Implied Lévy Volatility

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Motivation

- introduction of the concept of implied Lévy volatility (extension of Black-Scholes implied volatility)
    - Lévy implied time volatility
    - Lévy implied space volatility
- Study of the shape of implied Lévy volatilities
- Model performance ⇒ delta-hedging strategies (periodical rebalancing)
    - qualitatively (Greeks)
    - historical time-series of the S&P500
    - historical option prices of the Dow Jones

The Black-Scholes model

- Diffusion part of the log-return process: modelled by geometrical Brownian motion (W_t)

$$S_t = S_0 \exp((r - q - \sigma^2/2)t + \sigma W_t), \quad t \geq 0.$$

Definition

The **Black-Scholes implied volatility** is the volatility $\sigma = \sigma(K, T)$ such that the model and market option prices coincide.

- $\sigma = \sigma(K, T) \equiv$ volatility surface $\Rightarrow \sigma$ needs to be adjusted separately for each individual contract
- Historical stock returns: skewed and fatter tails than those of the normal distribution
- Development of a similar concept but now under a Lévy framework $\Rightarrow$based on more empirically founded distributions

The Lévy space model

- Lévy space stock price model:

$$S_t = S_0 \exp((r - q + \omega)t + \sigma X_t), \qquad t \geq 0,$$

where $\mathbb{E}[X_1] = 0$, $\text{Var}[X_1] = 1$ and

$$\omega = -\log(\phi(-\sigma \mathrm{i}))$$

where $\phi \equiv$ characteristic function of X_1: $\phi(u) = \mathbb{E}\left[\exp(iuX_1)\right]$

Note: $\mathbb{E}[X_t] = 0$ and $\text{Var}[X_t] = t \Rightarrow \text{Var}[\sigma X_t] = \sigma^2 t$

Definition

The volatility parameter $\sigma = \sigma(K, T)$ needed to match the model price with a given market price is called the **implied Lévy space volatility**.

The Lévy time model

- Lévy time stock price model:

$$S_t = S_0 \exp((r - q + \omega\sigma^2)t + X_{\sigma^2 t}), t \geq 0,$$

where $\mathbb{E}[X_1] = 0$, $\mathrm{Var}[X_1] = 1$ and

$$\omega = -\log(\phi(-\mathrm{i}))$$

Note: $\mathrm{Var}[X_{\sigma^2 t}] = \sigma^2 t$

Definition

The volatility parameter $\sigma = \sigma(K, T)$ needed to match the model price with the market price is called the **implied Lévy time volatility**.

Pricing Vanillas under Lévy Models

- Carr-Madan formula in combination with FFT gives a very fast evaluation of vanillas:

$$C(K,T) = \frac{\exp(-\alpha \log(K))}{\pi} \int_0^{+\infty} \exp(-\mathrm{i}v \log(K)) \varrho(v) \mathrm{d}v,$$

  where

$$\varrho(v) = \frac{\exp(-rT) E[\exp(\mathrm{i}(v - (\alpha+1)\mathrm{i}) \log(S_T))]}{\alpha^2 + \alpha - v^2 + \mathrm{i}(2\alpha+1)v}.$$

- Only dependence of the Carr-Madan formula on the model: risk neutral (i.e. under Q) characteristic function of the log-price process at maturity T:

$$\phi(u;T) = E_Q[\exp(\mathrm{i}u \log(S_T))].$$

- This characteristic function is available in closed-form for many popular Lévy processes.

Greeks under Lévy models

- Delta and many others Greeks can be calculated in a similar fashion:

$$\Delta = \frac{\partial C(K,T)}{\partial S_0} = \frac{\exp(-\alpha \log(K))}{\pi} \int_0^{+\infty} \exp(-\mathrm{i}v \log(K)) \varrho_\Delta \, \mathrm{d}v$$

where

$$\varrho_\Delta(v) = \frac{\exp(-rT)\phi(v-(\alpha+1)\mathrm{i};T)}{S_0(\alpha+\mathrm{i}v)}.$$

Option prices and Greeks computation

- COS method rests on Fourier-cosine series expansions and can be applied for any model if the characteristic function $\psi(u;T)$ is available where ψ is the characteristic function of the log-moneyness at maturity

$$\psi(u;T) = E_Q\left[\exp\left(\mathrm{i}u\log\left(\frac{S_T}{K}\right)\right)\right]$$

(see Fang, F. and Oosterlee, C.W. (2008) A novel pricing method for European Options based on Fourier-cosine Series Expansions. *SIAM Journal on Scientific Computing* **31-2**, 826-848.)

