Estimating the jump measure of a Lévy process

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Outline

Lévy-Khintchine formula

Lévy-Khintchine characterisation

The characteristic function of a Lévy process L is given by:

$$\varphi_t(u) := \mathbb{E}\left[e^{iuL_t}\right] = \exp(t\psi(u))$$
$$\psi(u) := -\frac{\sigma^2}{2}u^2 + ibu + \int_{-\infty}^{\infty}(e^{iux} - 1 - \tfrac{iux}{1+x^2})\,\nu(dx)$$

with volatility $\sigma \geqslant 0$, drift $b \in \mathbb{R}$ and jump measure ν.

(ν is σ-finite with $\int(1 \wedge x^2)\nu(dx) < \infty$)

Lévy-Itô decomposition if $\nu(\mathbb{R}) < \infty$:

$$L_t = \sigma W_t + \tilde{b}t + \sum_{k=1}^{N_t} Y_k$$

with Brownian motion W_t, $\tilde{b} = b - \int \frac{x}{1+x^2}\nu(dx)$, a Poisson process N_t of intensity $\lambda := \nu(\mathbb{R})$ and jumps $Y_k \sim \nu/\lambda$.

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Statistical problem

Observations

$(L_{t_0}, L_{t_1}, \dots, L_{t_n})$ with $t_{i+1} - t_i = \Delta$ equidistant.

Assumptions

$T = n\Delta \to \infty$ (long-time asymptotics),
$\Delta = \Delta_n \to 0$ fast (*high frequency*)
or $\Delta > 0$ fixed resp. $\Delta = \Delta_n \to 0$ slowly (*low frequency*).

Goal

Estimate the triplet (σ^2, b, ν) with

- $\sigma^2 \geqslant 0$;
- $b \in \mathbb{R}$;
- measure ν with interest in $\int f \, d\nu$ for different integrands f.

High-frequency estimation

Idea:

Large increments $X_k := L_{t_k} - L_{t_{k-1}}$ indicate *a* jump

(larger than diffusive oscillations, i.e. $\gg \sqrt{\Delta \log\log \Delta^{-1}}$)

Figueroa-Lopez, Houdré (2006):

$g_\nu \in L^2$ density of ν, $D \Subset \mathbb{R}\setminus\{0\}$, (φ_j) ONB in $L^2(D)$:

$$\hat{g}_\nu := \sum_{j=1}^{J} \hat{\beta}_j \varphi_j \text{ with } \hat{\beta}_j := T^{-1} \sum_{k=1}^{n} \varphi_j(X_k)$$

If g_ν has Sobolev-regularity s on D and $\Delta_T \to 0$ sufficiently fast:

$$\mathbb{E}[\|\hat{g}_\nu - g_\nu\|_{L^2(D)}^2] \lesssim T^{-2s/(2s+1)}$$

$$\mathbb{E}\Big[\Big(\int_D f\hat{g}_\nu - \int_D fg_\nu\Big)^2\Big] \lesssim T^{-1} \text{ for all } f \in L^4$$

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If g_ν has Sobolev-regularity s on D and $\Delta_T \to 0$ sufficiently fast:

$$\mathbb{E}[\|\hat{g}_\nu - g_\nu\|^2_{L^2(D)}] \lesssim T^{-2s/(2s+1)}$$

$$\mathbb{E}\Big[\Big(\int_D f\hat{g}_\nu - \int_D f g_\nu\Big)^2\Big] \lesssim T^{-1} \text{ for all } f \in L^4$$

Idea for low-frequency estimation

Transformation to density estimation

$X_k := L_{t_k} - L_{t_{k-1}}$, $k = 1, \ldots, n$, are i.i.d.
with characteristic function

$$\varphi(u) = \exp\Big(\Delta\Big(-\frac{\sigma^2}{2}u^2 + ibu + \int_{-\infty}^{\infty}(e^{iux} - 1 - \tfrac{iux}{1+x^2})\,\nu(dx)\Big)\Big)$$

