Maximum-Likelihood-Estimation of Lévy driven Ornstein-Uhlenbeck Processes

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Ornstein-Uhlenbeck (OU) Process

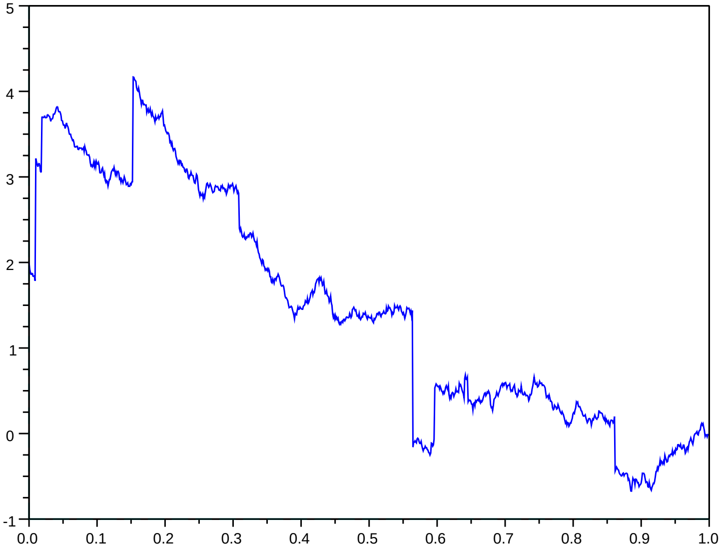
Let $(L_t, t \geq 0)$ be a Lévy process on $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$. For every $a \in \mathbb{R}$

$$dX_t = -aX_t\,dt + dL_t, \quad t \in \mathbb{R}_+, \quad X_0 = x, \tag{1}$$

defines an Ornstein-Uhlenbeck process driven by the Lévy process L with initial distribution $\pi = \mathcal{L}(X_0)$.
Equivalently,

$$X_t = e^{-at} X_0 + \int_0^t e^{-a(t-s)} dL_s. \tag{2}$$

Sample path from compound Poisson plus Wiener process driver



Recent Literature on Lévy OU Inference

- Stochastic volatility modelling: Barndorff-Nielsen and Shephard [2001]
- (Non-)Parameteric estimation for driving subordinators Jongbloed, van der Meulen and van der Vaart [2005]
- Maximum-likelihood-estimation from discrete observations Valdivieso, Schoutens and Tuerlinckx [2009]
- Least squares estimation from discrete observations for α-stable driver Hu and Long [2009]

Setting

Problem: Estimation of a from continuous observations X_t, $0 \leq t \leq T$ and know Lévy-Khintchine triplet of L.

We work throughout in the canonical setting:

- $\Omega = D(\mathbb{R}_+) = \{f : \mathbb{R}_+ \to \mathbb{R};\ f \text{ càdlàg}\}$
- $X(\omega, t) = \omega(t)$ for all $\omega \in \Omega$ coordinate process
- Filtration generated by X

$$\mathcal{F}_t = \bigcap_{s>t} \sigma(X_u : u \leq s) \text{ and } \mathcal{F} = \bigvee_t \mathcal{F}_t$$

For every $a \in \mathbb{R}$ we obtain a solution measure P^a of the OU equation on $D(\mathbb{R}_+)$.

Absolute Continuity/Singularity (ACS) Problem

$P_t^a := P^a_{|\mathcal{F}_t}$ denotes the restriction of P^a to $\mathcal{F}_t$.
Local absolute continuity:

$$P^{a'} \overset{loc}{\ll} P^a \Longleftrightarrow P_t^{a'} \ll P_t^a \quad \forall t \in \mathbb{R}_+$$

In order to define an MLE for the statistical experiment $(\Omega, \mathcal{F}, (\mathcal{F}_t), (P^a)_{a\in\mathbb{R}})$ we need:

1. Does
$$P^{a'} \overset{loc}{\ll} P^a \text{ hold for all } a, a' \in \mathbb{R}?$$
2. Can we derive $Z_t = \frac{dP_t^{a'}}{dP_t^a}$ explicitly?

Hellinger Process and ACS Problems

Let P, P' be two probability measures on $(\Omega, \mathcal{F}, (\mathcal{F}_t))$.

Theorem (Jacod and Mémin (1979))

Let $h(\alpha)$, $\alpha \in (0,1)$ be a version of the Hellinger process $h(\alpha; P, P')$. Then for every stopping time T there is equivalence between

1. $P'_T \ll P_T$;
2. $\exists \alpha \in (0,1)$ *such that* $P'(h(\alpha)_T < \infty) = 1$ *and* $P'_0 \ll P_0$ *and* $P'(h(0)_T = 0) = 1$.

Semimartingale characteristics of X

Let (b, σ^2, μ) denote the Lévy-Khintchine triplet of L.

Then the semimartingale characteristics (B, C, ν) of X are given by

$$\begin{aligned} B(\omega, t) &= bt - a \int_0^t X_{s-}(\omega) ds, \\ C(\omega, t) &= \sigma^2 t, \\ \nu(\omega, dt, dx) &= \mu(dx)\, \lambda(dt), \end{aligned}$$

where λ denotes the Lebesgue measure on $\mathbb{R}$.

The Hellinger Process

Proposition

A version of the Hellinger process of two solution measures $P^a, P^{a'}$ is

$$h_t(\alpha; a, a') = \frac{\alpha(1-\alpha)}{2\sigma^2} \int_0^t \left[\int_0^u \left(a' e^{-a'(u-s)} - a e^{-a(u-s)} \right) L(ds) \right]^2 du.$$

for $a, a' \in \mathbb{R}$.