NIG

- Characteristic function of the normal inverse Gaussian distribution $\textbf{NIG}(\alpha, \beta, \delta, \mu)$ with parameters $\alpha > 0$, $\beta \in]-\alpha, \alpha[$, $\delta > 0$ and $\mu \in \mathbb{R}$:

$$\phi_{\texttt{NIG}}(u; \alpha, \beta, \delta, \mu) = \exp\Big(\mathrm{i}u\mu - \delta\big(\sqrt{\alpha^2 - (\beta + \mathrm{i}u)^2} - \sqrt{\alpha^2 - \beta^2}\big)\Big), \quad u \in \mathbb{R}.$$

- If the parameter β is equal to zero the distribution is symmetric around μ whereas negative and positive values of β result in negative and positive skewness

	$\textbf{NIG}(\alpha, \beta, \delta, \mu)$	$\textbf{NIG}(\alpha, 0, \delta, \mu)$
mean	$\mu + \frac{\delta\beta}{\sqrt{\alpha^2-\beta^2}}$	μ
variance	$\alpha^2\delta\left(\alpha^2 - \beta^2\right)^{-3/2}$	$\frac{\delta}{\alpha}$
skewness	$3\beta\alpha^{-1}\delta^{-1/2}\left(\alpha^2 - \beta^2\right)^{-1/4}$	0
kurtosis	$3\left(1 + \frac{\alpha^2+4\beta^2}{\delta\alpha^2\sqrt{\alpha^2-\beta^2}}\right)$	$3\left(1 + \frac{1}{\alpha\delta}\right)$

Meixner

- Characteristic function of the Meixner distribution **Meixner**$(\alpha, \beta, \delta, \mu)$ with parameters $\alpha > 0$, $\beta \in]-\pi, \pi[$, $\delta > 0$ and $\mu \in \mathbb{R}$:

$$\phi_{\texttt{Meixner}}(u; \alpha, \beta, \delta, \mu) = \exp(iu\mu)\left(\frac{\cos\left(\frac{\beta}{2}\right)}{\cosh\left(\frac{\alpha u - i\beta}{2}\right)}\right)^{2\delta}, \quad u \in \mathbb{R}.$$

- A parameter β equal to zero indicates a symmetric distribution around μ whereas negative and positive values of β lead to negative and positive skewness

	Meixner$(\alpha, \beta, \delta, \mu)$	**Meixner**$(\alpha, 0, \delta, \mu)$
mean	$\mu + \alpha\delta \tan\left(\frac{\beta}{2}\right)$	μ
variance	$\frac{\alpha^2\delta}{2\cos^2\left(\frac{\beta}{2}\right)}$	$\frac{\alpha^2\delta}{2}$
skewness	$\sin\left(\frac{\beta}{2}\right)\sqrt{\frac{2}{\delta}}$	0
kurtosis	$3 + \frac{2-\cos(\beta)}{\delta}$	$3 + \frac{1}{\delta}$

Lévy waves

- we compute the implied Lévy space and time volatility for the NIG Lévy model for various Black-Scholes implied volatility shape and vice-versa ($T = 1, r = q = 0, S_0 = 100$).
- The symmetric cases:

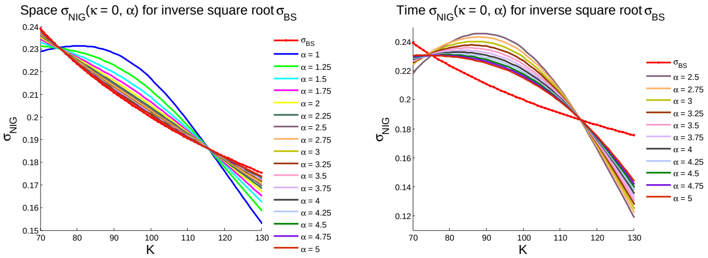


Figure: Implied volatility for the symmetric space (left) and time (right) NIG models.

Lévy Waves

- Some asymmetric cases:

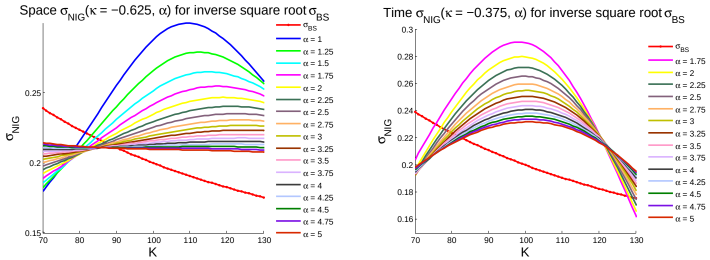


Figure: Implied volatility for inverse square root BS volatility for asymmetric asymmetric NIG space models ($\kappa = -0.625$) (left) and asymmetric NIG time models ($\kappa = -0.375$) (right).

