Stability Properties of Networks with Interacting TCP Flows

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Joint work with

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A single long TCP flow

Congestion:

Adaptive Algorithm to regulate data transfers

Evolution of Throughput:

— **Linear Increase**

— **Multiplicative Decrease**

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Literature
Various asymptotic formulas for throughput

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Literature

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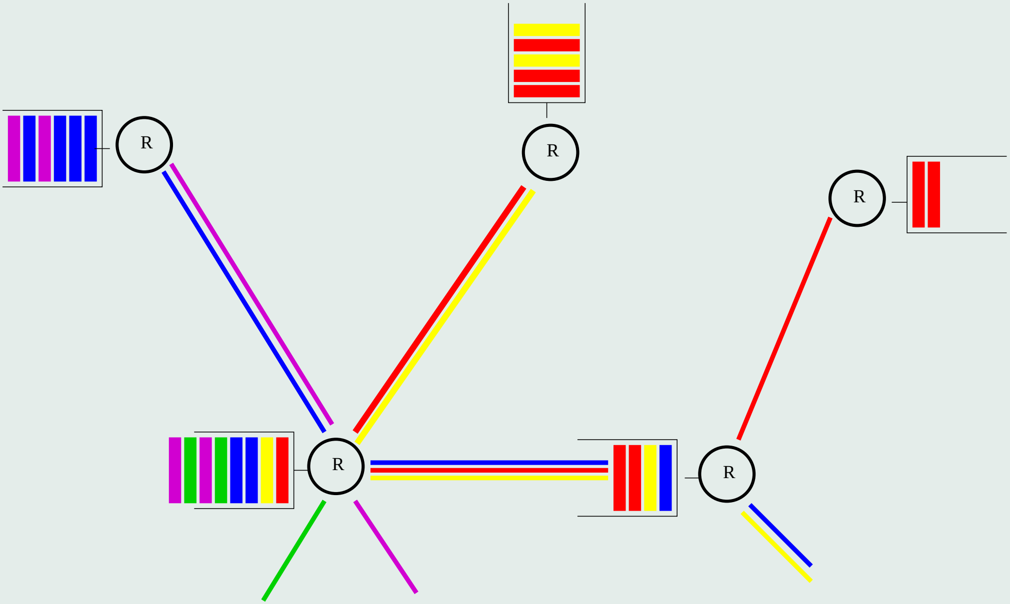
The Internet with ONE node is fully understood !

Coexistence of TCP Flows



Coexistence of TCP Flows

Literature

Mathematical Description of Coexistence:

Few rigorous results.

Deterministic Models

Coexistence of TCP Flows: an approach

As an optimisation problem

x_k routes of class $k \in 1, \ldots, K$:
receive throughput λ_k^0, such that (λ_k^0) achieves

$$\max_{\lambda \in \mathcal{C}} \sum_{k=1}^{K} x_k U_k(\lambda_k / x_k)$$

U_k utility function.

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Kelly, Maulloo and Tan (1998)
Massoulié and Roberts (1999)
Kelly and Williams (2004), Massoulié (2007).

Limits from Microscopic Dynamics

— (W_n) successive congestion window sizes of a single flow

— α loss rate.

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Theorem Dumas *et al.*(2002)

$$\lim_{\alpha \to 0} \left(\sqrt{\alpha} W_{\lfloor t/\sqrt{\alpha} \rfloor}, 0 \le t \le +\infty \right) = \left(\overline{W}(t), 0 \le t \le +\infty \right)$$

$(\overline{W}(t))$ is a Markov process with generator

$$\Omega(f)(x) = f'(x) + x(f(rx) - f(x))$$

A Dynamic Picture of a Single Connection

The throughput $W(t) = w$ of a single connection

— a rate of increase.

— bw loss rate at time t.

