Non-parametric Estimation of the Lévy Copula

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1. Introduction

- Stochastic processes with jumps are becoming increasingly popular, especially in financial mathematics.
- Among jump processes Lévy processes constitute a fundamental class, which is mainly due to their analytical tractability and their flexibility.

Introduction [2]

In many instances in finance and insurance one has to do with multiple sources of risk and it is of vital importance to model the dependence between these risks.

Examples:

- risk taxonomy for financial conglomerates (market, credit, insurance, operational, liquidity, concentration)
- asset classes (stocks, bonds, real estate, ...; between and within)
- credit risk (between sectors, intra-sectorial)
- etc.

Multivariate probabilistic and statistical modeling is much more complicated than univariate modeling because the number of degrees of freedom ramifies rapidly with dimension.

Introduction [3]

Quoting the CEO of Van Lanschot while explaining the main causes of the current credit crunch in The Dutch House of Representatives:

"Over the past few years, we have invested tremendously in improving our risk management systems, but our models appeared to be unable to appropriately capture the interdependences between risks." (NRC Handelsblad, November 27, 2008)

Dependence in multi-dimensional Lévy processes

- The dependence among components of a multi-dimensional Lévy process is characterized by:
  1. A Gaussian copula for the continuous Brownian part; and
  2. A Lévy copula for the discontinuous jump part.
- Relatively little is known about the corresponding inference problem for multi-dimensional Lévy processes.

The problem and its relevance

- I would like to enable statistical inference on the dependence function within a multi-dimensional Lévy process.
- It would allow a scale-free measurement of dependence.
- It would also reveal important information for the construction of multi-dimensional Lévy processes.

Contribution of the paper

- This paper proposes two non-parametric estimators for the Lévy copula.
- Under natural conditions I prove (weak and strong) consistency and asymptotic normality of the estimators.
- A test for independent jumps is also constructed.
- Finally, the estimators and test are implemented on Monte Carlo simulations and on asset returns data.
- The write-up is for the bivariate case but extensions to a multi-dimensional ($n > 2$) setting are feasible.

Outline

2. Preliminaries for Lévy Processes

- A bivariate stochastic process initialized at $\mathbf{X}_0 = \mathbf{0}$ is a bivariate Lévy process if it has stationary and independent increments and is continuous in probability.
- The law of the bivariate Lévy process $(\mathbf{X}_t)_{\{t \geq 0\}}$ is uniquely determined by the law of $\mathbf{X}_t$ for some $t > 0$.
- The characteristic function of $\mathbf{X}_t$ is given by the Lévy-Khintchine formula.

Preliminaries for Lévy Processes [2]

- The triplet $(\gamma, \mathbf{A}, \nu)$, called the characteristic triplet of the Lévy process, completely describes the probabilistic behavior of the process; γ is the drift rate, $\mathbf{A}$ the covariance matrix of the continuous Brownian component and ν the Lévy measure of the pure jump component.
- Sample paths of the process are continuous if and only if $\nu \equiv 0$.
- Note that $\nu(\mathbb{R}^2)$ may be finite or infinite and hence that ν is not a probability measure.

Preliminaries for Lévy Copulas

- Rather than modeling the dependence structure implicit in the bivariate law of $\mathbf{X}_t$ for some $t > 0$ by means of a (time-dependent) regular copula C_t, the Lévy copula models the dependence structure implicit in the (time-invariant) Lévy measure ν.
- Jointly with the correlation coefficient of the continuous Brownian component, the Lévy copula completely characterizes the dependence structure among the elements of a bivariate Lévy process.

