COEFFICIENTS RECONSTRUCTION OF A SEMIMARTINGALE WITH JUMPS GIVEN DISCRETE OBSERVATIONS, AND INTEREST RATE MODELING

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Overlay of the presentation

- The problem & the model
- The proposed solution: asymptotic results
- Monte Carlo simulations
- Empirical application: short rate modeling
- Comparisons
- Conclusions

The problem and the model

A state variable R (e.g. spot interest rate) follows

$$dR_t = \mu_t dt + \sigma(R_t) dW_t + dJ_t, \quad t \in [0,T]:$$

⋆ level-dependent diffusion coefficient $\sigma(R_t)$

Goal

reconstruct the function $\sigma^2(r)$, fixed T, given observations

$$\{R_0, R_{t_1}, R_{t_2}, R_{t_3}, ..., R_{t_n}\}, \quad t_i = i\delta, \quad i = 1...n, \quad n\delta = T$$

⋆ any progressively measurable cadlag drift μ_t

$$J = J_1 + \tilde{J}_2$$

$\star$ J_1 any pure jump finite activity (FA) semimartingale: *a.s. finite number of jumps in each finite time interval*

$$J_1 = \int_0^{\cdot} \int_{\mathbb{R}} x \ m_1(dx, du) = \sum_{j=1}^{N_t} \gamma_j$$

m_1 jump measure of J_1 with a.s. $N_t = \int_0^t \int_{\mathbb{R}} 1 m_1(dx, du) < \infty, \forall\, t \leq T$
$N =$ jumps counting process, jump intensity λ_t

e.g. J_1 doubly stochastic compound Poisson with $E[\int_0^T \lambda_t dt] < \infty$

$\star$ $\tilde{J}_2$ pure jump infinite activity (IA) SM: *there are paths with infinitely many jumps in some finite time interval*

$$\tilde{J}_2 = \int_0^{\cdot} \int_{|x| \leq 1} x \tilde{m}_2(dx, du),$$

$\tilde{m}_2 = m_2 - \nu_t(\omega, dx)dt$ compensated m_2

Furthermore

in the FA case, assuming $\mu_t \equiv \mu(R_t)$, $\lambda_t \equiv \lambda(R_t)$

we even want to reconstruct $\mu(r), \lambda(r)$ as $T \to \infty$

PRELIMINARIES

In the continuous case, $J \equiv 0$: kernel estimation

proposed by Florens-Zmirou (1993), and generalized by Stanton (1997), Jiang and Knight (1997), Bandi and Phillips (2003), Renò (2008)

$$\hat{\sigma}_n^2(r) = \frac{\sum_{i=1}^{n} K\left(\frac{R_{t_{i-1}} - r}{h}\right) (R_{t_i} - R_{t_{i-1}})^2}{\sum_{i=1}^{n} K\left(\frac{R_{t_i} - r}{h}\right) \delta}$$

K kernel, h bandwidth: $h \to 0$ at given speed as $\delta \to 0$.

Asymptotic properties of $\hat{\sigma}_n^2(r)$, for fixed $T < \infty$, are fully assessed in Florens-Zmirou (1993) using the local time properties of R

Why this works?

Notation:

$$\Delta_i R = R_{t_i} - R_{t_{i-1}}, \qquad K_{i-1} = K\left(\frac{R_{t_{i-1}} - r}{h}\right)$$

through K, we take a term $(\Delta_i R)^2$ only if observation $R_{t_{i-1}}$ is close to level r:

$$\hat{\sigma}_n^2(r) = \frac{\sum_{i=1}^n K_{i-1} \frac{(\Delta_i R)^2}{\delta}}{\sum_{i=1}^n K_{i-1}} \approx E\left[\frac{(\Delta_i R)^2}{\delta} \middle| R_{t_{i-1}} = r\right]$$

$$\approx E\left[\frac{\sigma^2(R_{t_{i-1}})\delta}{\delta} \middle| R_{t_{i-1}} = r\right] = \sigma^2(r)$$

However adding jumps is important

- better explains interest rates (macroeconomic news and announcement effects $\rightarrow$ abrupt changes, Das 02, Piazzesi 05), stock prices (endogenous crashes, e.g. Black Friday, 1987, or exogenous crashes, e.g. September 11), commodities prices (abrupt spikes due to shortages), electricity prices (supply are highly inelastic to demand $\rightarrow$ spikes)

- better explains option prices (flattening smile in short term options when underlying asset jumps)

- jump-diffusion models for bond and derivative pricing are used (e.g. Eraker et al. 03)

- IA jumps are used to model asset prices (Carr et al. 02, Ait-Sahalia and Jacod, 08)

But in presence of jumps, e.g. FA with $\lambda_t \equiv \lambda(R_t)$:

$$\frac{\sum_{i=1}^{n} K_{i-1} \frac{(\Delta_i R)^2}{\delta}}{\sum_{i=1}^{n} K_{i-1}} \approx E\left[\frac{(\Delta_i R)^2}{\delta} \middle| R_{t_{i-1}} = r\right] \approx \sigma^2(r) + \lambda(r)\sigma_J^2$$

information about σ (and μ) is carried only by the continuous part within $R_{t_i} - R_{t_{i-1}} \rightarrow$ discard the jump part

Literature in disentangling jumps from diffusion in nonparametric models, using discrete observations:

- Power, Bipower and multipower variation (Berman 1965, Barndorff-Nielsen and Shephard, 2004, 2006, Woerner 2005, 2006, Jacod 2008)
- Range theory (Christensen and Podolskij, 2006, Dobrev, 2007)
- Wavelets (Fan and Wang, 2007)
- Threshold technique (Mancini 2001, 2009, Jacod, 2008)

Threshold technique is efficient $\Rightarrow$

we use a combination of threshold and kernel techniques

♣ **alternative** kernel method to reconstruct the coefficients in presence of FA jumps with $\mu_t \equiv \mu(\mathbb{R}_t)$, $\lambda_t \equiv \lambda(R_t)$: Bandi and Nguyen, 2003, Johannes, 2004

$\star$ however threshold can also estimate jump sizes and times

$\star$ our estimator is the first kernel estimator of $\sigma^2(r)$ in presence of IA jumps

The intuition of the threshold approach: FA jumps

When $\delta \to 0$, diffusive variations go to zero, while jumps do not.

$$\text{Paul Lévy law} \Longrightarrow \text{a.s.} \quad \lim_{\delta \to 0} \sup_{i \in \{1,\dots,n\}} \frac{|\Delta_i W|}{\sqrt{2\delta \log \frac{1}{\delta}}} \leq 1.$$

The stochastic integral $\sigma.W$ is a time changed Brownian motion

$$\Downarrow$$

a.s. for small δ

$$\sup_i \frac{|\int_{t_{i-1}}^{t_i} \mu_s ds + \int_{t_{i-1}}^{t_i} \sigma_s dW_s|}{\sqrt{2\delta \log \frac{1}{\delta}}} \leq \Lambda < \infty$$

