau_i \in \mathbb{R}_+,$$

so that the position $X(t)$ of the random walker at time $t \in [t_n, t_{n+1})$ is

$$X(t) \overset{\text{def}}{=} S_{N(t)} \overset{\text{def}}{=} \sum_{i=1}^{N(t)} \xi_i, \quad \xi_i \in \mathbb{R}^d.$$

Sample paths of continuous-time random walks



The scale parameters are linked by $\gamma_x^\alpha = \gamma_t^\beta$ with $\gamma_t = 0.001$. The jumps become larger with smaller α and larger γ_x; the waiting times become longer with smaller β and larger γ_t.

Lévy property of continuous-time random walks

- The assumption that jumps and waiting times are IID means that the joint probability density function (PDF) of any pair of jumps and waiting times, $\varphi(\xi_i, \tau_i)$, does not depend on i.

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- The assumption that jumps and waiting times are IID means that the joint probability density function (PDF) of any pair of jumps and waiting times, $\varphi(\xi_i, \tau_i)$, does not depend on i.
- Because its increments are independent and time-homogeneous (stationary), a CTRW is a **Lévy process**.

Markov or semi-Markov property of uncoupled CTRWs

- A CTRW is called uncoupled if the joint PDF $\lambda(\xi, \tau)$ factorizes into marginal PDFs for jumps $\lambda(\xi)$ and waiting times $\psi(\tau)$, i.e., $\lambda(\xi, \tau) = \lambda(\xi)\psi(\tau)$.

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- An uncoupled CTRW is **Markovian** if and only if the waiting time distribution is exponential, i.e., $\psi(\tau) = \exp(-\tau/\gamma_t)/\gamma_t$.

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- An uncoupled CTRW belongs to the class of **semi-Markov processes**, i.e., for any $A \subset \mathbb{R}^d$ and $t > 0$ we have

$$\mathbb{P}(S_n \in A, \tau_n \le t \,|\, S_0, \dots, S_{n-1}, \tau_1, \dots, \tau_{n-1}) \\ = \mathbb{P}(S_n \in A, \tau_n \le t \,|\, S_{n-1}).$$

If we fix the position $S_{n-1} = y$ of the random walker at time t_{n-1}, the probability on the right will be independent of n.

Montroll-Weiss equation

In the generic coupled case, where the law of (ξ_i, τ_i) is given by a joint PDF $\varphi(\xi, \tau)$, we can rewrite $S_n = S_{n-1} + \xi_n$ as

$$\mathbb{P}(S_n \in A, \tau_n \leq t \,|\, S_{n-1}) = \int_A \int_0^t \varphi(x - S_{n-1}, \tau)\, d\tau dx.$$

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Montroll and Weiss (1965) wrote this as an integral equation for the PDF $p_X(x,t)$ of finding the random walker in position x at time t in terms of the joint PDF $\varphi(\xi, \tau)$,

$$p_X(x,t) = \delta(x)\Psi(t) + \int_{\mathbb{R}^d} \int_0^t \varphi(\xi, \tau) p_X(x - \xi, t - \tau)\, d\tau d\xi,$$

where $\Psi(t) = 1 - \int_0^t \psi(\tau)\, d\tau$ is the complementary cumulative distribution function for the waiting times, also called survival function. This equation can be solved in the Fourier-Laplace domain, but the inverse transforms are possible only in the uncoupled case, and yield a series.

Choice of waiting-time and jump marginal densities

The marginal jump PDF is a symmetric **Lévy α-stable** function with order $\alpha \in (0, 2]$ and scale parameter $\gamma_x \in \mathbb{R}_+$:

$$\lambda(\xi) = L_\alpha(\xi).$$

The marginal waiting-time PDF is the derivative of a **Mittag-Leffler** function with order $\beta \in (0, 1]$ and scale parameter $\gamma_t \in \mathbb{R}_+$:

$$\psi(\tau) = -\frac{d}{d\tau}\Psi(\tau) = -\frac{d}{d\tau}E_\beta(-(\tau/\gamma_t)^\beta)$$

A motivation is the behaviour in the **diffusive limit**

$$\gamma_x \to 0, \ \gamma_t \to 0 \quad \text{with} \quad \gamma_x^\alpha/\gamma_t^\beta = D.$$

Standard diffusion equation

The well-known solution of the Cauchy problem

$$\begin{aligned} \frac{\partial}{\partial t} u_X(x,t) &= D \frac{\partial^2}{\partial x^2} u_X(x,t) \\ u_X(x,0^+) &= \delta(x), \quad x \in \mathbb{R}, \quad t \in \mathbb{R}_+, \end{aligned}$$

is the one-point PDF of the Wiener process $X(t) = W(t)$,

$$u_W(x,t) = \frac{1}{\sqrt{4\pi D t}} e^{-x^2/(4Dt)},$$

i.e. a normal distribution $N(\mu, \sigma^2)$ with $\mu = 0$ and $\sigma^2 = 2Dt$.