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A Dynamic Picture of a Single Connection

Stochastic Differential Equation for $(W(t))$

$$dW(t) = a\,dt + (1-r)W(t-)\mathcal{N}_{W(t-)b}(dt),$$

$\mathcal{N}_x$: Poisson process with rate x.

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Throughput λ at Equilibrium: Ott et al. (1996)

$$\lambda = \psi\sqrt{\frac{a}{b}}$$

with

$$\psi = \sqrt{\frac{2}{\pi}}\prod_{n=1}^{+\infty}\frac{1-r^{2n}}{1-r^{2n-1}}.$$

A Dynamic Picture: Network Context

— J **nodes.**

— K **classes of connections.**

— N_k **class** k **permanent connections.**

$$N = N_1 + \cdots + N_K.$$

A Dynamic Picture

— $W_{k,\ell}(t)$: **throughput of ℓth connection of class k.**

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— $u(t) = (u_j(t), 1 \leq j \leq J)$, utilization of nodes

$$u_j(t) = \sum_{k=1}^{K} A_{k,j} \sum_{\ell=1}^{N_k} W_{k,\ell}(t)$$

where (for example)

$$A_{k,j} = \begin{cases} 1 & \text{if class } k \text{ connections use node } j \\ 0 & \text{otherwise.} \end{cases}$$

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$u_j(t)$: level of congestion of node j.

A Dynamic Picture

Generator of $(W_{k,\ell}(t))$

$$\Omega_k(f)(w_k) = a_k f'(w_k) + w_k b_k(u)(f(r_k w_k) - f(w_k))$$

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— $u \longrightarrow b_k(u)$ depends of the coordinates u_j for which $A_{k,j} \neq 0$.

A Dynamic Picture: Stochastics

Stochastic Differential Equation for $(W_{k,\ell}(t))$

$$dW_{k,\ell}(t) = a_k W_{k,\ell}(t)\,dt - (1 - r_k) W_{k,\ell}(t-) \mathcal{N}^{k,\ell}_{W_{k,\ell}(t-) b_k(U(t-))}(dt),$$

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with $U(t) = (U_j(t), 1 \leq j \leq J)$ **and**

$$U_j(t) = \sum_{k=1}^{K} A_{jk} \sum_{i=1}^{N_k} W_{k,i}(t),$$

and $\mathcal{N}^{k,\ell}_x$ **Poisson process with rate** x**.**

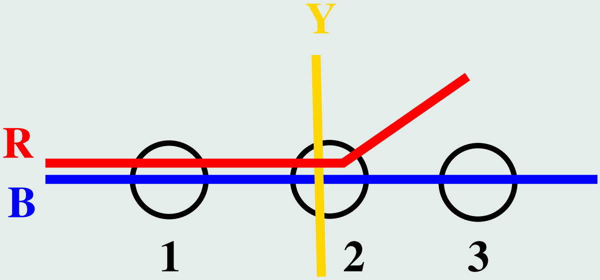
A Markovian Picture

The process

$$(\boldsymbol{W}(t)) = [(\boldsymbol{W}_{k,\ell}(t), 1 \leq \ell \leq \boldsymbol{N}_k), 1 \leq \boldsymbol{k} \leq \boldsymbol{K}]$$

has the Markov property.

An Example



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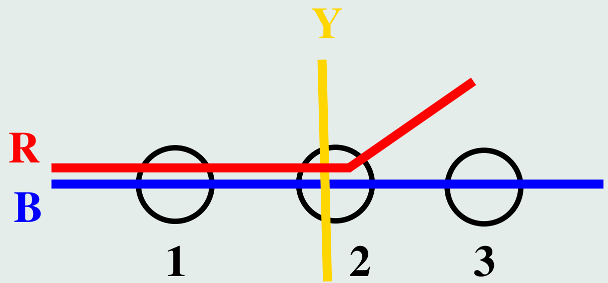