Empirical characteristic function

$$\varphi_n(u) := \frac{1}{n}\sum_{k=1}^{n} e^{iuX_k} \xrightarrow{n\to\infty} \varphi(\sigma^2, b, \nu; u) \text{ under } \mathbb{P}_{(\sigma^2,b,\nu)}$$

Minimum-distance estimator

$$(\hat\sigma_n^2, \hat b_n, \hat\nu_n) := \operatorname{arginf}_{(\sigma^2,b,\nu)} d\big(\varphi_n, \varphi(\sigma^2, b, \nu; \bullet)\big)$$

Lévy-Ornstein-Uhlenbeck-type process, option calibration:
Jongbloed/van der Meulen/van der Vaart 2005, Cont/Tankov 2004

Consistency for $\Delta > 0$ fixed

Assumptions on the distance

$$\lim_{n\to\infty} d\left(\varphi_n, \varphi(\sigma^2, b, \nu; \bullet)\right) = 0 \quad \mathbb{P}_{(\sigma^2,b,\nu)}\text{-a.s.}$$

$$\lim_{m\to\infty} d(\varphi^{(m)}, \varphi) = 0 \Rightarrow \lim_{m\to\infty} \int_s^t \varphi^{(m)}(u)\,du = \int_s^t \varphi(u)\,du \quad \forall s,t \in \mathbb{R}\,.$$

Example: $d(f,g) = (\int |f-g|^p w)^{1/p}$ with $w \in L^1$, $w > 0$

Then:

$$d(\varphi(\hat\sigma_n^2, \hat b_n, \hat\nu_n; \bullet), \varphi(\sigma^2, b, \nu; \bullet)) \leqslant d(\varphi(\hat\sigma_n^2, \hat b_n, \hat\nu_n; \bullet), \hat\varphi_n) + d(\hat\varphi_n, \varphi(\sigma^2, b, \nu; \bullet)) \leqslant 2d(\hat\varphi_n, \varphi(\sigma^2, b, \nu; \bullet)) \xrightarrow{\text{a.s.}} 0$$

and therefore $\int_s^t \varphi(\hat\sigma_n^2, \hat b_n, \hat\nu_n; u)du \xrightarrow{\text{a.s.}} \int_s^t \varphi(\sigma^2, b, \nu; u)du$.
This implies $\mathbb{P}_{(\hat\sigma_n^2, \hat b_n, \hat\nu_n)} \Rightarrow \mathbb{P}_{(\sigma^2,b,\nu)}$ a.s.

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$$\begin{aligned} d(\varphi(\hat\sigma_n^2, \hat b_n, \hat\nu_n; \bullet), &\varphi(\sigma^2, b, \nu; \bullet)) \\ &\leqslant d(\varphi(\hat\sigma_n^2, \hat b_n, \hat\nu_n; \bullet), \hat\varphi_n) + d(\hat\varphi_n, \varphi(\sigma^2, b, \nu; \bullet)) \\ &\leqslant 2d(\hat\varphi_n, \varphi(\sigma^2, b, \nu; \bullet)) \xrightarrow{\text{a.s.}} 0 \end{aligned}$$

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Consistency II

Does $\mathbb{P}_{(\hat\sigma_n^2,\hat b_n,\hat\nu_n)} \overset{n\to\infty}{\Longrightarrow} \mathbb{P}_{(\sigma^2,b,\nu)}$ imply $\hat\sigma_n^2 \to \sigma^2$, $\hat b_n \to b$, $\hat\nu_n \to \nu$?

Proposition (Gnedenko/Kolmogorov 1949)

$\mathbb{P}_{(\sigma_n^2,b_n,\nu_n)} \Rightarrow \mathbb{P}_{(\sigma^2,b,\nu)} \iff b_n \to b$ and $\nu_{\sigma,n} \Rightarrow \nu_\sigma$ weakly

with the finite measure $\nu_\sigma(dx) := \sigma^2\delta_0(dx) + \frac{x^2}{1+x^2}\nu(dx)$.

