Limit theorems for bipower variation of semimartingales

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Outline of the talk

- Outline of the talk
- Estimation of integrated volatility
- Two central limit theorems
- References

Setting

- We are given an Ito semimartingale X on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}_+}, P)$, thus

$$X_t = X_0 + \int_0^t b_u \, du + \int_0^t \sigma_u \, dW_u + (\delta 1_{\{|\delta| \le 1\}}) \star (\underline{\mu}_t - \underline{\nu}_t) + (\delta 1_{\{|\delta| > 1\}}) \star \underline{\mu}_t,$$

where W is a Brownian motion and $\underline{\mu}$ and $\underline{\nu}$ are a Poisson random measure on $\mathbb{R}_+ \times E$ and its compensator $\underline{\nu}(du, dx) = du \times \lambda(dx)$.

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where W is a Brownian motion and $\underline{\mu}$ and $\underline{\nu}$ are a Poisson random measure on $\mathbb{R}_+ \times E$ and its compensator $\underline{\nu}(du, dx) = du \times \lambda(dx)$.
- Standard assumptions:
  1. The drift b is optional and càglàd.
  2. The volatility σ is an Ito semimartingale itself and satisfies $\sigma > 0$ almost surely.
  3. The function δ is predictable and locally bounded by a family (γ_k) of non-negative functions such that $\int_E (1 \wedge \gamma_k^s(z)) \, \lambda(dz) < \infty$.

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  3. The function δ is predictable and locally bounded by a family (γ_k) of non-negative functions such that $\int_E (1 \wedge \gamma_k^s(z)) \, \lambda(dz) < \infty$.
- Aim: Being on a fixed time span $[0, T]$ and observing $X_{\frac{i}{n}}$ for $i = 0, \ldots, \lfloor nT \rfloor$ we are interested in estimating the quadratic variation

$$\int_0^t \sigma_u^2 \, ds + \sum_{u\leq t} |\Delta X_u|^2$$

  or parts thereof, typically the integrated volatility $\int_0^t \sigma_u^2 \, du$.

Realized volatility

- In the continuous case

$$X_t = X_0 + \int_0^t b_u \, du + \int_0^t \sigma_u \, dW_u$$

the standard estimator for the integrated volatility is the realized variance $RV(X)_t^n = \sum_{i=1}^{\lfloor nt \rfloor} |\Delta_i^n X|^2$, where $\Delta_i^n X = X_{\frac{i}{n}} - X_{\frac{i-1}{n}}$.

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- We have

$$RV(X)_t^n \xrightarrow{P} \int_0^t \sigma_u^2 \, du$$

and an associated central limit theorem

$$\sqrt{n}\Big(RV(X)_t^n - \int_0^t \sigma_u^2 \, du\Big) \xrightarrow{\mathcal{D}_{st}} \sqrt{2} \int_0^t \sigma_u^2 \, dW_u',$$

where W' is defined on an extension of $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}_+}, P)$ and independent of $\mathcal{F}$.

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- Mancini has proposed to use a truncated version of $RV(X)_t^n$, namely

$$TRV(X)_t^n = \sum_{i=1}^{\lfloor nt \rfloor} |\Delta_i^n X|^2 1_{\{|\Delta_i^n X| \le \alpha n^{-\varpi}\}}$$

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- Intuition: We are cutting off large increments as these are likely due to a jump within $[\frac{i-1}{n}, \frac{i}{n}]$.
- $TRV(X)_t^n$ is consistent for the integrated volatility and we obtain the same central limit theorem as before, as long as $s \leq \frac{4\varpi - 1}{2\varpi}$ (so in particular $s < 1$):

$$\sqrt{n}\Big(TRV(X)_t^n - \int_0^t \sigma_u^2 \, du \Big) \xrightarrow{\mathcal{D}_{st}} \sqrt{2} \int_0^t \sigma_u^2 \, dW_u'.$$

Multipower variation I

- Alternatively, one uses multipower variations, which are defined as

$$MV(X,\mathbf{r})_t^n = n^{|\mathbf{r}|/2-1} \sum_{i=1}^{\lfloor nt \rfloor - q+1} \prod_{j=1}^{q} |\Delta_{i+j-1}^n X|^{r_j},$$

with $\mathbf{r} = (r_1, \ldots, r_q)$ having non-negative components. We set $|\mathbf{r}| = r_1 + \ldots + r_q$, $\mathbf{r}_+ = \max(r_1, \ldots, r_q)$ and $\mathbf{r}_- = \min(r_1, \ldots, r_q)$.

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- Intuition: One pairs intervals containing jumps with those that do not contain jumps, and typically (depending on $\mathbf{r}$) these intervals do not play a role in the asymptotics.
- Let m_p denote the p-th absolute moment of a standard normal distribution and set $m_{\mathbf{r}} = \prod_{j=1}^{q} m_j$. Then

$$MV(X,\mathbf{r})_t^n \xrightarrow{P} m_{\mathbf{r}} \int_0^t \sigma_u^{|\mathbf{r}|} \, du,$$

  as long as $\mathbf{r}_+ < 2$.

Multipower variation II

- Central limit theorems have only been shown for specific choices of $\mathbf{r}$: Suppose that $\frac{s}{2-s} < \mathbf{r}_- \leq \mathbf{r}_+ < 1$. Then we have

$$\sqrt{n}\Big(MV(X,\mathbf{r})_t^n - m_{\mathbf{r}} \int_0^t \sigma_u^{|\mathbf{r}|}\ du\Big) \xrightarrow{\mathcal{D}_{st}} \sqrt{p(\mathbf{r})} \int_0^t \sigma_u^{|\mathbf{r}|}\ dW'_u$$

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  for some known function p.
- The two most prominent estimators for the integrated volatility in this context are $MV(X,(1,1))_t^n$ and $MV(X,(2/3,2/3,2/3))_t^n$. For both, we have convergence in probability (of a rescaled version) towards the integrated volatility, but only for the latter one we have a central limit theorem, as long as $\frac{s}{2-s} < \frac{2}{3}$.