Preliminaries for Lévy Copulas [2]

The function $F : (-\infty, +\infty]^2 \to (-\infty, +\infty]$ is a Lévy copula if:

i. F is 2-increasing:

$$F(v_1, v_2) - F(v_1, u_2) - F(u_1, v_2) + F(u_1, u_2) \geq 0, \tag{1}$$

for all u_1, u_2, v_1, v_2 satisfying $u_1 \leq v_1$, $u_2 \leq v_2$.

ii. F is grounded: $F(u_1, 0) = F(0, u_2) = 0$.

iii. $F(u_1, u_2) < +\infty$ whenever $(u_1, u_2) \neq (+\infty, +\infty)$.

iv. $F_j(u) = u$, $j = 1, 2$, $u \in \mathbb{R}$. Here, $F_1(u) := F(u, +\infty) - \lim_{a \to -\infty} F(u, a)$ and F_2 is obtained from this by symmetry.

Preliminaries for Lévy Copulas [3]

There exists an analogue of Sklar's Theorem: Let $\mathbf{X}$ be a bivariate Lévy process. Then $\exists F$ satisfying i.-iv. above such that

$$U(x_1, x_2) = F(U_1(x_1), U_2(x_2)), \qquad (x_1, x_2) \in \mathbb{R}^2 \backslash \{\mathbf{0}\}, \tag{2}$$

where the so-called tail integrals $U : \mathbb{R}^2 \backslash \{\mathbf{0}\} \to \mathbb{R}$, $U_j : \mathbb{R} \backslash \{0\} \to \mathbb{R}$, $j = 1, 2$, are defined by

$$U(x_1, x_2) := \prod_{j=1}^{2} \operatorname{sgn}(x_j) \nu \left(\prod_{l=1}^{2} \mathcal{I}(x_l) \right),$$

$$U_j(x_j) := \operatorname{sgn}(x_j) \nu_j \left(\mathcal{I}(x_j) \right),$$

with

$$\mathcal{I}(x) := \begin{cases} (x, +\infty), & x \geq 0; \\ (-\infty, x], & x < 0; \end{cases}$$

and where ν and ν_j are the Lévy and marginal Lévy measure of $\mathbf{X}$, respectively. The Lévy copula is unique on $\prod_{j=1}^{2} \overline{\mathrm{Ran}\ U_j}$.

Preliminaries for Lévy Copulas [4]

Analogue of Sklar's Theorem [Continued]: Conversely, if F is a bivariate Lévy copula and U_1, U_2 are the tail integrals of two univariate Lévy processes then there exists a bivariate Lévy process $\mathbf{X}$ with tail integral (2) and marginal tail integrals U_1, U_2. The Lévy measure of $\mathbf{X}$ is uniquely determined by F and U_1, U_2.

Outline

Statistical setting

The setting of our statistical problem is as follows:

- A 2-dimensional Lévy process $\mathbf{X}$ is observed at $i = 1, \ldots, n$ discrete instants separated by Δ_n units of time.
- Since $\mathbf{X}_0 = \mathbf{0}$ this amounts to observing the n increments $\mathbf{X}_{i\Delta_n} - \mathbf{X}_{(i-1)\Delta_n}$.
- So when $\Delta_n = \Delta$ is fixed, we observe n i.i.d. random vectors distributed as $\mathbf{X}_\Delta$.

Intuition

- The non-parametric estimators are inspired by the fact that the Lévy copula $F : (-\infty, +\infty]^2 \to (-\infty, +\infty]$ of a 2-dimensional Lévy process $\mathbf{X}$ can be obtained as the following limit, involving the regular copula C_t corresponding to the joint distribution of $\mathbf{X}_t$ at a given time t:

$$F(u_1, u_2) = \lim_{t \to 0} \frac{1}{t} C_t^{(\mathrm{sgn}\ u_1, \mathrm{sgn}\ u_2)}(t|u_1|, t|u_2|) \prod_{j=1}^{2} \mathrm{sgn}\ u_j, \qquad (3)$$

$$(u_1, u_2) \in \prod_{j=1}^{2} \overline{\mathrm{Ran}\ U_j},$$

with $C_t^{(\alpha_1, \alpha_2)}$ a survival copula of $(\alpha_1 X_{1,t}, \alpha_2 X_{2,t})$ and

$$\mathrm{sgn}\ u_j := \begin{cases} 1, & u_j \geq 0; \\ -1, & u_j < 0. \end{cases}$$

- From (3) it becomes apparent that the Lévy copula is determined only by the behavior of C_t in the corners of its domain $[0, 1]^2$.