Properties of diffusion processes

Let $u(x,t)$ be the solution of a second order parabolic partial differential equation. Its properties are:

1. Conservation of the total quantity:
   $\int_{-\infty}^{+\infty} u(x,t)\,dx = \int_{-\infty}^{+\infty} u(x,0^+)\,dx,\ \forall t \in \mathbb{R}_+$.

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   $u(x,0^+) \geq 0, \forall x \in \mathbb{R} \Rightarrow u(x,t) \geq 0, \forall x \in \mathbb{R},\ \forall t \in \mathbb{R}_+$.
3. Spreading law for $t \to \infty$:
   $\sigma^2(t) = \int_{-\infty}^{+\infty} x^2 u(x,t)\,dx \sim 2Dt$,
   or more generally, if there is a drift $\mu(t) = \int_{-\infty}^{+\infty} x u(x,t)\,dx$,
   $\sigma^2(t) = \int_{-\infty}^{+\infty} x^2[u(x,t) - \mu(x,t)]\,dx \sim 2Dt$.

Anomalous vs. standard diffusion

- Some processes have only properties 1 and 2;
  their variance does not exhibit linear growth for $t \to \infty$.
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Anomalous vs. standard diffusion

- Some processes have only properties 1 and 2; their variance does not exhibit linear growth for $t \to \infty$. This is called anomalous diffusion.
- In *sub-diffusion*, the variance grows more slowly than linearly.
- In *super-diffusion*, the variance grows faster than linearly, or is infinite.
- Classes of sub- and super-diffusive processes can be described by fractional diffusion equations, that generalize the standard diffusion equation solved by the one-point PDF of the Wiener process.

Space-time fractional diffusion equation

The standard diffusion equation can be generalized to

$$\begin{aligned} \frac{\partial^\beta}{\partial t^\beta} u_X(x,t) &= D \frac{\partial^\alpha}{\partial |x|^\alpha} u_X(x,t) \\ u_X(x,0^+) &= \delta(x), \quad x \in \mathbb{R}, \quad t \in \mathbb{R}_+. \end{aligned}$$

Riesz space-fractional derivative of order $\alpha \in (0,2]$:

$$\frac{d^\alpha}{d|x|^\alpha} f(x) = \mathcal{F}_k^{-1}[-|k|^\alpha \widehat{f}(k)](x).$$

Caputo time-fractional derivative of order $\beta \in (0,1]$:

$$\frac{d^\beta}{dt^\beta} f(t) = \mathcal{L}_s^{-1}[s^\beta \widetilde{f}(s) - s^{\beta-1} f(0^+)](t).$$

Symmetric Lévy α-stable distribution

The Lévy α-stable function is a generalization of a Gaussian, the latter being a special case for $\alpha = 2$, and is best defined as the inverse Fourier (or cosine) transform of its characteristic function $\exp(-|\gamma_x k|^\alpha)$:

$$L_\alpha(\xi) = \mathcal{F}_k^{-1}\left(e^{-|\gamma_x k|^\alpha}\right)(\xi) = \frac{1}{\pi}\int_0^\infty e^{-(\gamma_x k)^\alpha}\cos(\xi k)\,dk.$$

However, there are series expressions for the Lévy function too:

$$L_\alpha(\xi) = -\frac{1}{\pi\xi}\sum_{n=1}^\infty \frac{\Gamma(n/\alpha+1)}{n!}\sin\left(\frac{n\pi}{2}\right)(-\xi)^n, \qquad \alpha \in (1,2]$$

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One-parameter Mittag-Leffler function

$$E_\beta(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta n + 1)}$$

with

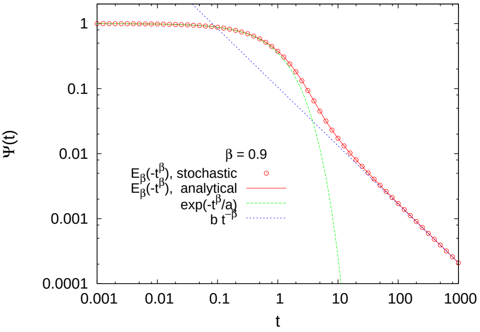
$$E_\beta(-Dt^\beta) = \mathcal{L}_s^{-1}\left[\frac{s^{\beta-1}}{D + s^\beta}\right](t), \quad t \in \mathbb{R}_+.$$