Remark: $\Lambda = \Lambda\Big(\sup_{s \in [0,T]} |\sigma(R_s)| \Big)$

choose a threshold $\vartheta(\delta) > 2\delta \log \frac{1}{\delta}$
if $(\Delta_i R)^2 > \vartheta(\delta) > 2\delta \log \frac{1}{\delta} \Rightarrow$ some jumps occurred

Theorem (Mancini, 2001 and extensions) Suppose T fixed, FA jumps, $\vartheta(\delta)$ deterministic function

$$\lim_{\delta \to 0} \vartheta(\delta) = 0 \quad \text{and} \quad \lim_{\delta \to 0} \frac{\delta \log \frac{1}{\delta}}{\vartheta(\delta)} = 0$$

$$\Downarrow$$

a.s. $\exists \bar{\delta}(\omega) > 0$ such that $\forall \delta < \bar{\delta}(\omega)$ we have $\forall i = 1, ..., n$,

$$I_{\{\Delta_i N \neq 0\}}(\omega) = I_{\{(\Delta_i R)^2 > \vartheta(\delta)\}}(\omega).$$

•

The proposed estimator of $\sigma^2(r)$: when J is FA

Theorem Let R be a FA jump-diffusion process, assume:

- regularity of $\sigma(r)$;
- as $\delta \to 0$, $\vartheta(\delta) \to 0$ and $\left(\delta \ln \frac{1}{\delta}\right)/\vartheta(\delta) \to 0$;
- conditions on how $h, \delta \to 0$ and $n \to \infty$;
- regularity of K (e.g. Gaussian)

$$\hat{\sigma}_n^2(r) \doteq \frac{\sum_{i=1}^n K\left(\frac{R_{t_{i-1}}-r}{h}\right) (\Delta_i R)^2 I_{\{(\Delta_i R)^2 \le \vartheta(\delta)\}}}{\sum_{i=1}^n K\left(\frac{R_{t_{i-1}}-r}{h}\right) \delta}$$

for all r visited by R

$$\sqrt{nh}\Big(\hat{\sigma}_n^2(r) - \sigma^2(r)\Big) \xrightarrow{st} M\mathcal{N}\left(0, 2\, \frac{\sigma^6(r) L_T^{\star\star}(r)}{(L_T^{\star})^2(r)}\right)$$

$M\mathcal{N}(0, U^2)$ mixed normal law with stochastic variance U^2

$L_T^{\star}(r), L_T^{\star\star}(r)$ estimable modified versions of the local time of R

Remark

$$\vartheta_t(\delta) = c_t \cdot \bar{\vartheta}(\delta) \text{ is allowed,}$$

with c_t stoch proc. bdd and bdd away from zero

Extension when J is IA

We need $\nu_t(\omega, dx)$ is stable-like for x around the origin, i.e.

$$\nu_t(\omega, dx) = \Big(\frac{A}{x^{1+\alpha}} I_{\{x>0\}} + \frac{B}{|x|^{1+\alpha}} I_{\{x<0\}}\Big)$$

and $\tilde{J}_2$ has finite variation: a.s. $\sum_{s \le T} |\Delta \tilde{J}_{2s}| < \infty \Rightarrow \exists\ L_t(r)$ local time for R for which occupation time formula still holds true

$\star$ assumptions satisfied e.g. if $\tilde{J}_2$ is

Gamma or Variance Gamma process,
α-stable process with $\alpha < 1$,
CGMY model with $Y < 1$.

Theorem Same asymptotic result for $\widehat{\sigma}_n^2(r)$ as soon as

- $\vartheta(\delta) = \delta^\eta, h = \delta^\phi$ with proper range for η, ϕ

Threshold setting

Any $\vartheta(\delta)$ satisfying the given conditions gives consistent $\widehat{\sigma}^2(r)$

However in practice $\delta > 0$ is fixed, and the particular choice of the function $\vartheta(\delta)$ can make a difference in the performance of the estimator.

An auxiliary model for the threshold

We use a GARCH(1,1) model. Even if misspecified, it asymptotically provides the optimal volatility forecast (Nelson and Foster, 1994). We fit:

$$R_{t_i} - R_{t_{i-1}} = \bar{R} + \sqrt{h_{t_i}} \cdot \varepsilon_{t_i} \forall i$$

$$h_{t_i} = \omega + \alpha(R_{t_{i-1}} - R_{t_{i-2}})^2 + \beta h_{t_{i-1}}$$

then we set

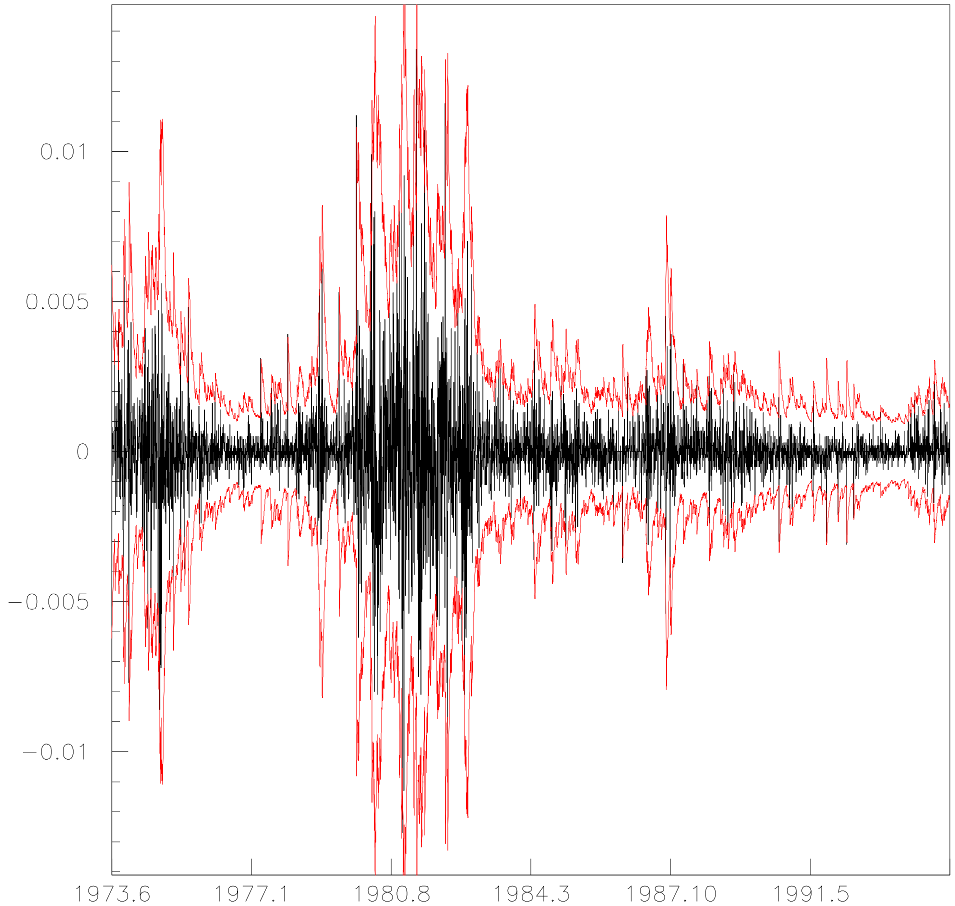
$$\vartheta_{t_i} = c \cdot h_{t_i}$$

with c = 9 (three standard deviations)