Intuition [2]

- The first estimator then, is based on (3), treating the limit relation as an approximate equality for t small enough:

$$\hat{F}(u_1, u_2) := \frac{1}{k}\sum_{i=1}^{n} \mathbf{1}_{\left\{\forall j=1,2: \begin{cases} R^{X_j}_{i\Delta_n} > n+1-ku_j, & \operatorname{sgn}\ u_j \geq 0; \\ R^{X_j}_{i\Delta_n} \leq k|u_j|, & \operatorname{sgn}\ u_j < 0; \end{cases}\right\}} \prod_{j=1}^{2} \operatorname{sgn}\ u_j, \tag{4}$$

  where $R^{X_j}_{i\Delta_n}$ is the rank of $X_{j,i\Delta_n} - X_{j,(i-1)\Delta_n}$ among $\{X_{j,i\Delta_n} - X_{j,(i-1)\Delta_n}, i = 1, \dots, n\}$, and where $k = k_n$ is an intermediate sequence of integers; that is, $k \to +\infty$, $k/n \to 0$, as $n \to +\infty$.
- The estimator (4) seems natural since it is essentially the empirical tail (survival) copula at a sampling interval Δ_n; it can be viewed as a tail version of Deheuvels' (1979) empirical copula computed under a finer and finer microscope.
- However, the mathematical details of the derivations are delicate.

Notation

Before stating the main results we need to introduce some notation:

- Let Λ be a non-negative measure on $(-\infty, +\infty]^2 \setminus \{(+\infty, +\infty)\}$ induced by F via

$$\Lambda((u_1 \wedge 0, u_1 \vee 0] \times (u_2 \wedge 0, u_2 \vee 0]) = F(u_1, u_2) \prod_{j=1}^{2} \operatorname{sgn} u_j,$$

for all $(u_1, u_2) \in (-\infty, +\infty]^2 \setminus \{(+\infty, +\infty)\}$.

Notation [2]

- Furthermore, let W_Λ be a mean-zero Wiener process on $(-\infty, +\infty]^2 \setminus \{(+\infty, +\infty)\}$ with covariance function

$$\mathbb{E}[W_\Lambda(C) W_\Lambda(\tilde{C})] = \Lambda(C \cap \tilde{C}),$$

  for Borel sets C and $\tilde{C}$.
- Define, for $(u_1, u_2) \in (-\infty, +\infty]^2 \setminus \{(+\infty, +\infty)\}$,

$$W(u_1, u_2) := W_\Lambda((u_1 \wedge 0, u_1 \vee 0] \times (u_2 \wedge 0, u_2 \vee 0]),$$

  and for $u_1, u_2 \in (-\infty, +\infty)$ let its marginals W_1 and W_2 be defined by

$$W_1(u_1) := W_\Lambda((u_1 \wedge 0, u_1 \vee 0] \times (0, +\infty]) + W_\Lambda((u_1 \wedge 0, u_1 \vee 0] \times (-\infty, 0));$$
$$W_2(u_2) := W_\Lambda((0, +\infty] \times (u_2 \wedge 0, u_2 \vee 0]) + W_\Lambda((-\infty, 0) \times (u_2 \wedge 0, u_2 \vee 0]).$$

Asymptotics

- Let $k = k_n \leq n$ be an intermediate sequence of integers such that

$$k \to +\infty \qquad \text{and} \qquad k/n \to 0 \qquad \text{as} \qquad n \to +\infty. \tag{5}$$