Corollary

The minimum-distance estimator is consistent for (b,ν_σ).

Proposition

The volatility σ^2 cannot be estimated uniformly consistently: no estimator $\tilde\sigma_n^2$ can satisfy for any $\sigma^2 \geqslant 0$

$$\forall \varepsilon > 0 : \lim_{n\to\infty} \sup_{b,\nu} \mathbb{P}_{(\sigma^2,b,\nu)}(|\tilde\sigma_n^2 - \sigma^2| > \varepsilon) = 0.$$

The same is true, e.g., for the Blumenthal-Getoor index.

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Kolmogorov representation

Assumption: $\mathbb{E}[L_t^2] < \infty$ or equivalently $\int x^2 \nu(dx) < \infty$.
Canonical representation (Kolmogorov 1932):

$$\varphi(u) = \exp\left(\Delta\left(-\frac{\sigma^2}{2}u^2 + iub + \int_{-\infty}^{\infty}(e^{iux} - 1 - iux)\,\nu(dx)\right)\right)$$
$$= \exp\left(\Delta\left(iub + \int_{-\infty}^{\infty}\frac{e^{iux} - 1 - iux}{x^2}\,\nu_\sigma(dx)\right)\right)$$

with $\nu_\sigma(dx) = \sigma^2\delta_0(dx) + x^2\nu(dx)$.
Note: here $\mathbb{E}[L_1] = b$ and $\mathrm{Var}(L_1) = \nu_\sigma(\mathbb{R})$ hold.

Important observation:

$$\Delta^{-1}\frac{d^2}{du^2}\log(\varphi(u)) = \int_{-\infty}^{\infty}\frac{d^2}{du^2}\frac{e^{iux} - 1 - iux}{x^2}\,\nu_\sigma(dx) = -\mathcal{F}\nu_\sigma(u)$$

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Two constructions of estimators

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1. Plug-in + spectral cut-off at $U > 0$: (cf. Belomestny/Reiß 2006)

$$\mathcal{F}\hat{\nu}_\sigma(u) := -\Delta^{-1}\frac{d^2}{du^2}\log(\varphi_n(u))\mathbf{1}_{[-U,U]}(u), \quad u \in \mathbb{R}$$

2. Minimum-distance estimator with a C^2-distance to control

$$\left|\frac{\varphi_n''(u)\varphi_n(u)-\varphi_n'(u)^2}{\varphi_n(u)^2} - \frac{\varphi''(u)\varphi(u)-\varphi'(u)^2}{\varphi(u)^2}\right|$$

Two-step estimation procedure

Preliminary estimator:

$$\tilde{b}_n = T^{-1} L_T , \quad \mathcal{F}\tilde{\nu}_{\sigma,n} = -\Delta^{-1} \log(\varphi_n)'' \mathbf{1}(|\varphi_n| \geqslant \kappa n^{-1/2})$$

Minimum-distance estimator:

optimize locally around $(\tilde{b}_n, \tilde{\nu}_{\sigma,n})$ to find

$$(\hat{b}_n, \hat{\nu}_{\sigma,n}) := \operatorname{arginf}_{(b,\nu_\sigma)} d_{C^2}\big(\varphi_n, \varphi(b, \nu_\sigma; \bullet)\big)$$

where (weighting $w(u) = \log(e + |u|)^{-1/2-\varepsilon}$)

$$d_{C^2}(g, h) = \max_{j=0,1,2} \Delta^{-j/2} \sup_{u\in\mathbb{R}} \Big(|g^{(j)}(u) - h^{(j)}(u)| w(u)\Big)$$

Estimation error

How fast can we estimate $\int f\,d\nu_\sigma$?