So that

high volatility forecast (persistency) $\Rightarrow$ high threshold

less diffusive variations are mis-interpreted as jumps


0.01
0.005
0
−0.005
−0.01
1973.6
1977.1
1980.8
1984.3
1987.10
1991.5

Checking the performance on simulations: FA jumps

$$dR_t = \kappa_1(b_t - R_t)dt - \bar{\kappa}\lambda R_t dt + \gamma\sqrt{R_t}dW_{1,t} + (e^{Z_t} - 1)R_t dN_t$$

$$db_t = \kappa_3(\theta - b_t)dt + \eta_2\sqrt{b_t}dW_{2,t}$$

where

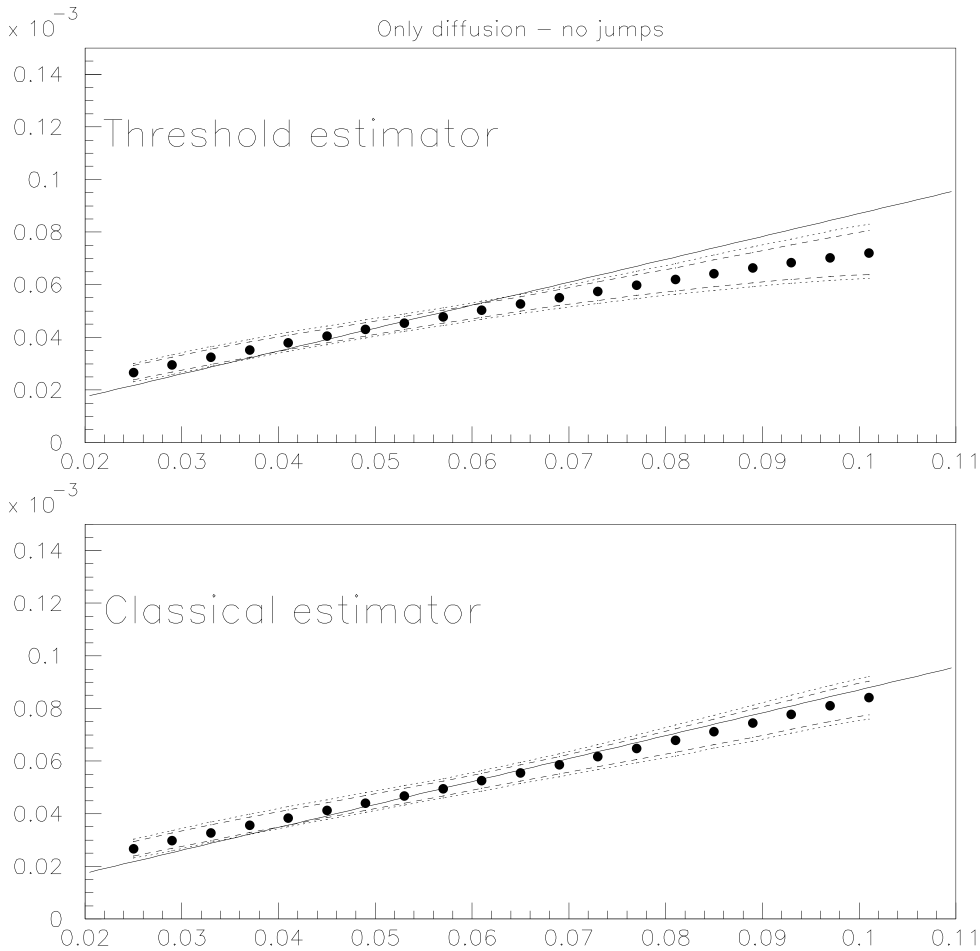
$$Z_t \simeq \mathcal{N}(\mu_J, \sigma_J^2), \quad \bar{\kappa} = E[e^{Z_t} - 1] = e^{\mu_J + \frac{1}{2}\sigma_J^2}$$

This is a modified version of the models estimated by Andersen, Benzoni and Lund (2004) on a time series of 3-months Treasury Bills annualized rates.

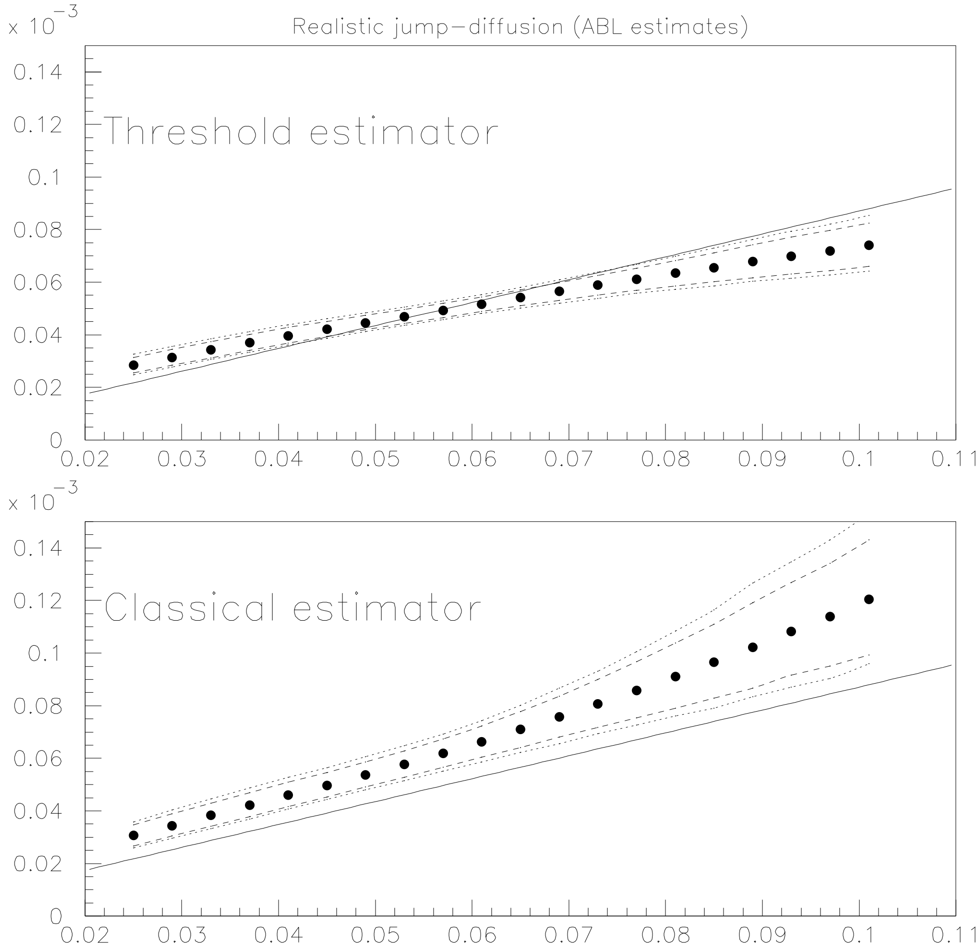
We use their estimates of the $SV_1J - SD$ model.