For $\beta = 1$ the Mittag-Leffler function is a standard exponential:

$$E_1(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n+1)} = \sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z.$$

Other special cases:

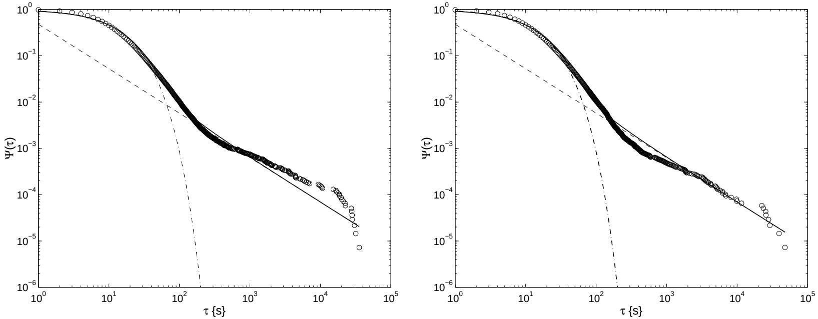
$$E_{1/2}(z) = \exp(z^2)\mathrm{erfc}(-z), \quad E_0(z) = (1-z)^{-1}, \quad E_2(z) = \cosh(\sqrt{z}).$$



The one-parameter Mittag-Leffler function is halfway between a stretched exponential (Weibull function) and a power law with index β:

$$E_\beta\left(-t^\beta\right) \sim \begin{cases} \exp\left(-t^\beta/\Gamma(1+\beta)\right) & \text{for } t \to 0^+ \\ t^{-\beta}/\Gamma(1-\beta) & \text{for } t \to \infty \end{cases} .$$

Empirical evidence of ML waiting times in finance



Survival functions for BTP futures traded at LIFFE with delivery date June (left) and September (right) 1997; in both cases $\beta = 0.96$, $\gamma_t = 13\,\mathrm{s}$. From M. Raberto, E. Scalas, R. Gorenflo, F. Mainardi, “The waiting time distribution of LIFFE bond futures”, APFA2, Liège, 13–15/07/2000, `arXiv:cond-mat/0012497`; see also same authors, *Physica A* **287**, 468 (2000).

Lévy α-stable PDF $L_\alpha(\xi/\gamma_x)$

Chambers, Mallows, Stuck, *J. Am. Stat. Assoc.* **71**, 340 (1976):

$$\xi = \gamma_x \left(\frac{-\log u \cos\phi}{\cos((1-\alpha)\phi)} \right)^{1-1/\alpha} \frac{\sin(\alpha\phi)}{\cos\phi}, \quad \phi = \pi \left(v - \frac{1}{2} \right).$$

For $\alpha = 2$ this gives Box-Muller: $\xi = 2\gamma_x \sqrt{-\log u} \sin\phi$.

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Mittag-Leffler PDF $-dE_\beta\big(-(\tau/\gamma_t)^\beta\big)/d\tau$

Kozubowski, Rachev, *J. Comput. Anal. Appl.* **1**, 177 (1999):

$$\tau = -\gamma_t \log u \left(\frac{\sin(\beta\pi)}{\tan(\beta\pi v)} - \cos(\beta\pi) \right)^{1/\beta}.$$

For $\beta = 1$ this gives the exponential distribution: $\tau = -\gamma_t \log u$.

Solution of the space-time fractional diffusion equation

In the Fourier-Laplace domain

$$\widehat{\widetilde{u}}_X(k,s) = \frac{s^{\beta-1}}{D|k|^\alpha + s^\beta}.$$

Because

$$\mathcal{L}_s^{-1}\left[\frac{s^{\beta-1}}{D|k|^\alpha + s^\beta}\right](t) = E_\beta(-D|k|^\alpha t^\beta)$$

in the space-time domain

$$u_X(x,t) = t^{-\beta/\alpha} G_{\alpha,\beta}(xt^{-\beta/\alpha}),$$

with the time-independent Green function ($\kappa = kt^{\beta/\alpha}$)

$$G_{\alpha,\beta}(\xi) = \mathcal{F}_\kappa^{-1}\left[E_\beta(-D|\kappa|^\alpha)\right](\xi).$$

where $\alpha \in (0,2]$ and $\beta \in (0,1]$.

Monte Carlo approximation of the Green function

- A stochastic solution of the FDE can be obtained from the *diffusive limit* of a properly scaled CTRW with a symmetric Lévy α-stable distribution of jumps and a one-parameter Mittag-Leffler distribution of waiting times.