Evolution of $(W_R(t), W_B(t), W_Y(t))$

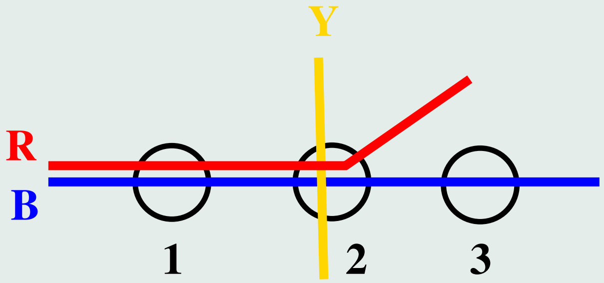
Loss Rates:

— **For** B: $w_B b_B(w_R + w_B, w_R + w_B + w_Y, w_B)$

— **For** R: $w_R b_R(w_R + w_B, w_R + w_B + w_Y)$

— **For** Y: $w_Y b_Y(w_R + w_B + w_Y)$

An Example (II)



Loss Rates:

— For B:
$w_B[b_1(w_R + w_B) + b_2(w_R + w_B + w_Y) + b_3(w_B)]$

— For R: $w_R[b_1(w_R + w_B) + b_2(w_R + w_B + w_Y)]$

— For Y: $w_Y b_2(w_R + w_B + w_Y)$

A Mean Field Setting

— $N = N_1 + \cdots + N_K \longrightarrow +\infty.$

$$\frac{N_k}{N} \longrightarrow p_k$$

— **The functions b_k are scaled by $1/N$.**

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Capacity of a node $\sim N$.

If $W_{k,\ell} = w$, the loss rate of connection is

$$w \sum_{j=1}^{J} A_{jk} b_{jk} \left(\frac{u_j}{N} \right)$$

A Multi-Class Mean-Field Convergence Result

$$(\boldsymbol{W}^N(t)) = [(\boldsymbol{W}^N_{k,\ell}(t), 1 \leq \ell \leq N_k), 1 \leq k \leq K]$$

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Theorem [Graham and Robert (2008)] As $N \to +\infty$ $(W^N_{k,*}(t), 1 \leq k \leq K)$ converges in distribution to $(\overline{W}_k(t), 1 \leq k \leq K)$.

The $(\overline{W}_k(t), t \geq 0)$, $k = 1, \ldots K$ are independent.

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Proofs:

— Iterative scheme for estimation.

— Decomposition in boxes for local averaging.

The limiting process $(\overline{W}_k(t), 1 \leq k \leq K)$

Solution of a SDE such that

$$d\overline{W}_k(t)) = a_k\, dt + (1 - r_k)\overline{W}_k(t)\mathcal{N}_{\overline{W}_k(t) b_k(\ )}(dt)$$

$\mathcal{N}_x$ **Poisson with rate** x.

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$$u_{\overline{W}}(t) = (u_{\overline{W},j}(t), 1 \le j \le J)$$

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Theorem:

A unique solution to (unconventional) SDE.

The Limiting process $(\overline{W}_k(t), 1 \leq k \leq K)$

$\overline{W}_k(t)$ **depends**

— on the past $\overline{W}_k(s), s \leq t$

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A Markov process non-homogeneous with respect to time.

The Equilibrium of the Limiting process

Theorem

Equilibrium Distributions

are in one to one correspondence with $u = (u_j)$

solution of fixed point equation:

$$u_j = \sum_{k=1}^{K} A_{jk} \psi_k p_k \sqrt{\frac{a_k}{b_k(u)}}, \quad 1 \leq j \leq J,$$

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Interaction of flows at equilibrium
$\Rightarrow$ Fixed Point Equation.

Throughput at Equilibrium

Throughput $\boldsymbol{\lambda_k}$ for class $\boldsymbol{k}$ connection

$$\boldsymbol{\lambda_k} = \mathbb{E}(\overline{\boldsymbol{W}}_k) = \psi_k \sqrt{\frac{a_k}{b_k(u)}}.$$

with $\boldsymbol{u}$ solution of $\boldsymbol{H(u) = 0}$.