Assume $\mathbb{E}[|L_t|^{4+\varepsilon}] < \infty$. For integrands f of regularity s ($(1+x^2)^{s/2}\mathcal{F}f \in L^1$) we have

$$\left|\int f\,d\hat{\nu}_{\sigma,n} - \int f\,d\nu_\sigma\right| \lesssim_P v(T,\Delta) \vee T^{-1/2}$$

with

$$\begin{aligned} &\sigma > 0: && v(T,\Delta) = (\Delta^{-1}\log(T))^{-s/2} \\ &\sigma = 0,\ |\varphi(u)| \gtrsim e^{-\Delta\alpha|u|}: && v(T,\Delta) = (\Delta^{-1}\log(T))^{-s} \\ &\sigma = 0,\ |\varphi(u)| \gtrsim (1+|u|)^{-\beta\Delta}: && v(T,\Delta) = T^{-s/(2\beta\Delta)} \end{aligned}$$

(times a log-factor)

These rates are *optimal* in a minimax-sense.

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Estimation vs. discretisation error

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$$|\int f\, d\hat{\nu}_{\sigma,n} - \int f\, d\nu_\sigma| \lesssim_P T^{-1/2}$$

provided

$$\sigma > 0: \qquad \Delta \lesssim T^{-1/s}$$

$$\sigma = 0,\ |\varphi(u)| \gtrsim e^{-\Delta\alpha|u|}: \qquad \Delta \lesssim T^{-1/(2s)}$$

$$\sigma = 0,\ |\varphi(u)| \gtrsim (1+|u|)^{-\beta\Delta}: \qquad \Delta < s/\beta$$

Example:
Gamma-processes ($\varphi(u) = (1 - iu/\lambda)^{-p\Delta}$) require $\Delta < s/p$; compound Poisson processes do not require Δ to be small.

Main tool in the proof ($\Delta = 1$)

Consider the k-th derivative of the *empirical characteristic process*

$$C_n^{(k)}(u) = \frac{d^k}{du^k}\Big(n^{-1/2}\sum_{j=1}^{n}(e^{iuX_j} - \mathbb{E}[e^{iuX_j}])\Big).$$

Then with the empirical process $\mathbb{G}_n = n^{1/2}(\mathbb{F}_n^{X_1,\dots,X_n} - F^X)$

$$\sup_{u\in\mathbb{R}}|C_n^{(k)}(u)|w(u) = \sup_{h\in H}\int h \, d\,\mathbb{G}_n$$

with

$$H = \Big\{x \mapsto w(u)\frac{d^k}{du^k}e^{iux} : \; u \in \mathbb{R}\Big\}.$$

Main tool in the proof (ctd.)

$$H = \left\{ x \mapsto w(u) \frac{d^k}{du^k} e^{iux} \, : \, u \in \mathbb{R} \right\}$$

Consequently, empirical process theory yields

$$\boxed{\sup_n \mathbb{E}\left[\sup_{u \in \mathbb{R}} |C_n^{(k)}(u)| w(u) \right] \lesssim \mathbb{E}[(X_j)^{2k}] + J_{[]}(H)}$$

with $J_{[]}$ the *bracketing entropy* of H. For suitable brackets

$$h_j^{\pm}(x) = \left(w(u_j) \tfrac{d^k}{du^k} e^{iu_j x} \pm \varepsilon |x|^k \right) \mathbf{1}_{[-M,M]}(x) + \|w\|_\infty |x|^k \mathbf{1}_{[-M,M]^c}(x)$$

this $J_{[]}$ is finite for $w(u) \sim (\log(u))^{-1/2-\varepsilon}$ and $\mathbb{E}[|X_j|^{2k+\varepsilon}] < \infty$.