What is flat here is not flat there

- A flat Black-Scholes implied volatility curve corresponds to a reversed smiling Lévy implied volatility curve under each symmetric NIG model.

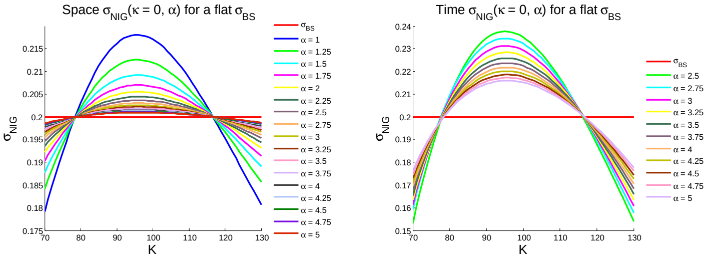


Figure: Implied space (left) and time (right) volatility for a flat BS volatility for some symmetric NIG distributions.

What is flat here is not flat there

- A flat NIG implied volatility curve corresponds to a smiling implied Black-Scholes volatility curve.

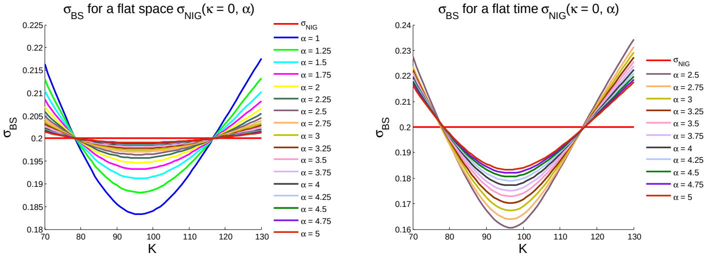


Figure: Implied BS volatility for a flat space (left) and time (right) volatility for some symmetric NIG distributions.

Vanna

$$\text{Vanna} = \frac{\partial^2 C}{\partial S_0 \partial \sigma} = \frac{\partial \Delta}{\partial \sigma}$$

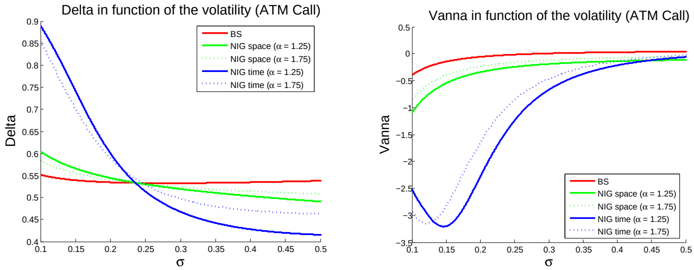


Figure: Delta (left) and Vanna (right) as a function of the volatility for a maturity of 1 month (ATM option).

Charm

$$\text{Charm} = \frac{\partial \Delta}{\partial T}$$

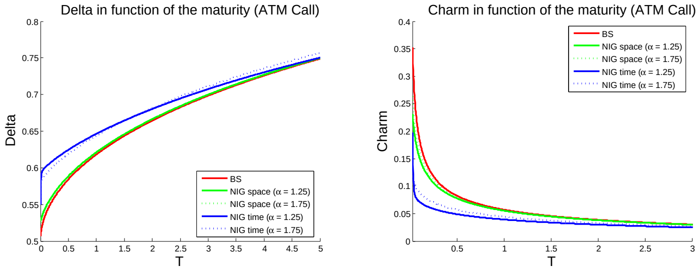


Figure: Delta (left) and Charm (right) as a function of the option maturity for a volatility of 0.2 (ATM option)

Hedging error

- Each day, we delta-hedge an ATM one-month Call option for one single day. The next day, we start with a new ATM option with again one whole month as a lifetime.
- Hedging indicator:

$$\mathtt{HE}(t_0+\Delta t) = C_{t_0+\Delta t}(K, T-\Delta t) - \Delta_{t_0} S_{t_0+\Delta t} + (\Delta_{t_0} S_{t_0} - C_{t_0}(K, T))\, e^{r\Delta t}.$$

- By considering an implied Lévy model different from the Black-Scholes model, different values of the free parameters will lead to different distributions of the hedging error (HE).
- Optimal free parameter set $\vec{p^*} = \{p_1^*, \ldots p_n^*\}$:

$$\text{abs}(\mu_{\text{HE}}(\vec{p^*})) + \sigma_{\text{HE}}(\vec{p^*}) \leq \text{abs}(\mu_{\text{HE}}(\vec{p})) + \sigma_{\text{HE}}(\vec{p}).$$

- Data set: VIX and S&P500 from the 2nd January 1990 to the 9th October 2008.