- Furthermore, let the sequence $\Delta = \Delta_n$ satisfy for some $\alpha \geq 1$,

$$\frac{k}{n} - \Delta = o\left(\left(\frac{k}{n}\right)^{\alpha}\right) \qquad \text{as} \qquad n \to +\infty. \tag{6}$$

Theorem (Consistency)

1. *Weak Consistency: Suppose that (5) and (6) hold. Then, as* $n \to +\infty$,

$$\hat{F}(u_1, u_2) \xrightarrow{\mathbb{P}} F(u_1, u_2), \qquad (u_1, u_2) \in \prod_{j=1}^{2} \overline{\mathrm{Ran}\ U_j}. \tag{7}$$

2. *Strong Consistency: Suppose in addition that, as* $n \to +\infty$, $k/\log\log n \to +\infty$. *Then, as* $n \to +\infty$,

$$\hat{F}(u_1, u_2) \xrightarrow{\text{a.s.}} F(u_1, u_2), \qquad (u_1, u_2) \in \prod_{j=1}^{2} \overline{\mathrm{Ran}\ U_j}. \tag{8}$$

Theorem (Asymptotic Normality)

ASSUMPTION (CD) *$F(u_1, u_2)$ has continuous first partial derivatives*

$$F^1(u_1, u_2) := \frac{\partial F(u_1, u_2)}{\partial u_1}; \qquad F^2(u_1, u_2) := \frac{\partial F(u_1, u_2)}{\partial u_2}. \tag{9}$$

ASSUMPTION (SO) *Second order condition: For some $\alpha, c > 0$, as $t \to 0$,*

$$\frac{1}{t} C_t^{(\mathrm{sgn}\ u_1, \mathrm{sgn}\ u_2)}(t|u_1|, t|u_2|) \prod_{j=1}^{2} \mathrm{sgn}\ u_j - F(u_1, u_2) = O(t^\alpha), \tag{10}$$

holds uniformly on the set

$$\left\{ u_1^2 + u_2^2 = c, \qquad (u_1, u_2) \in \prod_{j=1}^{2} \overline{\mathrm{Ran}\ U_j} \right\}.$$

In addition, suppose that (5) holds, and that, as $n \to +\infty$,
$\frac{k}{n} - \Delta = O\left(\left(\frac{k}{n}\right)^{1+\alpha}\right)$, and $k = o(n^{2\alpha/(1+2\alpha)})$.

Theorem (Asymptotic Normality [Continued])

Then, as $n \longrightarrow +\infty$,

$$\sqrt{k}(\hat{F}(u_1, u_2) - F(u_1, u_2)) \xrightarrow{d} B(u_1, u_2), \qquad (u_1, u_2) \in \prod_{j=1}^{2} \overline{\mathrm{Ran}\ U_j}, \tag{11}$$

where

$$\begin{aligned} B(u_1, u_2) :=& W(u_1, u_2) \prod_{j=1}^{2} \mathrm{sgn}\ u_j \\ & - F^1(u_1, u_2) W_1(u_1)\ \mathrm{sgn}\ u_1 - F^2(u_1, u_2) W_2(u_2)\ \mathrm{sgn}\ u_2. \end{aligned} \tag{12}$$

Discussion

- Strengths: natural and simple.
- A drawback of the estimator (4) is that it is not a Lévy copula itself. In particular, it does not satisfy a positive homogeneity property.
- To remedy this problem I will also study an estimator based on the so-called Lévy spectral measure introduced in this paper. This second estimator does belong to the class of Lévy copulas.
- Having at our disposal an estimator that takes values in the class of Lévy copulas is important for several reasons: it makes simulation from estimated multi-dimensional Lévy processes feasible and is likely to exhibit superior efficiency.