The starting values R_0 and b_0 are sampled from the unconditional distribution.

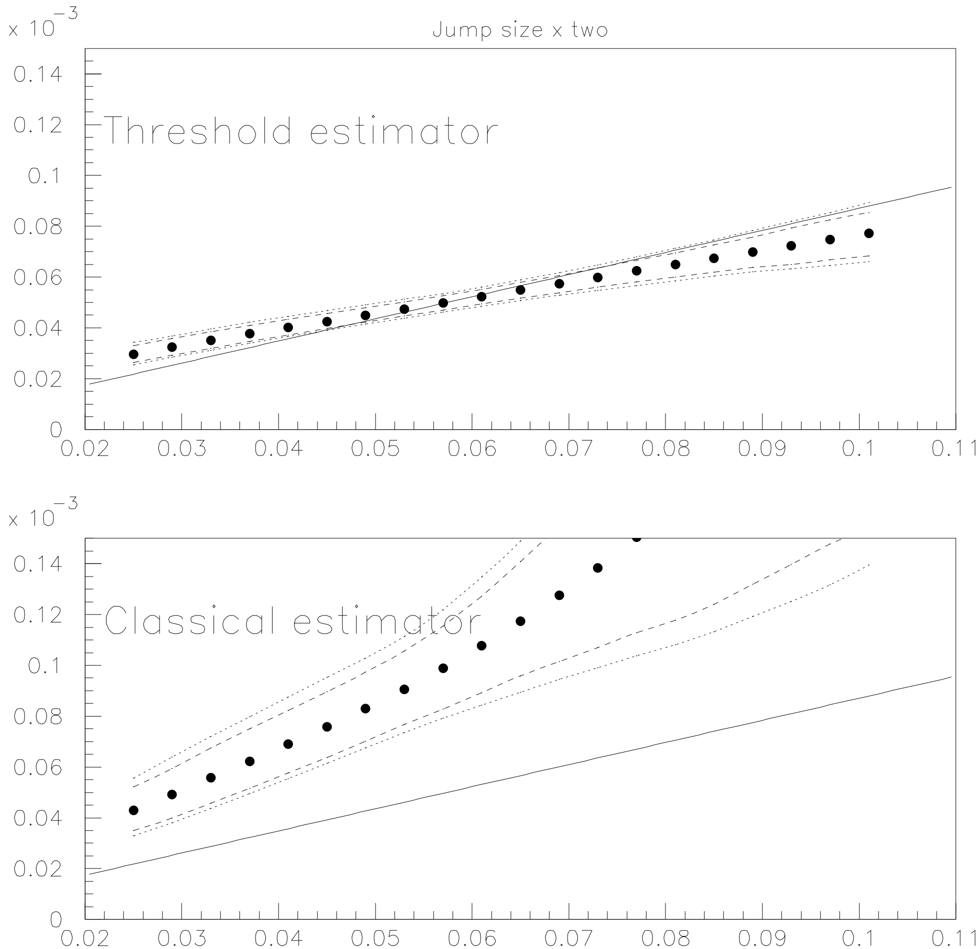
Simulations - no jumps ($\lambda = 0$)



Simulations - realistic jumps: $\hat{\lambda} = 5.3$ per year, $\hat{\sigma}_J = 0.026 \approx 5\%$ of current rate level



Simulations - doubled jump size $2\hat{\sigma}_J$



Estimating μ and λ: FA jumps, $\mu_t = \mu(R_t), \lambda_t = \lambda(R_t)$

estimate the drift and the jump intensity functions by letting $T \to \infty$ while $\delta = T/n \to 0$.

Estimator for the drift:

$$\widehat{\mu}_n(r) = \frac{\sum_{i=1}^{n} K_{i-1} \Delta_i R \cdot I_{\{(\Delta_i R)^2 \leq \vartheta(\delta)\}}}{\sum_{i=1}^{n} K_{i-1} \delta}$$

Estimation of the intensity function:

$$\widehat{\lambda}_n(r) = \frac{\sum_{i=1}^{n} K_{i-1} \cdot I_{\{(\Delta_i R)^2 > \vartheta(\delta)\}}}{\sum_{i=1}^{n} K_{i-1} \delta},$$

Theorem. Analogous assumptions as above plus $T \to \infty$

- R is Harris recurrent;
- $\forall \varepsilon > 0 \quad P\{|\gamma_\ell| < \varepsilon\} \leq c\varepsilon$ and $\{\gamma_\ell\}_\ell$ independent on N;
- conditions on how $\delta, h \to 0, n \to \infty$ now depend ALSO on how $L_T(r) \to \infty$ (as in Bandi & Nguyen 03).

For each r visited by R we have

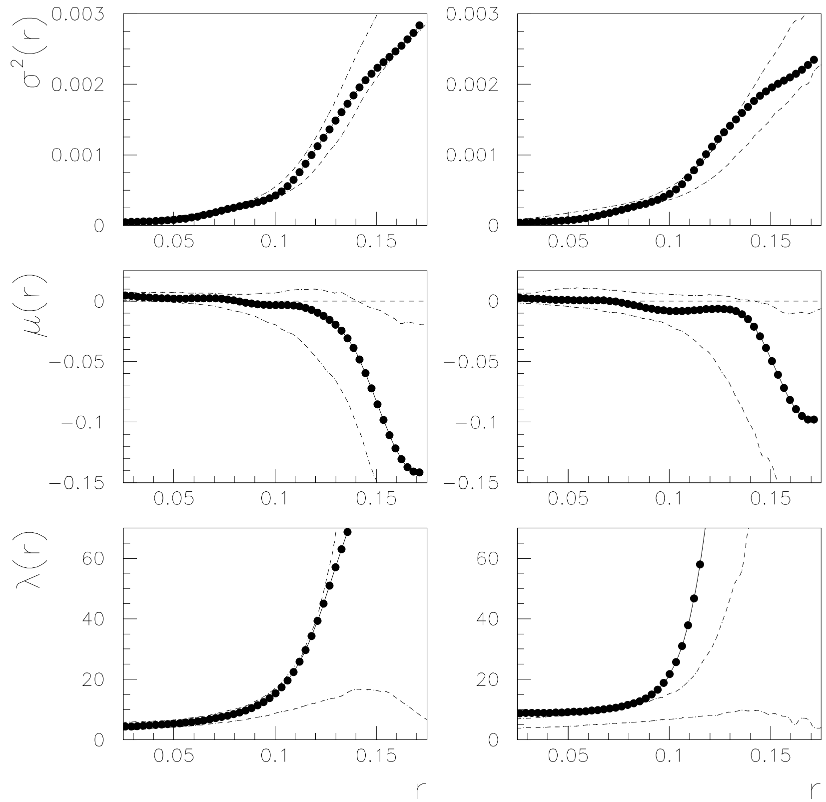
$$\sqrt{h\hat{L}_T^\star(r)}\Big(\hat{\mu}_n(r) - \mu(r)\Big) \xrightarrow{d} M\mathcal{N}(0, K_2\sigma^2(r)),$$

where $K_2 := \int_{\mathbb{R}} K^2(u)du$;

$$\sqrt{h\hat{L}_T^\star(r)}\Big(\hat{\lambda}_n(r) - \lambda(r)\Big) \xrightarrow{d} M\mathcal{N}(0, K_2\lambda(r)).$$