Decay of φ versus singularity of ν

Lemma:

Consider a Lévy process with a characteristic function of at most polynomial decay: $|\varphi_t(u)| \gtrsim (1+u^2)^{-\beta t}$. Then the Lévy measure ν satisfies for any $\varepsilon > 0$

$$\int_{[-1,1]} \log(|x|^{-1})^{-2-\varepsilon}\nu(dx) < \infty.$$

This implies $\sup_{u\in\mathbb{R}}|\varphi'(u)/\varphi(u)| < \infty$ provided $\varphi \in C^1$.

Example:

The Lévy measure of the Gamma process has a density of order x^{-1} around zero and $\int_{[-1,1]} \log(|x|^{-1})^{-2-\varepsilon}|x|^{-1}dx < \infty$.

Intuition:

The more singular the Lévy measure at zero, the rougher the sample paths, but the smoother the transition density.

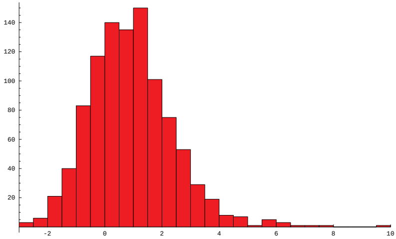
Example

Simulation:

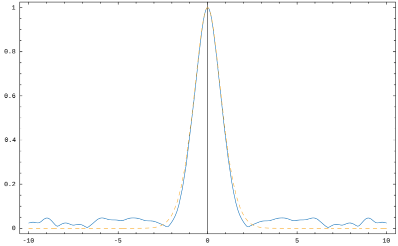
Superposition of Brownian motion and Exp(1)-process

$\sigma = 1$, $b = 1$, $\nu(dx) = x^{-1}e^{-x}\,\mathbf{1}_{\{x>0\}}\,dx$

$\Delta = 1$, $n = 1000$



Histogram of the data.

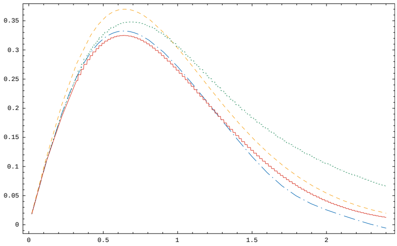


modulus of the empirical (solid blue) and true (dashed orange) characteristic function.

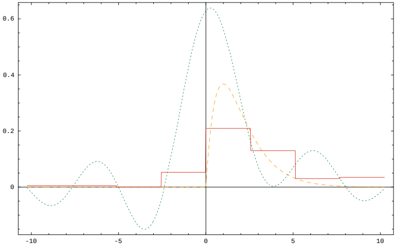
Estimation for the example

$\widehat{b}_n = 0.922$ (true $b = 1$), $\widehat{\nu}_{\sigma,n}(\{0\})^{1/2} = 1.092$ (true $\sigma = 1$)

Estimate of $\nu([1,\infty)) = \int_1^\infty x^{-2}\nu_\sigma(dx)$ is 0.16 (true: 0.22).
The HF-estimator $\frac{1}{n}\sum_{i=1}^n \mathbf{1}_{\{L_i - L_{i-1} \geqslant 1\}}$ yields 0.46.



Im(φ_n) (blue), Im(φ) (orange), Im($\tilde{\varphi}_n$) (green) and Im($\hat{\varphi}_n$) (red).



density of $\tilde{\nu}_\sigma$ (green), $\hat{\nu}_\sigma$ (solid) and ν_σ (orange);
$\tilde{\nu}_\sigma$ has no point mass in zero.

Summary

- High. vs. low-frequency observations of a Lévy process
- LF-estimation based on minimum-distance fit of characteristic function.
- LF-estimation of σ^2 cannot be uniformly consistent.
- Adjust to any observation distance Δ.
- LF-rate for $b = \mathbb{E}[L_1]$ is parametric.
- LF-rate for $\int f\, d\nu_\sigma$ depends on the decay of φ.
- LF-rate $T^{-1/2}$ for $\int f\, d\nu_\sigma$ if $\Delta < \overline{\Delta}(\varphi, s; T)$ (automatic).

Thank you very much for your attention!

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