Discussion [2]

- Consider the related statistical problem of testing the simple null hypothesis $\mathcal{H}_0 : F = F_0$ for a given Lévy copula F_0, against the alternative $\mathcal{H}_a : F \neq F_0$.
- Adequately chosen functionals (test statistics) of the empirical process

$$\nu_n(u_1, u_2) := \sqrt{k_n}\left(\hat{F}(u_1, u_2) - F_0(u_1, u_2)\right), \tag{13}$$

  can be used for this purpose.
- While (13) can serve as a basis for testing a simple null hypothesis, it is not an appropriate basis for testing for independent jump components. This is so because under the null of independent jump components its limit process is degenerate.
- I therefore propose a different basis to test for independence, and derive its asymptotic properties.

Outline

4. Independence Test

- A bivariate Lévy process has independent jump components if and only if its Lévy measure is supported on the coordinate axes.
- This entails that the corresponding Lévy copula is not unique.
- It entails furthermore that the limit process B is degenerate for any such Lévy copula.
- Therefore the empirical process ν_n defined in (13) is not an appropriate basis for constructing tests for independence.

Theorem (Asymptotic Normality under Independence)

Suppose that, as $n \to +\infty$, $k/\sqrt{n} \to +\infty$, *and for some* $\alpha > 0$, $\frac{k}{n} - \Delta = O\left(\left(\frac{k}{n}\right)^{2+\alpha}\right)$, *and* $k = o\left(n^{\frac{1+2\alpha}{2+2\alpha}}\right)$. *Then, under the null hypothesis of independent jump components, as* $n \to +\infty$,

$$\frac{k}{\sqrt{n}}\left(\frac{n}{k}\hat{F}(u_1, u_2) - u_1 u_2\right) \xrightarrow{d} W^{\perp}(u_1, u_2)\prod_{j=1}^{2} \mathrm{sgn}\ u_j, \tag{14}$$

$$(u_1, u_2) \in \prod_{j=1}^{2} \overline{\mathrm{Ran}\ U_j},$$

where the mean-zero Wiener process $W^{\perp}$ *on* $(-\infty, +\infty]^2 \setminus \{(+\infty, +\infty)\}$ *has covariance function*

$$\mathbb{E}[W^{\perp}(u_1, u_2)W^{\perp}(v_1, v_2)] = |u_1| \wedge |v_1||u_2| \wedge |v_2| 1_{\{\forall j=1,2,\ \mathrm{sgn}\ u_j \mathrm{sgn}\ v_j = 1\}}, \tag{15}$$

for $(u_1, u_2), (v_1, v_2) \in (-\infty, +\infty]^2 \setminus \{(+\infty, +\infty)\}$.

Corollary (Asymptotic Distribution of Independence Test Statistic)

Suppose that the conditions of the previous Theorem hold. Then, under the null hypothesis of independent jump components, as $n \to +\infty$,

$$\int_{(u_1,u_2)\in[-c,c]\cap\prod_{j=1}^{2}\overline{\mathrm{Ran}\ U_j}} \frac{k^2}{n}\left(\frac{n}{k}\hat{F}(u_1,u_2) - u_1 u_2\right)^2 du_1 du_2 \qquad (16)$$

$$\xrightarrow{d} \int_{(u_1,u_2)\in[-c,c]\cap\prod_{j=1}^{2}\overline{\mathrm{Ran}\ U_j}} W^{\perp}(u_1,u_2)^2 du_1 du_2,$$

for an arbitrary $c > 0$.

Discussion

- A bivariate Lévy process has independent jump components if and only if the marginal processes never jump at the same time. This means that the independence test may also be regarded as a test for (non-)co-jumping.
- Test statistics are very easy to compute.