Back to Kelly's Picture

A route of class $k \in 1, \ldots, K$:
receives throughput λ_k^0, such that (λ_k^0) achieves

$$\max_{\lambda \in \mathcal{C}} \sum_{k=1}^{K} p_k U_k(\lambda_k / p_k)$$

or equivalently (under some conditions)

$$\nabla G(\lambda^0 / p) = 0,$$

for some convenient function G.

How Many Solutions for Fixed Point Equations ?

Uniqueness holds for several topologies

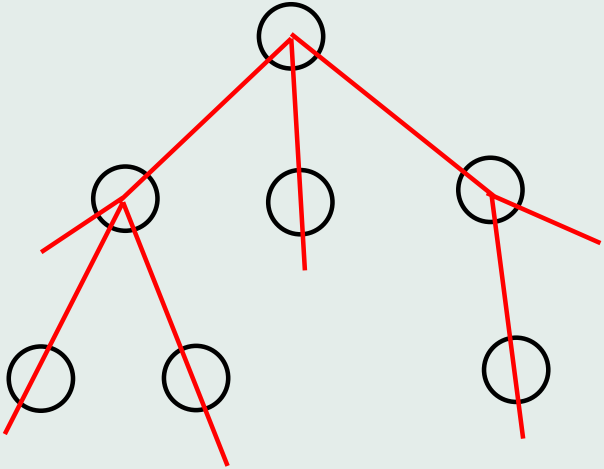
— Linear Network

— Torus

— Trees

under some assumptions.

Examples



Trees

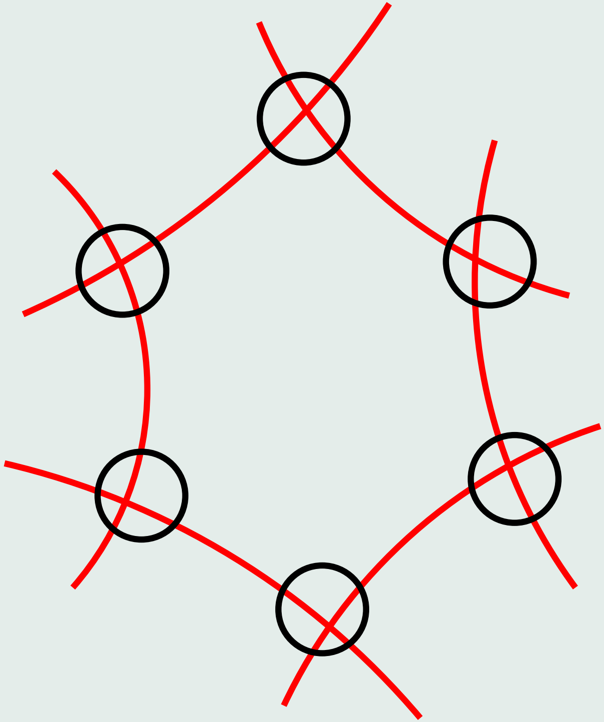
Proposition:

If the functions b_H, $H \in \mathcal{T}$, are continuous and non-decreasing, then there exists a unique solution for fixed point equations.

Proof:

Recursion (starting from leaves).