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- In the diffusive limit γ_x and γ_t vanish with $\gamma_x^\alpha / \gamma_t^\beta = D$; the histogram of the PDF $p_X(x, t; \alpha, \beta, \gamma_x, \gamma_t)$ of finding the CTRW X in position x at time t converges weakly to the Green function of the FDE $u_X(x, t; \alpha, \beta)$, weakly because a singularity at $x = 0$ is always present in $p_X(x, t; \alpha, \beta, \gamma_x, \gamma_t)$ for any finite value of γ_x and γ_t.

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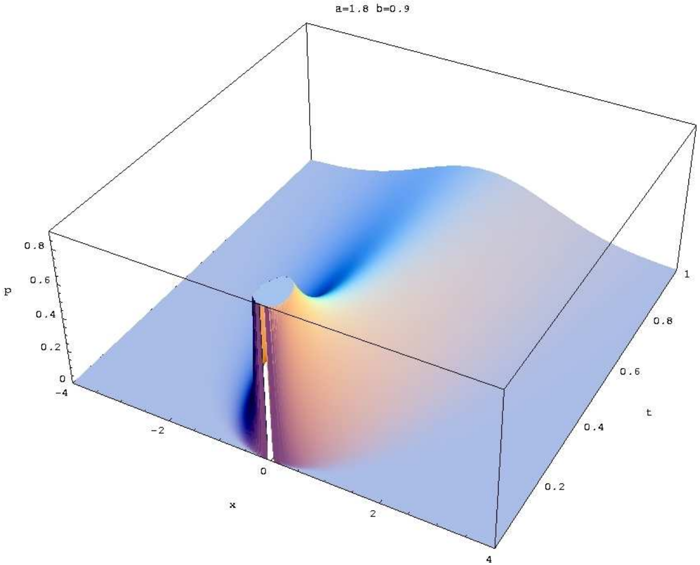
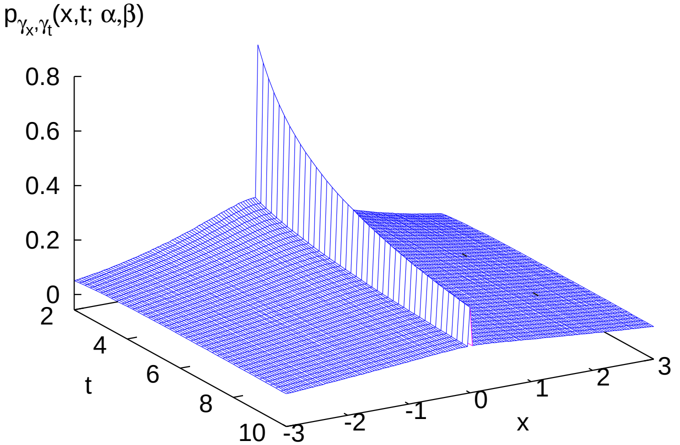
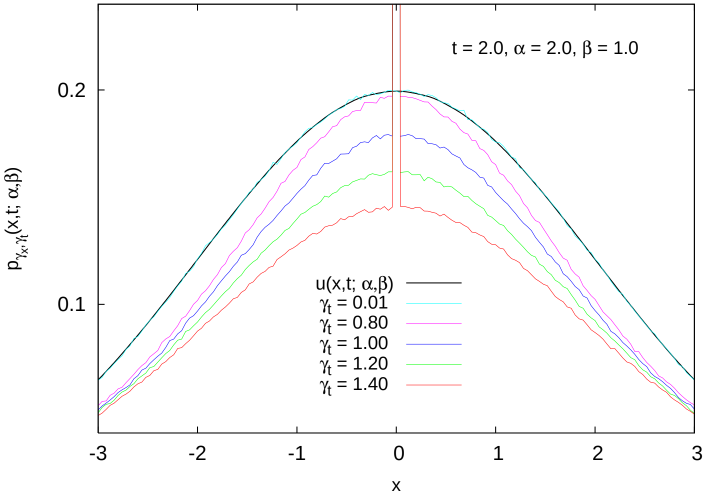
- For $\alpha = 2$ and $\beta = 1$, one recovers the Green function $u_W(x, t)$ of the standard diffusion equation, i.e. the one-point PDF of the Wiener process.