$\star$ as $T \to \infty$ it holds that $\forall r,\ L_T^\star(r) \to \infty$ (rate T if R is stationary)

Estimates on simulations constructed on Johannes (2004) data



Left: GARCH(1,1) thershold   right: iterating smoothing

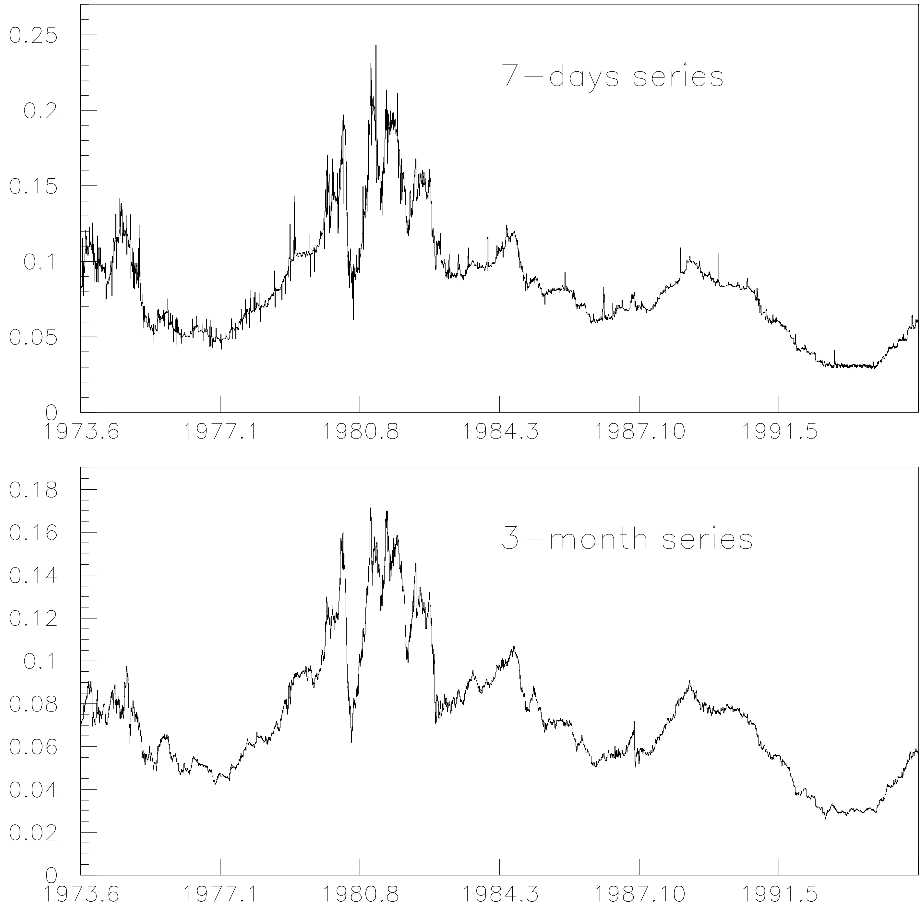
Empirical application: short rate modeling

Problem: which (term) rate to choose to proxy the (unobservable) spot rate?

$\star$ in the literature both 7-days rate (Eurodollar deposit, inter-bank rate) and 3-months rates (US Treasury Bills, market rate) are used

BUT

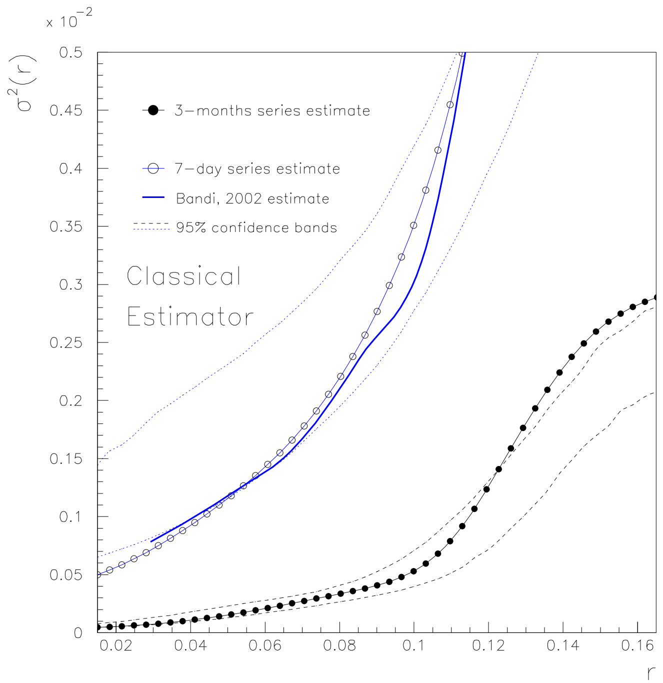
we show that they HAVE IMPORTANT DIFFERENCES!!



daily obs

⋆ 7-days rates: higher in level, more (periodic) spikes (liquidity effects)