Evolution of the implied volatility

Evolution of the implied volatility through time:

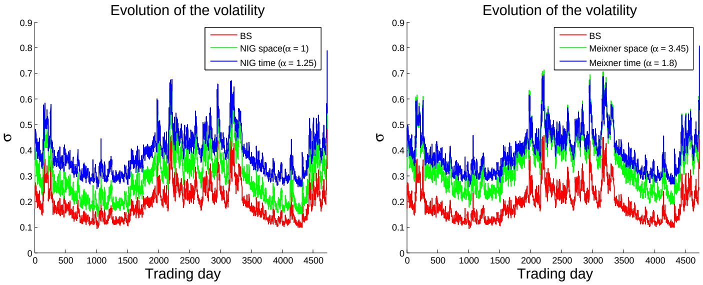


Figure: Evolution of the implied volatility for the NIG (upper) and Meixner (lower) volatility models

The implied Lévy space and time volatility moves in line with the implied Black-Scholes volatility.

Evolution of the Delta

Evolution of the Delta through time:

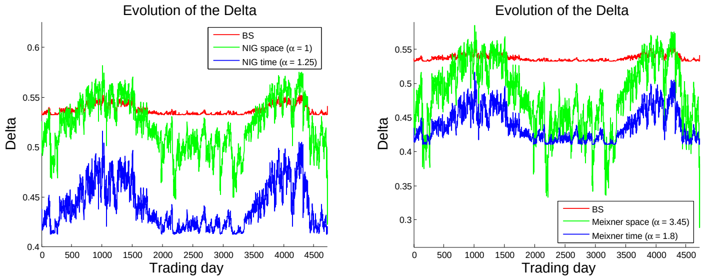


Figure: Evolution of the Delta for the NIG (upper) and Meixner (lower) volatility models

The higher slope of the Vanna curve under the Lévy models explains the higher volatility of the Lévy Deltas through time in comparison with the Black-Scholes Deltas.

The Historical Optimal Implied NIG Volatility

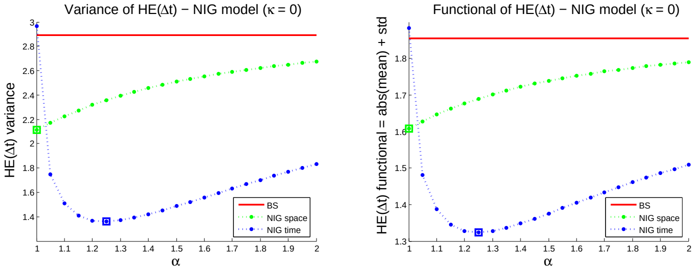


Figure: Variance (left) and functional (right) of the hedging error distribution for the NIG volatility models.

The Historical Optimal Implied Meixner Volatility

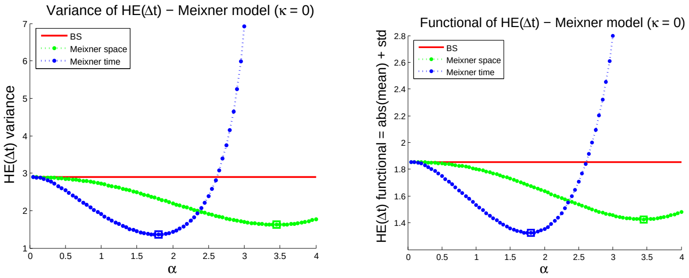


Figure: Variance (left) and functional (right) of the hedging error distribution for the Meixner volatility models.

The historical optimal model (NIG time model with $\alpha = 1.25$) leads to a reduction of the HE variance and functional from around 3 to less than 1.5 and from around 1.85 to 1.3, respectively.