Outline

Theorem (Lévy Spectral Measure)

Let $\mathbf{X}$ *be a 2-dimensional Lévy process with Lévy copula* F*. Then there exists a finite measure* μ_F *on* $[0,+\infty]^2 \setminus \{(0,0)\}$ *such that*

$$\frac{1}{t}\mathbb{P}\left[t(Z_{1,t}, Z_{2,t}) \in \cdot\right] \xrightarrow{v} \mu_F(\cdot), \qquad t \to 0, \tag{17}$$

with

$$Z_{j,t} := \frac{1}{1 - G_{j,t}(X_{j,t})}, \qquad j = 1, 2,$$

and where μ_F *and* F *are connected through*

$$F(u_1, u_2) = \mu_F\left(\left\{(z_1, z_2) \in [0,+\infty]^2 : \forall j = 1,2 \quad z_j \geq \frac{1}{u_j}\right\}\right), \tag{18}$$

with $(u_1, u_2) \in [0,+\infty]^2 \setminus \{(+\infty,+\infty)\}$*.*

Theorem (Lévy Spectral Measure [Continued])

Defining

$$\Phi_F(\cdot) := \mu_F\left(\{(z_1, z_2) \in [0, +\infty)^2 : r \geq 1, \theta \in \cdot\}\right) \tag{19}$$

with the polar coordinates

$$\begin{aligned} r &:= \|(z_1, z_2)\|_p, &\in (0, +\infty); \\ \theta &:= \arctan(z_1/z_2), &\in [0, \pi/2]; \end{aligned}$$

(17) implies that

$$\frac{1}{t}\mathbb{P}\left[\|(Z_{1,t}, Z_{2,t})\|_p \geq \frac{1}{t}, \arctan(Z_{1,t}/Z_{2,t}) \in \cdot\right] \xrightarrow{v} \Phi_{p,F}(\cdot), \qquad t \to 0. \tag{20}$$

We coin μ_F and $\Phi_{p,F}$ the Lévy exponent measure and the Lévy spectral measure (corresponding to the L_p-norm), respectively.

Results

- I propose a non-parametric estimator for Φ_F that is essentially a weighted version of its empirical counterpart.
- Consistency (relatively easy) and asymptotic normality (delicate!) results are established.

Outline

6. A Monte Carlo Study

- Two prominent examples of (bivariate) Lévy processes are
  1. the bivariate (Poisson) jump diffusion; and
  2. the bivariate Cauchy process with Brownian noise.
- We adopt a Clayton Lévy copula to specify the dependence in the bivariate (Poisson) jump diffusion case.

Figure: Scatter plots for samples of $n = 1000$ observations of the bivariate Cauchy process (top left) and a bivariate (Poisson) jump diffusion with exponential jump sizes and Clayton Lévy copula.

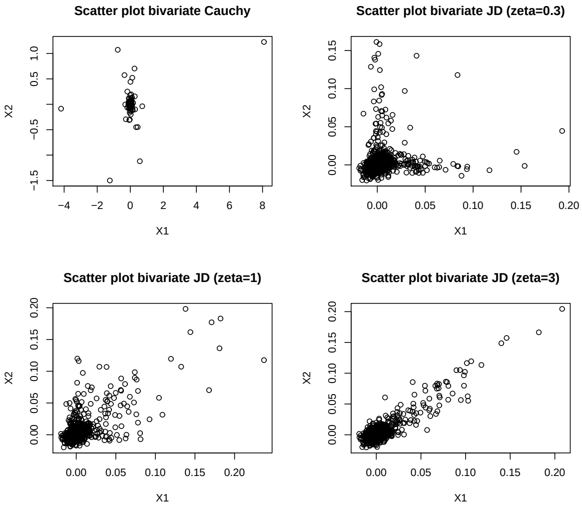


Figure: Empirical Lévy copula bivariate Cauchy with Brownian noise.