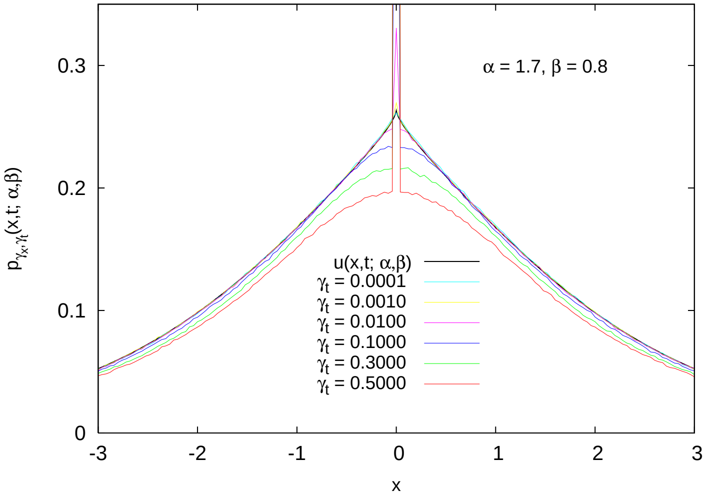
FDE solution $u_X(x,t)$ for $\alpha = 1.8,\ \beta = 0.9$.



Pdf $p_X(x, t; \alpha, \beta, \gamma_x, \gamma_t)$ with $\alpha = 1.7$, $\beta = 0.8$, $\gamma_t = 0.1$, $\gamma_x = \gamma_t^{\beta/\alpha}$. The crest at $x = 0$ is the survival function $\Psi(t) = E_\beta\left(-(t/\gamma_t)^\beta\right)$.



Convergence of $p_{\gamma_x,\gamma_t}(x,t;\alpha,\beta)$ to $u_X(x,t,\alpha,\beta)$ at $t=2.0$.



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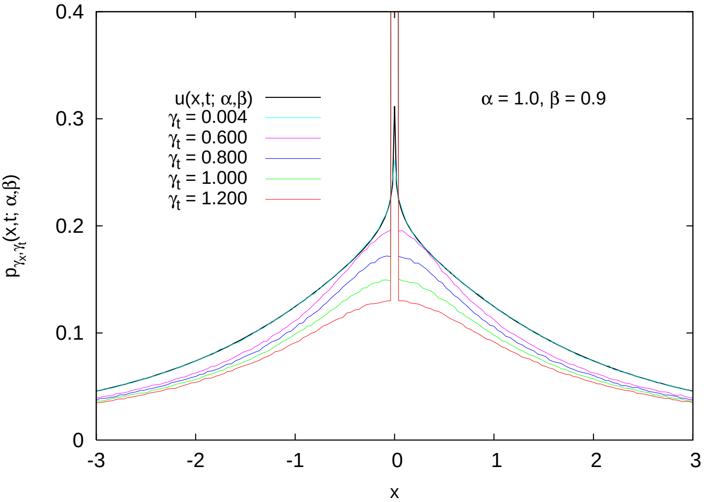
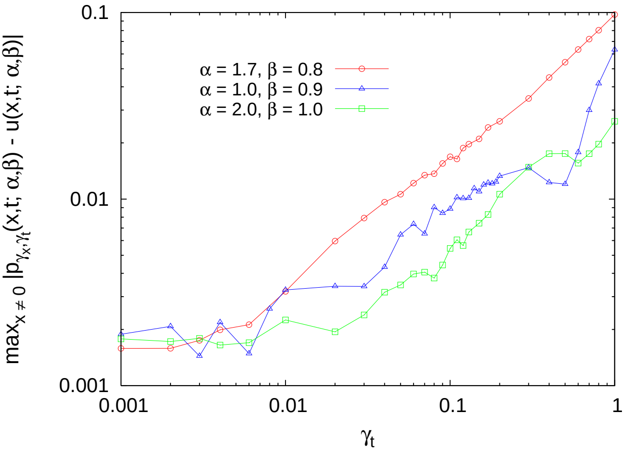


Convergence of $p_{\gamma_x,\gamma_t}(x, t; \alpha, \beta)$ to $u_X(x, t, \alpha, \beta)$ at $t = 2.0$.



Convergence of $\max_{x\neq 0} |p_{\gamma_x,\gamma_t}(x,t;\ \alpha,\beta) - u_X(x,t;\ \alpha,\beta)|$ for selected values of α and β when $\gamma_x,\ \gamma_t \to 0$ with $\gamma_x^\alpha = \gamma_t^\beta$.

CPU time for 100 million samples

	Pentium	Athlon	Opteron	Power4+
Gaussian	16	12	11	19
Lévy	73	66	52	95
Exponential	16	11	12	20
Mittag-Leffler	52	44	36	72

CPU time in seconds needed to generate 10^8 pseudorandom numbers with different probability distributions on different architectures: an Intel Pentium IV operating at 2.4 GHz, an AMD Athlon 64 X2 “Toledo” Dual-Core at 2.2 GHz, an AMD Opteron 270 at 2.0 GHz, and an IBM Power4+ at 1.7 GHz. On the first three architectures we used the Intel C++ compiler with the -O3 optimisation option; on the fourth, we used the IBM xlC compiler with the -O5 option.