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- Question: Is there no central limit theorem for bipower variation? Or has is simply not been proven yet? And if there is one, does it depend on the jumps?

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- Question: Is there no central limit theorem for bipower variation? Or has is simply not been proven yet? And if there is one, does it depend on the jumps?
- Notation:

$$V(X)^n_t = \sum_{i=1}^{\lfloor nt \rfloor} |\Delta^n_i X||\Delta^n_{i+1} X|.$$

Prerequisites

- Suppose we have $s \leq 1$, thus

$$X_t = X_0 + B_t + \int_0^t \sigma_u \, dW_u + \sum_{u \leq t} \Delta X_u$$

with $B_t = \int_0^t b_u \, du - \delta 1_{\{|\delta| \leq 1\}} \star \underline{\nu}_t$.

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- We define an appropriate extension of $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}_+}, P)$ supporting two independent sequences (U_{m+}) and (U_{m-}) of standard normally distributed random variables and an independent Brownian motion W', all independent of $\mathcal{F}$. Furthermore, (T_m) is a sequence of stopping times exhausting the jumps of X. We set

$$U'_t = \sum_{m: T_m \leq t} |\Delta X_{T_m}| \cdot \Big(\sigma_{T_m -} |U_{m-}| + \sigma_{T_m} |U_{m+}| \Big)$$

and

$$U''_t = \sqrt{1 + 2m_1^2 - 3m_1^4} \int_0^t \sigma_u^2 \, dW'_u.$$

CLT for bipower variation I

- Then we have

$$\sqrt{n}\Big(V(X)_t^n - m_1^2 \int_0^t \sigma_u^2 \, du\Big) \xrightarrow{\mathcal{D}_{st}} U_t' + U_t''.$$

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- Intuition: If there are only finitely many jumps on $[0, T]$, all jump times T_m and $T_{m'}$ satisfy $|T_m - T'_m| > \frac{2}{n}$ for large n. Thus, if a jump lies within $[\frac{i-1}{n}, \frac{i}{n}]$, it occurs in $V(X)^n_t$ as

$$|\Delta^n_i X| \cdot (|\Delta_{\frac{i-1}{n}} X| + |\Delta_{\frac{i+1}{n}} X|),$$

and the neighbouring increments are not affected by jumps. Using the approximations

$$\Delta^n_i X \approx \Delta_{T_m} X \quad \text{and} \quad \Delta_{\frac{i-1}{n}} X \approx \sigma_{T_m -} \Delta_{\frac{i-1}{n}} W,$$

we end up with the finite sum

$$\sqrt{n} \sum_{m: T_m \le t} |\Delta_{T_m} X| \cdot (\sigma_{T_m -} |\Delta_{\frac{i-1}{n}} W| + \sigma_{T_m} |\Delta_{\frac{i+1}{n}} W|) \rightarrow U'_t.$$

CLT for bipower variation II

- U_t' is not a martingale (unless X is continuous, of course), and thus plays the role of a bias in the limit. How can one get rid of the bias?

CLT for bipower variation II

- U'_t is not a martingale (unless X is continuous, of course), and thus plays the role of a bias in the limit. How can one get rid of the bias?
- By subtracting an estimator for it: Quite naturally, we estimate

$$U'_t = \sum_{m: T_m \leq t} |\Delta X_{T_m}| \cdot \Big(\sigma_{T_m -} |U_{m-}| + \sigma_{T_m} |U_{m+}| \Big)$$

by $\sqrt{n} V^*(X, \alpha, \varpi)^n_t$ with

$$V^*(X, \alpha, \varpi)^n_t = \sum_{i=1}^{\lfloor nt \rfloor - 1} |\Delta^n_i X| |\Delta^n_{i+1} X| \cdot \Big(1_{\{|\Delta^n_i X| \geq \alpha n^{-\varpi}\}} 1_{\{|\Delta^n_{i+1} X| < \alpha n^{-\varpi}\}} + 1_{\{|\Delta^n_i X| < \alpha n^{-\varpi}\}} 1_{\{|\Delta^n_{i+1} X| \geq \alpha n^{-\varpi}\}} \Big)$$

and look at the asymptotics of

$$\sqrt{n} \Big((V(X)^n_t - V^*(X, \alpha, \varpi)^n_t) - m_1^2 \int_0^t \sigma_u^2 \, du \Big).$$

CLT for truncated bipower variation

- Alternatively, we can look at the direct analogue of $TRV(X)_t^n$, namely

$$TV^*(X,\alpha,\varpi)_t^n = \sum_{i=1}^{\lfloor nt \rfloor -1} |\Delta_i^n X| 1_{\{|\Delta_i^n X| < \alpha n^{-\varpi}\}} |\Delta_{i+1}^n X| 1_{\{|\Delta_{i+1}^n X| < \alpha n^{-\varpi}\}}.$$

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$$\sqrt{n}\Big((V(X)^n_t - V^*(X,\alpha,\varpi)^n_t) - m_1^2 \int_0^t \sigma_u^2 \, du \Big) \xrightarrow{\mathcal{D}_{st}} U''_t,$$

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$$\sqrt{n}\Big(TV^*(X,\alpha,\varpi)_t^n - m_1^2 \int_0^t \sigma_u^2 \, du \Big) \xrightarrow{\mathcal{D}_{st}} U_t''.$$

- Note that this results holds for all $s \leq 1$ and irrespectively of the choice of ϖ.

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