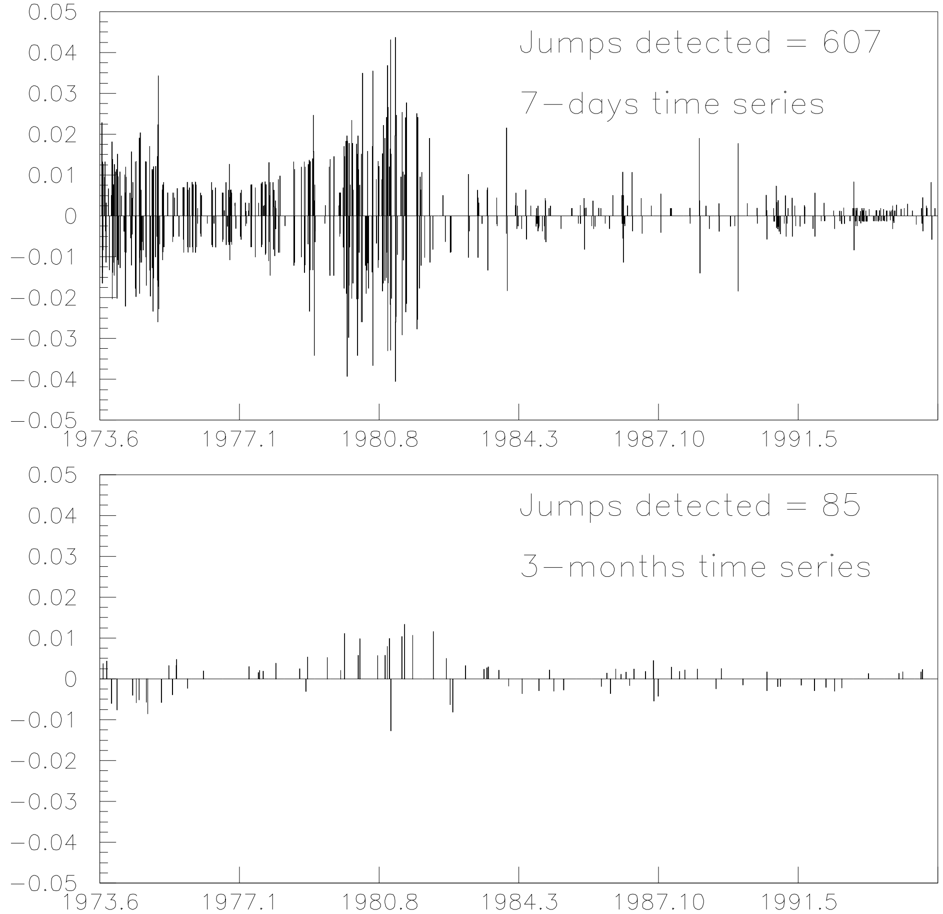
Estimation with Florens-Zmirou technique (assuming a diffusion underlying model)

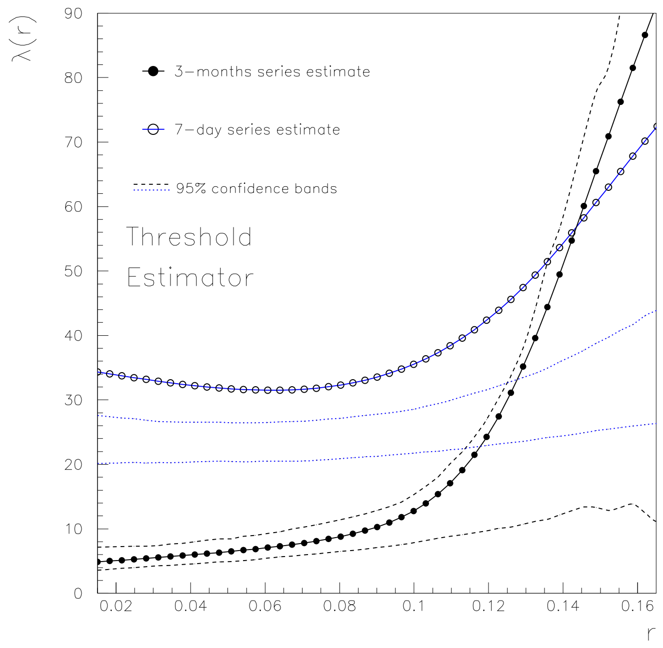


⋆ 7-days rates: spikes make variance increase

⋆ 7-days rates: bias (presence of jumps)

Estimation with threshold technique: jumps detected

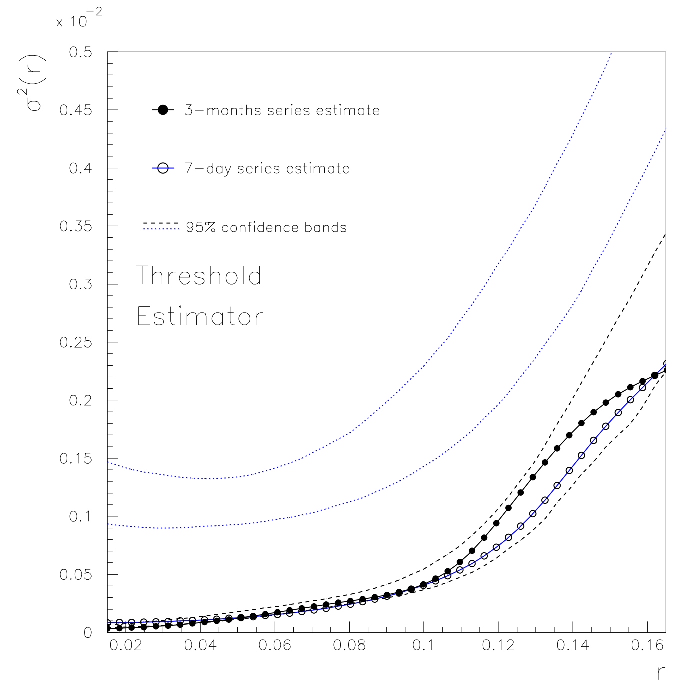




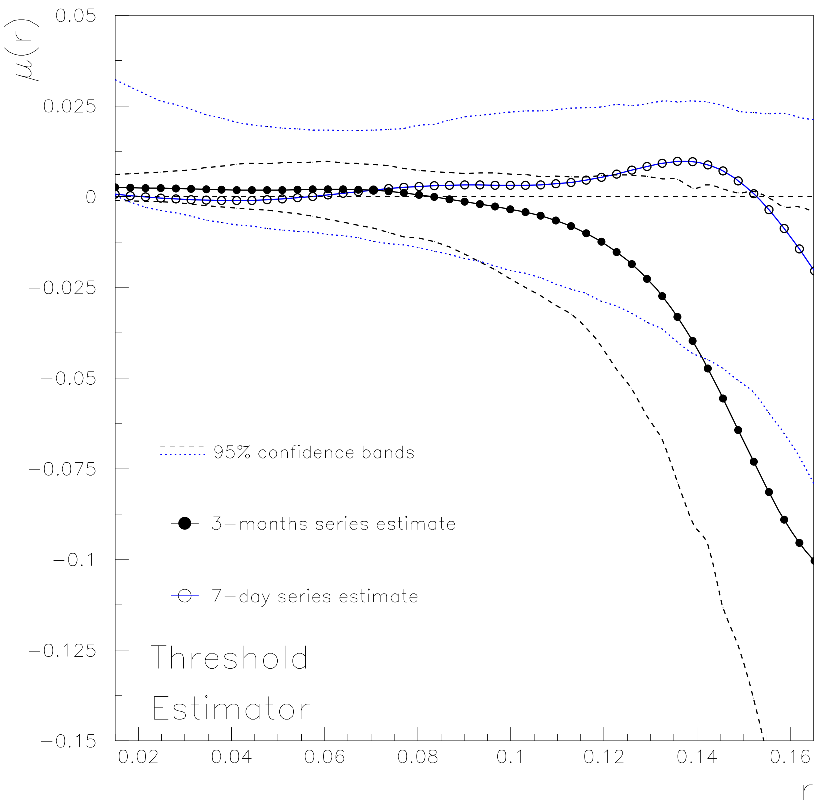
⋆ 7-days rates: our estimate (35 jumps/year) is quite consistent with the parametric estimates by Das (02) and correspond to need of liquidity