CPU times for 10 million Monte Carlo runs

α	β	γ_t	$\bar{n}$	t_{CPU}/sec
2.0	1.0	0.010	200	337
2.0	1.0	0.001	2000	3362
1.7	0.8	0.010	74	437
1.7	0.8	0.001	470	2895

Average number $\bar{n}$ of jumps per run and total CPU time t_{CPU} in seconds for 10^7 runs with $t \in [0, 2]$ on an AMD Athlon 64 X2 Dual-Core at 2.2 GHz using the Intel C++ compiler and the -O3 -static optimization options.

References

- D. Fulger, E. Scalas, G. Germano, “Monte Carlo simulation of uncoupled continuous-time random walks and stochastic solution of the space-time fractional diffusion equation”, *Phys. Rev. E* **77**, 021122 (2008).

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Definition of a stochastic integral driven by a CTRW

The result of a stochastic integral depends on where the integrand is evaluated with respect to the increment. This can be expressed with a parameter $a \in [0,1]$ that interpolates linearly between $Y(t_i^-) = Y(t_{i-1})$ and $Y(t_i)$:

$$\begin{aligned} J_a(t) &\stackrel{\text{def}}{=} \int_0^t Y(s_a)\, dX(s) = \sum_{i=1}^{N(t)} Y(t_i^a)\xi_i \\ &= \sum_{i=1}^{N(t)} [(1-a)Y(t_i^-) + aY(t_i)][X(t_i) - X(t_i^-)]. \end{aligned}$$

Definition of a stochastic integral driven by a CTRW

The previous equation can be rearranged to

$$J_a(t) = J_{1/2}(t) + \left(a - \frac{1}{2}\right)[X, Y](t),$$

where

$$[X, Y](t) \stackrel{\text{def}}{=} \sum_{i=1}^{N(t)} [X(t_i) - X(t_i^-)][Y(t_i) - Y(t_i^-)]$$

is the covariation (or cross variation) of $X(s)$ and $Y(s)$ for $s \in [0, t]$. When $Y(s) = X(s)$, the covariation $[X, X](t)$ is called quadratic variation and written shorthand $[X](t)$.

Itō and Stratonovich integrals

Thus each member of the family of stochastic integrals with $a \in [0,1]$ can be obtained adding a “compensator” to the Stratonovich integral $J_{1/2}(t) = S(t) = \int Y(s) \circ dX(s)$:

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- The **Stratonovich integral** $J_{1/2}(t) = S(t)$ corresponds to the symmetric variant of Heaviside’s unit step function, $H(t) = (\operatorname{sgn} t + 1)/2$, and is particularly appealing because it can be computed according to the usual rules of calculus.

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- The **Itō integral** $J_0(t) = I(t) = \int Y(s^-)\, dX(s)$ $= S(t) - [X, Y](t)/2$, corresponding to the left-continuous variant of Heaviside's step function, has the advantage of being a **martingale**.

Monte Carlo simulation

- The definition of a stochastic integral on a CTRW is exact without the need for a limit: the number of jumps $N(t)$ between 0 and t is a random finite integer.

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- The following figures show histograms from 1 million Monte Carlo realizations of $X(t)$, $I(t)$, $S(t)$ and $[X](t)$, where $t = 1$ and $Y(t) = X(t)$ is a symmetric CTRW with jump and time scale parameters $\gamma_x^\alpha = \gamma_t^\beta$.

Relation between $X, [X], S, I$

The PDF of $S(t) = X^2(t)/2$ can be worked out from the PDF of $X(t)$ by the transformation

$$p_S(s,t) = \sum_i p_X(x_i(s),t) \left| \frac{dx_i(s)}{ds} \right|,$$

where the sum is over all x_i that yield the same s. For $s = x^2/2$ this is $x_{1,2} = \pm\sqrt{2s}$ and thus

