Biased random walks on percolation clusters of trees

Nina Gantert

EURANDOM

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based on joint work with Gérard Ben Arous, Alexander Fribergh and Alan Hammond

1. Motivation: Biased RW on perc. clusters

2. Tree instead of lattice

1. Motivation: biased RW on perc. clusters

Consider i. i. d. bond percolation with parameter p on $\mathbb{Z}^2$. We will always assume that $p > p_c = \frac{1}{2}$.
Take $\beta > 1$.
Condition on the event that $(0,0)$ is in the (unique) infinite cluster.
Put on each bond $(z, \widetilde{z})$ the weight $\beta^{x \wedge \widetilde{x}}$ (where $z = (x, y)$, $\widetilde{z} = (\widetilde{x}, \widetilde{y})$).

The random walk $Z_n = (X_n, Y_n)$ is the biased random walk with $Z_0 = (0,0)$ and transition probabilities proportional to the weights.
Behaviour of $Z_n = (X_n, Y_n)$?

1. Motivation: biased RW on perc. clusters

M. Barma, D. Dhar (1982)
Directed Diffusion in a percolation network.
J. Phys. C: Solid state Physics.

A. Bunde, S. Havlin (1991)
Fractals and Disordered Systems.

Variants of the model:
– different weights
– site - instead of bond percolation.

1. Motivation: biased RW on perc. clusters

Transience to the right:

Theorem

For all $\beta > 1$ and all $p \in (\frac{1}{2}, 1)$,

$$X_n \to \infty, \quad P^\beta - a.s.$$

Theorem

(N. Berger, NG, Y. Peres, 03, A. S. Sznitman 03)
For each $p \in (\frac{1}{2}, 1)$ there are two constants $\beta_\ell = \beta_\ell(p)$ and $\beta_u = \beta_u(p)$ such that

$1 < \beta_\ell(p) \leq \beta_u(p) < \infty$ and

(i) $\frac{X_n}{n} \to 0 \quad P^\beta$-a.s. if $\beta > \beta_u$,

(ii) $\frac{X_n}{n} \to v_{\beta,p} > 0 \quad P^\beta$-a.s. if $\beta < \beta_\ell$.

(iii) There is a CLT for β close enough to 1. See A. S. Sznitman, "On the anisotropic walk on the supercritical percolation cluster".

1. Motivation: biased RW on perc. clusters

Remark

We took i.i.d. bond percolation but of course the results is believed to hold for variants of this model. However, as a new result of M. Deijfen and O. Häggström shows, the theorem does **not** hold for any translation invariant percolation!

Consider "dead ends" A in the right half plane. Compute $E^\beta(T_A)$ = expected time spent in A before return to the line $\{(0, y) : y \in \mathbb{Z}\}$. Let $\Gamma(\beta, p) := \sum_A p_A E^\beta(T_A)$ and define β_u such that
$\Gamma(\beta, p) < \infty$ for $\beta < \beta_u$,
$\Gamma(\beta, p) = \infty$ for $\beta > \beta_u$.
Then, (i) is satisfied with this value of β_u.

5. Open questions

(1) Existence of the critical value:

Conjecture: $\beta_\ell = \beta_u = \beta_c$.

(2) Conjecture: For β fixed, $p \to v_{\beta,p}$ is continuous and increasing.

New results by Alex Fribergh for p close to 1.

(3) Conjecture: For p fixed, $\beta \to v_{\beta,p}$ is continuous and unimodular.

(4) **Distribution** of Z_n if $\beta > \beta_u$??

Want to address Question (4).

2. Tree instead of lattice

Take a regular tree, consider i. i. d. bond percolation with parameter $p > p_c$ and condition on the event that the root is in an infinite cluster.
More general, consider a supercritical Galton-Watson tree with $p_0 = P[Z = 0] > 0$, conditioned on survival.
Run a β-biased random walk on this tree.

5. Tree instead of lattice

In this case, the critical value can be computed, see R. Lyons, R. Pemantle, Y. Peres: Let f be the generating function of the Galton-Watson tree and $q = f(q)$ its extinction probability. Then $\beta_c = 1/f'(q)$.

Let $|X_n|$ denote the distance of the walker to the root.

Theorem R. Lyons, R. Pemantle, Y. Peres 96
Let $\beta_c = 1/f'(q)$. Then

(i) $\frac{|X_n|}{n} \to 0 \quad P^\beta$-a.s. if $\beta \geq \beta_c$,

(ii) $\frac{|X_n|}{n} \to v_{\beta,p} > 0 \quad P^\beta$-a.s. if $\beta < \beta_c$.

5. Tree instead of lattice

Reason: **Decomposition** of the supercritical Galton-Watson tree conditioned on survival: it consists of a **backbone**, which is a Galton-Watson tree with generating function g, and independently attached **traps**, which are (subcritical) Galton-Watson trees with generating function h.

g and h are given by

$$g(s) = \frac{f((1-q)s+q)-q}{1-q}, \quad h(s) = \frac{f(qs)}{q}$$

5. Tree instead of lattice

Indeed, the expectation of the time spent in a trap is
- infinite if $\beta \geq \beta_c$
- finite if $\beta < \beta_c$.

Note that the traps have generating function h and hence their expected k-th generation size is $f'(q)^k$.

What about the **distribution** of $|X_n|$ in case (i) ?

5. Tree instead of lattice

Let $\gamma = \frac{\log \beta_c}{\log \beta}$. Let Z be the number of children of the root. Denote T_n the hitting time of the n-th level: $T_n := \min\{k \geq 1 : |X_n| = n\}$.

Theorem

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Assume $E[Z^2] < \infty$ and $\beta > \beta_c$. Then

$$\lim_{n\to\infty} \frac{\log |X_n|}{\log n} = \gamma, \quad P^\beta - \text{ a.s}$$

Further, the sequence $(|X_n|/n^\gamma)$ is tight, but does **not** converge in distribution (at least if β is large enough). For $\lambda > 0$, considering the subsequences $n_\lambda(k) = \lfloor \lambda \beta^{\gamma k} \rfloor$, $T_{n_\lambda(k)}/n_\lambda(k)^{1/\gamma}$ converges in distribution to an infinitely divisible law $\mathcal{L}_\lambda$.

2. Tree instead of lattice

To explain that convergence in distribution only takes place along subsequences, consider the following toy example:

Remark

Let $G_i, i = 1, 2, \ldots$ be i. i. d. random variables, geometrically distribution with parameter α, and $\beta > 1$. Let

$$S_n := \sum_{i=1}^{n} \beta^{G_i}$$

Then, taking $\gamma = |\log(1-\alpha)|/\log\beta$, if $\gamma < 2$, $S_n/n^{1/\gamma}$ is tight, but does **not** converge in distribution.

The reason is that β^{G_1} is not in the domain of attraction of a stable law.

2. Tree instead of lattice

Remark

The model is reminiscent of Bouchaud's trap model or of 1-dimensional RWRE. Recall that in the limit theorems for 1-dimensional RWRE (due to H. Kesten, M. V. Kozlov and F. Spitzer), there is a non-lattice assumption!