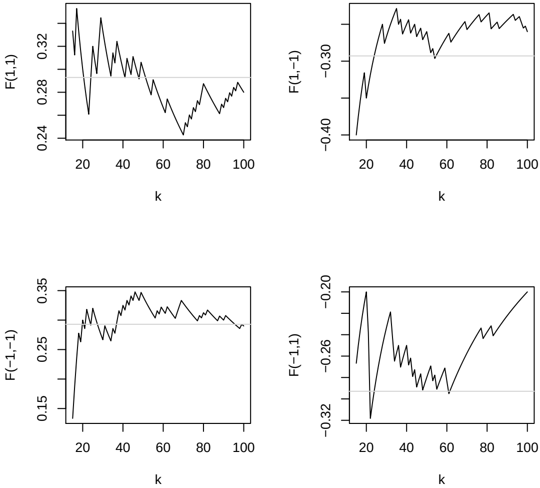


Figure: Empirical Lévy spectral measure bivariate Cauchy with Brownian noise ($p = 2$).

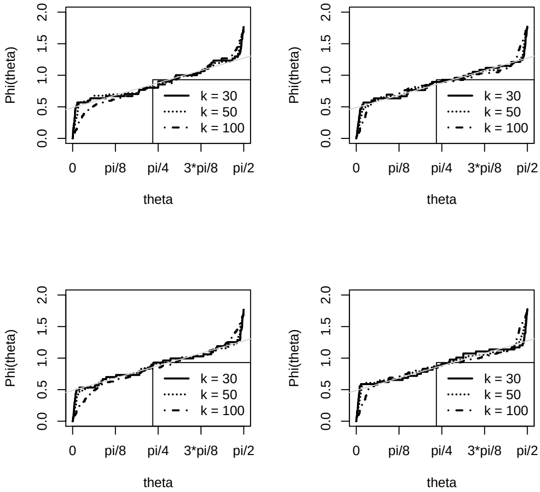


Figure: Empirical Lévy copula bivariate jump diffusion with Clayton Lévy copula.

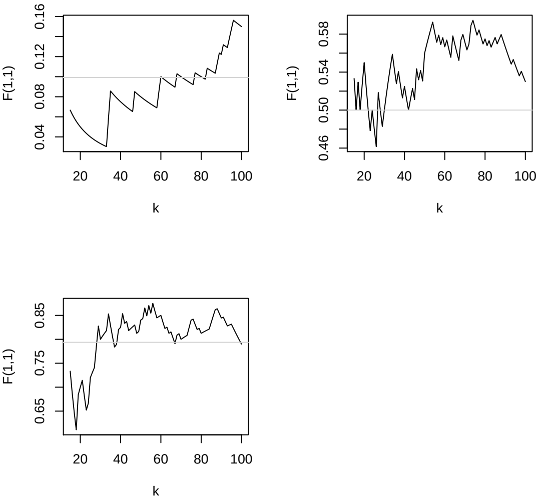
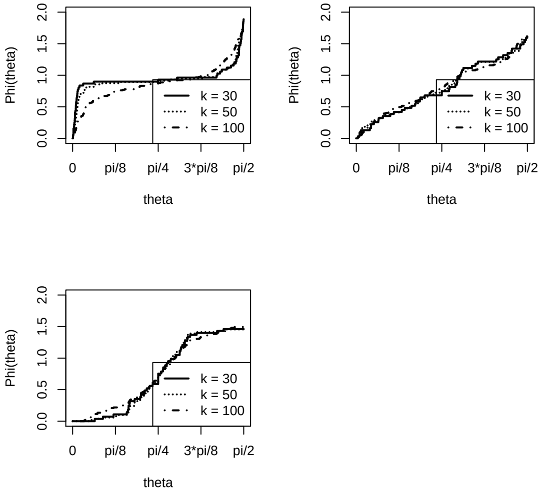


Figure: Empirical Lévy spectral measure bivariate jump diffusion with Clayton Lévy copula ($p = 2$).



7. Conclusion

- I have introduced two non-parametric estimators for the Lévy copula and derived the estimators' properties.
- In addition, I have constructed a test for independence.
- Extensive Monte Carlo evidence shows that the estimators and test are quite accurate on small samples.