$$p_S(s,t) = 2p_X(\sqrt{2s},t)/\sqrt{2s}, \quad s > 0.$$

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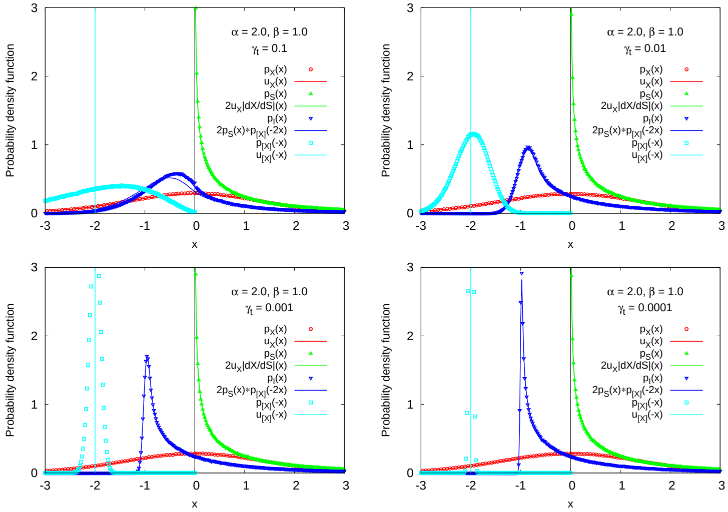
As seen before $I(t) = S(t) - [X](t)/2$; if the dependence of S and $[X]$ is small, the PDF of I can be approximated by the convolution of the PDF of S with the PDF of $[X]$ mirrored around zero and scaled to half its width:

$$p_I(x,t) \simeq 2\int_{-\infty}^{+\infty} p_S(x+2x',t)p_{[X]}(-2x',t)\,dx'.$$


α = 2.0, β = 1.0
γt = 0.1
γt = 0.01
γt = 0.001
γt = 0.0001
pX(x)
uX(x)
pS(x)
2uX|dX/dS|(x)
pI(x)
2pS(x)*p[X](-2x)
p[X](-x)
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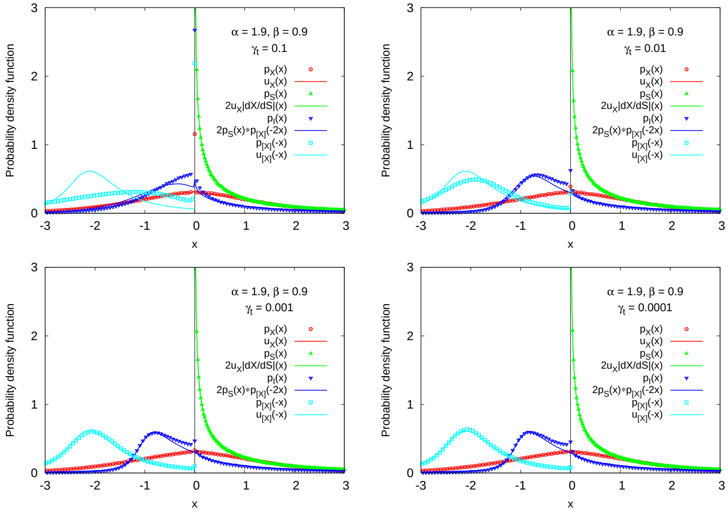
Probability density function
x


α = 1.9, β = 0.9
γt = 0.1
γt = 0.01
γt = 0.001
γt = 0.0001
pX(x)
uX(x)
pS(x)
2uX|dX/dS|(x)
pI(x)
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p[X](-x)
u[X](-x)
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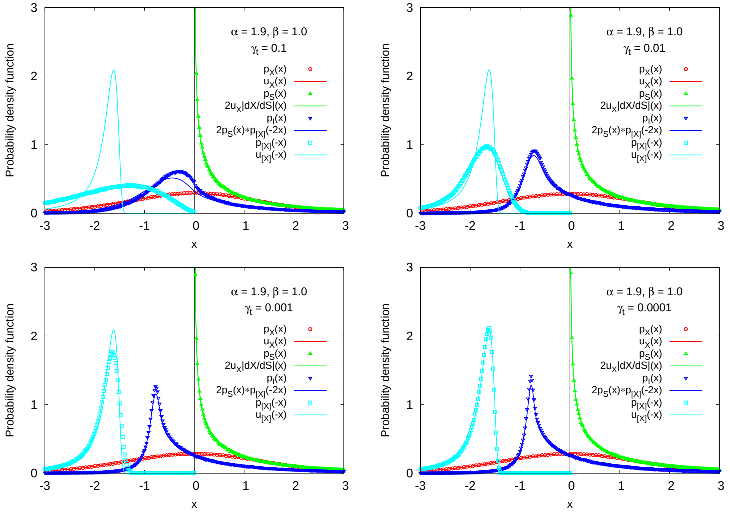
α = 1.9, β = 1.0
γ_t = 0.1
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$p_X(x)$
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$p_S(x)$
$2u_X|dX/dS|(x)$
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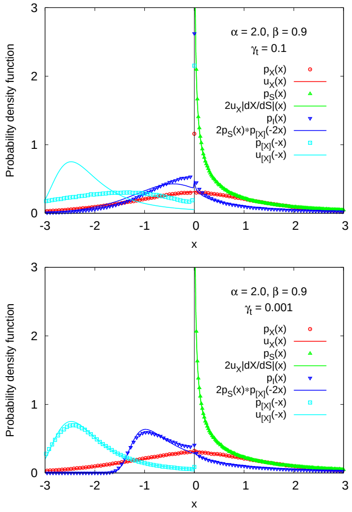
$p_{[X]}(-x)$
$u_{[X]}(-x)$
Probability density function
x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Probability density function
x