Dow Jones Profit and loss

- Delta-hedging strategy:
  - At time t_0: sell option + buy Δt_0 stocks
  - At time $0 < t_i < T$: buy $\Delta_{t_i} - \Delta_{t_{i-1}}$ stocks (rebalancing)
  - At time T: close the option and stock positions
- **balance** at time $t_i \equiv$ amount spent until time t_i to build the hedging portfolio:

$$\texttt{Balance}(t_0) = C_{t_0}(K, T) - \Delta_{t_0} S_{t_0}$$

and

$$\texttt{Balance}(t_i) = \texttt{Balance}(t_{i-1}) e^{r\Delta t} + \texttt{CF}(t_i), \quad 0 < t_i \geq T$$

where $\texttt{CF} \equiv$ rebalance cash flow:

$$\texttt{CF}(t_i) = -\left(\Delta_{t_i} - \Delta_{t_{i-1}}\right) S_{t_i}$$

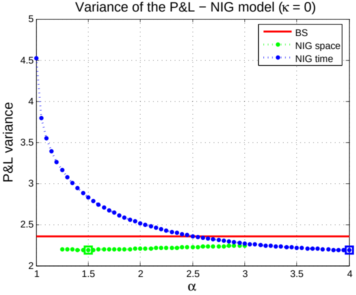
Dow Jones Profit and loss (cont.)

- Mark to Market at time $t_i \equiv$ amount spent until time t_i to build the hedging portfolio after the closing of the option and stock positions:

$$\mathtt{MtM}(t_i) = \mathtt{Balance}(t_i) + C_{t_i}(K, T - t_i) - \Delta t_i S_{t_i}$$

- $$\mathtt{P\&L} = -\mathtt{MtM}(T)$$
- Data : 491 liquid Put and Call option prices (different K and T) on the Dow Jones
- 2 cases:
  - sell each option once
  - sell each option for an amount of 1$
- ? Model which minimises the P&L variance

Dow Jones P&L results



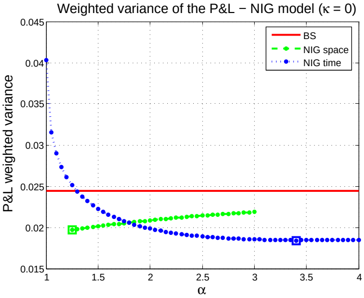
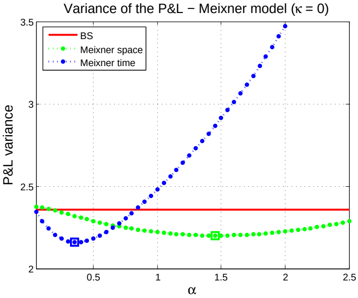


Figure: Variance (left) and weighted variance (right) of the P&L for the NIG volatility models.

Dow Jones P&L results



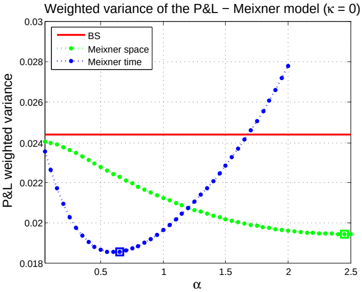
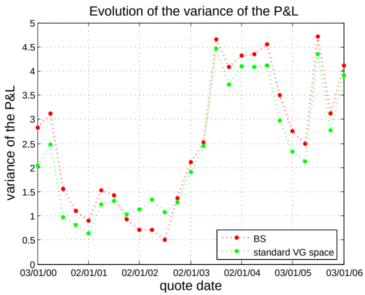


Figure: Variance (left) and weighted variance (right) of the P&L for the Meixner volatility models.

Dow Jones P&L results



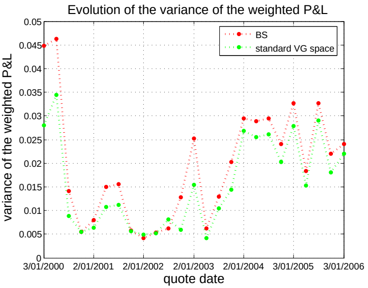


Figure: Variance (left) and weighted variance (right) of the P&L for the standard VG model volatility model.

Conclusion

- implied Lévy space and time models obtained by replacing the normal distribution of the Black-Scholes model by a more suitable Lévy distribution
- Switching from the BS world to the Lévy world ⇒ additional dof which can be used to
  - minimize the curvature of the volatility surface ⇒ any smiling or smirking BS volatility curve can be transformed into a flatter Lévy volatility curve under a well chosen parameter set ⇒ implied Lévy models could lead to flatter volatility curves for more practical datasets
  - minimise the absolute mean and the square root of the variance of the hedging error ⇒ using the historical optimal parameters leads to a significant reduction of the hedging error
  - minimise the variance of the P&L