⋆ 3-months rates: our estimate (5 jumps/year) is consistent with the parametric estimates by ABL (04) and correspond to macroeconomic announcements

Estimation with threshold technique:



⋆ for the 3-months rates: classical and threshold estimators are quite consistent



⋆ quite same drift for rates up to 0.1

Our interpretation:

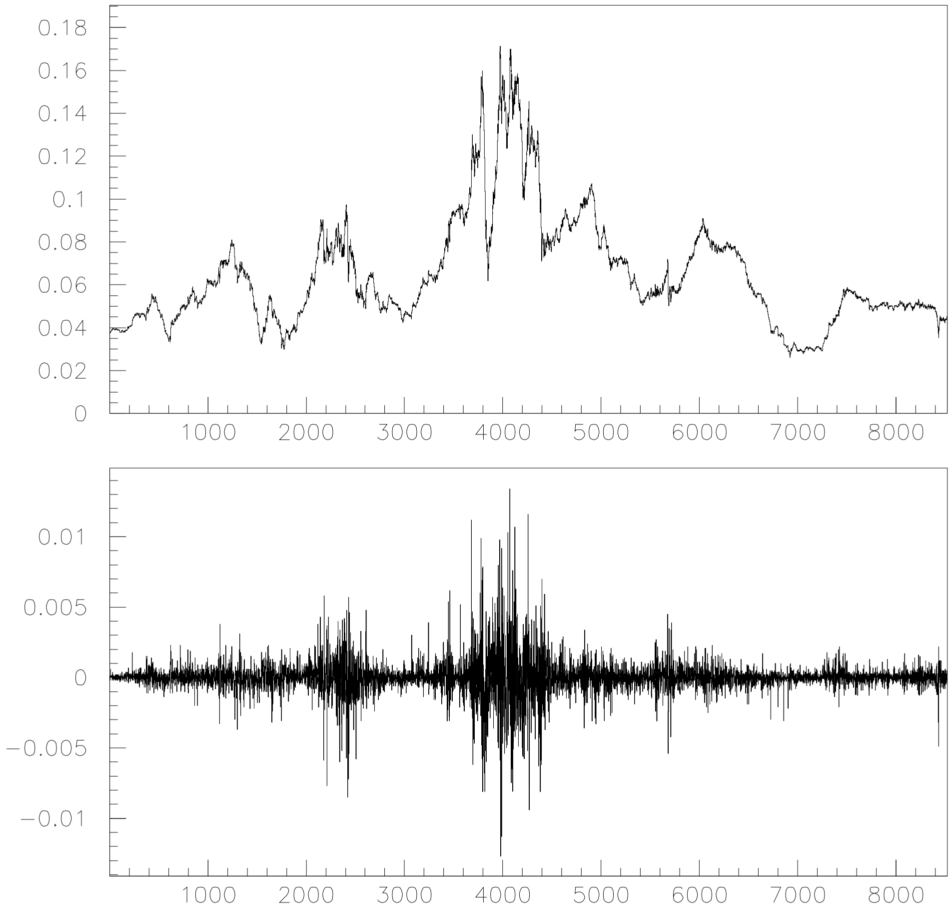
$\star$ weekly time series and monthly time series have the SAME underlying DIFFUSION PART
$\star$ however the JUMP COMPONENT of the 7-days time series is different, much more active

$$\Downarrow$$

use the 3month time series to proxy the spot rate: the 7-day rate is NOT a market rate

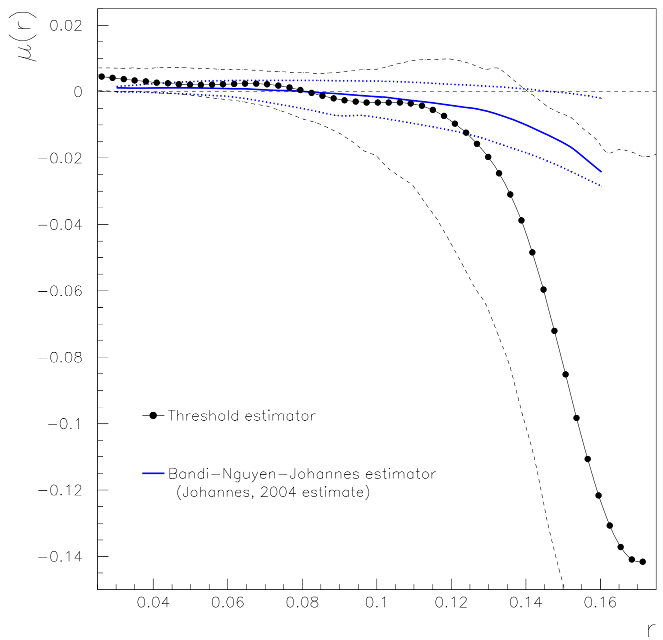
♣ Comparison with Bandi&Nguyen and Johannes

Johannes (2004) dataset (3 month rates 1965-1999, daily obs)