Quadratic variation of the solution of the FDE

$$\begin{aligned}\widehat{p}_{[X]}(k,t) &= \sum_{n=0}^{\infty} P(n,t)\widehat{p}^{\,n}_{\xi^2}(k) \\ &= E_\beta[-(t/\gamma_t)^\beta(1-\widehat{p}_{\xi^2}(k))].\end{aligned}$$

As the jumps ξ follow a Lévy α-stable distribution, for $x \to \infty$, $p_{\xi^2}(x) \sim x^{-\alpha/2-1}$, and the sum of ξ_i^2 converges to the positive stable distribution with index $\alpha/2$, whose characteristic function is

$$\widehat{L}^{+}_{\alpha/2}(k) = \exp\left(-(i\gamma_x k)^{\alpha/2}\right).$$

Inserting this distribution in the previous equation, the continuous limit yields the following characteristic function for the quadratic variation:

$$\widehat{u}_{[X]}(k,t) = E_\beta[-Dt^\beta(-ik)^{\alpha/2}].$$

Quadratic variation of the solution of the FDE

For $\alpha = 2$, inverting the Fourier transform, one gets

$$u_{[X]}(x,t) = t^{-\beta} M_\beta(xt^{-\beta}),$$

where $M_\beta(u)$ is the Mainardi-Wright function

$$M_\beta(u) = \mathcal{F}_\kappa^{-1}\left[E_\beta(iD\kappa)\right](u).$$

Related ongoing and future projects

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- Alternatives to CTRWs: autoregressive processes (GARCH-ACD).

The art of fitting financial time series with Lévy stable distributions

The binary program STABLE by J. Nolan, distributed on his web site www.robustanalysis.com, implements the following methods yielding $\alpha_X,\ \beta_X,\ \delta_X, \gamma_X$:

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Alternatives to CTRWs: autoregressive processes

According to UHF-GARCH (Engle 2000), the volatility σ_i of event i follows a GARCH(1,1)-ACD(1,1) process; it depends on the previous tick ξ_{i-1}, its own previous value σ_{i-1}, and the present duration τ_i, whose scale parameter θ_i depends in turn on the previous values τ_{i-1} and θ_{i-1}:

$$\begin{aligned}
\sigma_i^2 &= \omega + \alpha\xi_{i-1}^2 + \beta\sigma_{i-1}^2 + \gamma\tau_i^{-1} \\
\xi_i &= \sigma_i z_i, \quad z_i \sim \mathrm{N}(0,1) \text{ IID} \\
\theta_i &= \bar{\alpha}_0 + \bar{\alpha}_1\tau_{i-1} + \bar{\beta}_1\theta_{i-1} \\
\tau_i &= \theta_i \bar{z}_i, \quad \bar{z}_i \sim \mathrm{Exp}(1) \text{ IID}
\end{aligned}$$

R. Engle, “The econometrics of ultra-high-frequency data”, *Econometrica* **68**, 1–22 (2000).

GARCH(p,q)-ACD(p,q)

GARCH(p,q): Generalized AutoRegressive Conditional Heteroskedastic process (T. Bollerslev 1986; ARCH if $p = 0$, R. Engle 1982)

$$\sigma_i^2 = \alpha_0 + \sum_{j=1}^{q} \alpha_j \xi_{i-j}^2 + \sum_{j=1}^{p} \beta_j \sigma_{i-j}^2$$

$$\xi_i = \sigma_i z_i, \quad z_i \sim \mathrm{N}(0,1) \text{ iid}$$

ACD(p,q): Autoregressive Conditional Duration (R. Engle and J. Russell, 1998)

$$\theta_i = \bar{\alpha}_0 + \sum_{j=1}^{q} \bar{\alpha}_j \tau_{i-j} + \sum_{j=1}^{p} \bar{\beta}_j \theta_{i-j}$$

$$\tau_i = \theta_i \bar{z}_i, \quad \bar{z}_i \sim \mathrm{Exp}(1) \text{ iid}$$

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- We defined a class of class of stochastic integrals driven by a CTRW, that includes the Itō and Stratonovich cases. While the latter can be computed by the usual rules of calculus, the former is a martingale.
- We showed Monte Carlo calculations of a CTRW, its quadratic variation, its Stratonovich integral and its Itō integral, and highlighted the relation between them.