Theorem

Let $P^a, P^{a'}$ be two solution measures of the OU equation for the driving Lévy process L with characteristic triplet (b, σ^2, ρ) and initial distributions π and π'. Suppose that $\sigma^2 > 0$ and $\pi' \ll \pi$, then we have

$$P^{a'} \overset{loc}{\ll} P^a.$$

Proposition

There exists a P-local martingale $N : \Delta \to \mathbb{R}$ on a random interval $\Delta \subset \Omega \times \mathbb{R}_+$ such that the density process is given by

$$Z_t = \frac{dP_t^{a'}}{dP_t^a} = Z_0 \exp\left(N_t - \frac{(a'-a)^2}{2\sigma^2} \int_0^t X_{s-}^2 ds\right).$$

Furthermore, for every stopping time S such that $[0, S] \subset \Delta$ the stopped process N^S is of the form

$$N^S = \left(\frac{(a'-a)}{\sigma^2} X_{t-} 1_{[0,S]}\right) \cdot X^c,$$

where X^c denotes the continuous martingale part of X under P^a.

Maximum-Likelihood-Estimator (MLE)

For continuous observations of the Ornstein-Uhlenbeck process X the likelihood function $\mathcal{L}$ for the statistical experiment $(\Omega, \mathcal{F}, (\mathcal{F}_t), (P^a)_{a\in\mathbb{R}})$ takes the form

$$\mathcal{L}(a, X^T) = \frac{dP_t^a}{dP_t^0} = \exp\left(\frac{a}{\sigma^2}\int_0^T X_{s-}\, dX_s^c - \frac{a^2}{2\sigma^2}\int_0^T X_{s-}^2\, ds\right),$$

Hence, the MLE for a is explicitly given by

$$\hat{a}_T = \frac{\int_0^T X_{s-} dX_s^c}{\int_0^T X_{s-}^2 ds}.$$

The continuous martingale part X^c

By the Lévy-Itô decompostitition of L we can write X as

$$X_t = X_0 - a\int_0^t X_{s-}\, ds + W_t + J_t \quad , t \geq 0,$$

where W is a Wiener Process and J a quadratic pure jump process given by

$$J_t = \int_{\{|x|<1\}} x(N_t(dx) - t\mu(dx)) + bt + \sum_{0\leq s\leq t} \Delta X_s 1_{\{|\Delta X_s|\geq 1\}}.$$

N is the jump measure of L with compensator μ.

Under P^0 it follows that $X^c = W$, but under P^a

$$\tilde{W}_t = W_t - a \int_0^t X_{s-}\, ds$$

defines a Wiener process such that $X^c = \tilde{W}$ under P^a.
Hence, given observations $(X_t(\omega), t \in [0, T])$

$$X_t^c = X_t - \int_{\{|x|<1\}} x(N_t(dx) - t\mu(dx)) - bt - \sum_{0 \le s \le t} \Delta X_s 1_{\{|\Delta X_s| \ge 1\}}.$$

which can be reconstructed from continuous observations.
Hence, the MLE can be rewritten as

$$\hat{a}_T = \frac{\int_0^T X_{s-}(dW_s - aX_{s-} ds)}{\int_0^T X_{s-}^2 ds} = \frac{\int_0^T X_{s-} dW_s}{\int_0^T X_{s-}^2 ds} - a.$$

under P^a.

Curved Exponential Families

Let $\{P^\theta, \theta \in \Theta\}$ be a family of measures on $(\Omega, \mathcal{F}, (\mathcal{F}_t))$.

Definition (Küchler and Sørensen (1997))

We say that a statistical experiment $\{P^\theta, \theta \in \Theta\}$ forms a ***curved exponential family*** *if the likelihood function exists and is of the form*

$$\frac{dP_t^\theta}{dP_t^{\theta_0}} = \exp\left(\theta' A_t - \kappa(\theta) S_t\right)$$

where $\kappa : \Theta \to \mathbb{R}$, for $\theta_0 \in \Theta$ arbitrary but fixed and $A : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$ is a càdlàg process. Moreover, $S : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ is assumed to be a non-decreasing continuous process with $S_0 = 0$ and $S_t \stackrel{t\to\infty}{\longrightarrow} \infty$.

Strong Consistency

Theorem

Under the condition $\sigma^2 > 0$ the MLE $\hat{a}_T$ for any $a \in \mathbb{R}$ based on continuous observations X_t, $t \in [0, T]$ exists and is given by

$$\hat{a}_T = \frac{\int_0^T X_{s-} dX_s^c}{\int_0^T X_{s-}^2 ds}.$$

Furthermore, under P_a the MLE is unique for T sufficiently large and

$$\hat{a}_T \to a \quad \textit{almost surely}$$

as $T \to \infty$.

Asymptotic Normality

Theorem

Let X be a stationary OU process and $a > 0$ such that $c = E[X^2] < \infty$. Then under P^a

$$\sqrt{T}(\hat{a}_T - a) \to N(0, \frac{\sigma^2}{c}) \quad \textit{weakly}$$

as $T \to \infty$.

Summary and Outlook

- Under $\sigma^2 > 0$ the solution measures $\{P^a, a \in \mathbb{R}\}$ are locally equivalent.
- The MLE takes an explicit form and is consistent and asymptotically normal as well as efficient.
- Computation from discrete observations is straight forward.
- Asymptotics for $a < 0$ and without second moments of X?
- Delay estimation

Bibliography

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