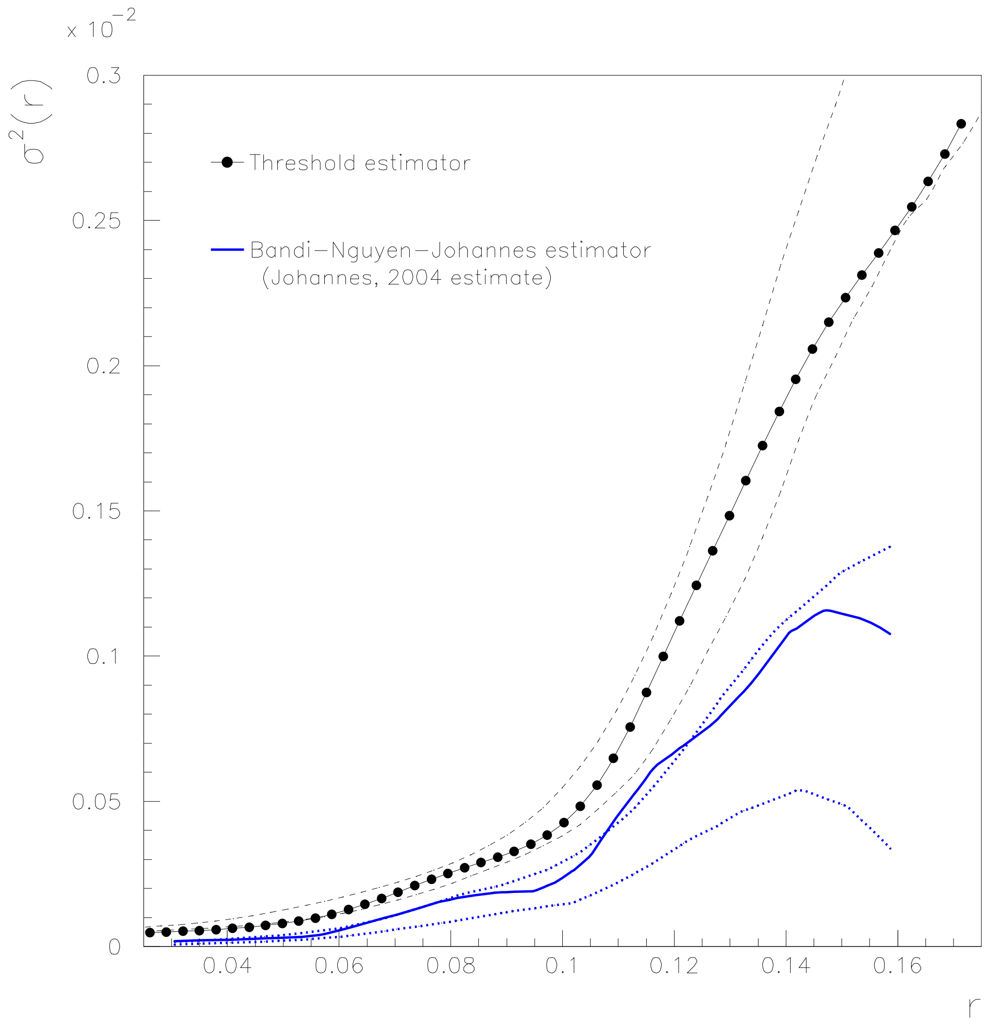
first differences

Comparison with Bandi-Nguyen-Johannes

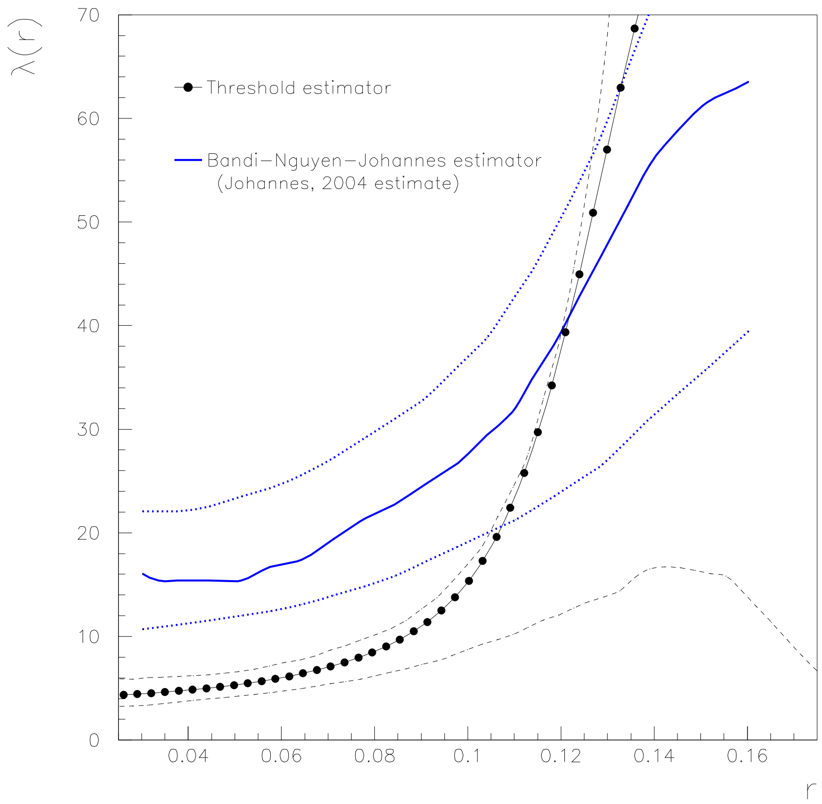


⋆ consistent

Comparison with Bandi-Nguyen-Johannes



Comparison with Bandi-Nguyen-Johannes



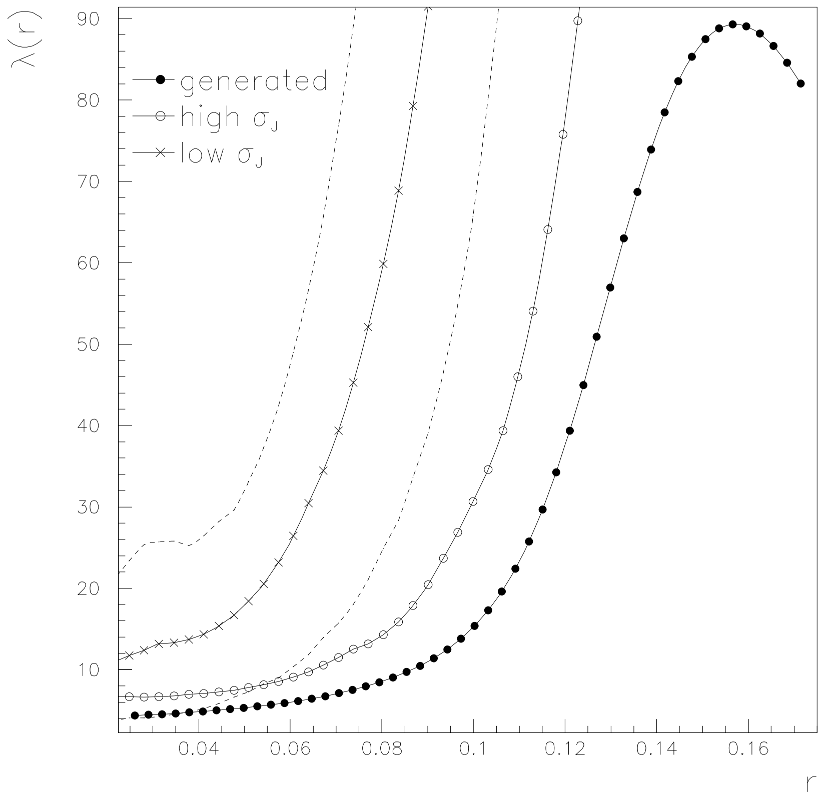
$\star$ to reach the same empirical second moment, BNJ attribute less to the diffusion part and more to the jump part than us, for levels of r up to 11%

BUT

⋆ 15-30 jumps/year are levels parametrically estimated by Das (02) for the 7-days rates, not for the 3-months

⋆ 5-10 jumps/year is consistent with parametric estimates in ABL (04) for the 3-months

⋆ Monte Carlo simulations show that Johannes estimated jump intensity is upper biased while the threshold estimator is not



$\Downarrow$

threshold is preferable

Conclusions

- We propose nonparametric estimators to reconstruct the coefficients in univariate jump-diffusion models, based on threshold-kernel estimation.

- The threshold can be set using standard econometric techniques (GARCH modeling).

- Monte Carlo simulations show that the proposed estimator is superior to estimators which do not consider jumps

- We show that the 7-day time series is characterized by a supplementary jump component (due to liquidity reasons) which is absent on the 3-months time series.
  The 7-days rate cannot be used as a proxy for the spot rate

- Our estimators provide different and more convincing results than those obtained so far on jump-diffusion models