Examples



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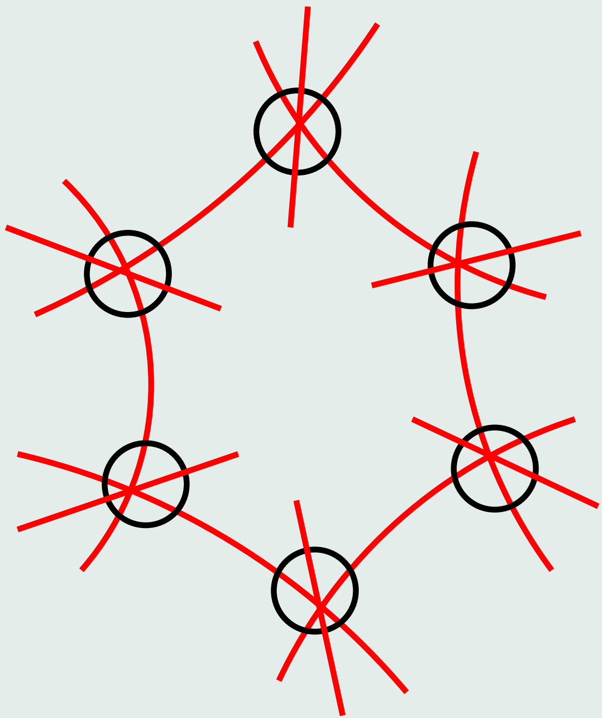

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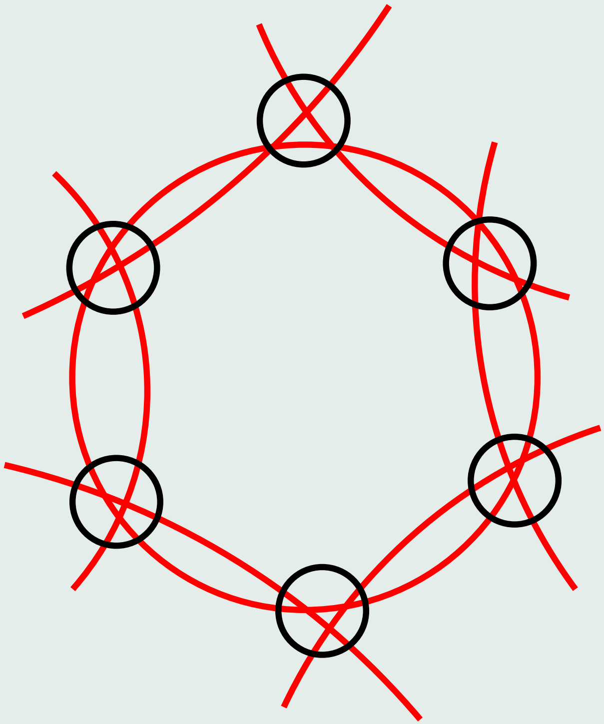

Ring Topologies

General Result:

If the functions b_k, $1 \leq k \leq K$, are Lipschtiz and non-decreasing, then there exists a unique solution for fixed point equations.

Two Methods for Proof:

Contraction Arguments.

Monotonicity Properties.

Uniqueness

Always true
as long as $u \longrightarrow b_k(u)$ is non-decreasing ?

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Counter-example of Raghunathan and Kumar (2007) in a wireless context.

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Counter-example of Raghunathan and Kumar (2007) in a wireless context.

If it fails:
Multi-stability phenomenon in these networks ?

Non-stable phenomena in IP networks

Bandwidth allocation algorithms

Non-stable phenomena in IP networks

Bandwidth allocation algorithms

may exhibit

— **Multiple Stable points ?**

— **Oscillations ?**

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How to prove with rigorous scaling results ?

Multiple stable phenomena in networks

Known for

— **Loss Networks**,
Gibbens *et al.* (1990), Marbukh (1993).

— **Wireless Networks**,
Antunes *et al.* (2008).

A Conclusion

Representation of the interaction of flows:

— **Instantaneous fluid picture of Kelly et al.:**
An optimisation problem
Data for Model: Utility function.

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— **Instantaneous fluid picture of Kelly et al.:**
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Data for Model: Utility function.

— **Starting from microscopic dynamics**
A fixed point equation
Data for Model: Function for loss rates.

On going work and open questions

— **A more dynamic setting:**
non-permanent connections.

On going work and open questions

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A different scaling.

On going work and open questions

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— **Convergence to equilibrium.**

On going work and open questions

— **A more dynamic setting:**
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— **Convergence to equilibrium.**
A difficult technical question.

The End