Recent Progress on the Ultrametricity Conjecture in Spin Glasses

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Outline

Gaussian Spin Glasses

We consider a Gaussian process on $\{-1,+1\}^N$ with

$$\mathbb{E}H_N(\sigma)H_N(\sigma') = N\sigma\cdot\sigma'$$

where $\sigma\cdot\sigma'$ is some covariance matrix on $\{-1,1\}^N$ with $\sigma\cdot\sigma = 1$.

Embed $\{-1,+1\}^N$ in the unit ball of Hilbert space $\mathcal{H}$.

$$d(\sigma,\sigma')^2 = 2(1-\sigma\cdot\sigma')$$

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Let $\mathcal{G}_{\beta,N}$ be the Gibbs measure of H on N spins

$$\mathcal{G}_{\beta,N}(\sigma) = \frac{\exp -\beta H_N(\sigma)}{Z_N(\beta)}$$

Random Probability Measure on $\mathcal{H}$

- Goal: Understand the limiting object $\lim_{N\to\infty} \mathcal{G}_{\beta,N}$.

Examples of Gaussian Spin Glasses

- GREM: Partition N spins into 2 blocks

$$\sigma \cdot \sigma' = a \; \delta_{\sigma^1 = \sigma^{1\prime}} + (1-a) \; \delta_{\sigma^1 = \sigma^{1\prime}} \delta_{\sigma^2 = \sigma^{2\prime}}$$

- Sherrington-Kirkpatrick model: $H_N(\sigma) = \frac{-1}{\sqrt{N}} \sum_{i,j=1}^{N} J_{ij} \sigma_i \sigma_j$ for J_{ij}'s iid standard Gaussians.

$$\sigma \cdot \sigma' = \left(\frac{1}{N} \sum_i \sigma_i \sigma_i' \right)^2$$

Parisi Theory

Let $\mathcal{G}_\beta = \lim_{N\to\infty} \mathcal{G}_{\beta,N}$.

Ultrametricity Conjecture

The metric on the support of $\mathcal{G}_\beta$ is ultrametric (tree-like)

$$d(\sigma,\sigma') \leq \max\{d(\sigma,\sigma''), d(\sigma'',\sigma')\} \ .$$

If so

- the law of $\mathcal{G}_\beta$ is a *Ruelle Cascade*
- for SK, it "gives" the free energy: the *Parisi Formula*

$$\lim_{N\to\infty} \frac{1}{N} \log Z_N(\beta) = \min_{\mu\sim RC} P(\beta;\mu)$$

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- The Conjecture is not true in general. Example at the end.
- BUT seems true for a large class of systems: Generic Systems. They are dense as far as the free energy is concerned.

The Infinite-Volume Gibbs Measure

- Let $(\sigma^{(k)})_{k\in\mathbb{N}}$ be iid $\mathcal{G}_{\beta,N}$-sampled configurations. replicas

Consider the covariance matrix

$$(Q_{\beta,N})_{kl} = \sigma^{(k)} \cdot \sigma^{(l)}$$

Randomness from sampling and H

- Extract a subsequence so that the law of $Q_{\beta,N}$ converges weakly to Q_β.
- Study Q_β.
- The Q_β matrix is weakly exchangeable: Law invariant under simultaneous permutation of rows and columns.

The Infinite-Volume Gibbs Measure

Theorem (Dovbysh-Sudakov '82, Aldous)

A w. e. covariance matrix Q, $Q_{ii} = 1$, *has a sampling measure* μ *(random) on the unit ball of* $\mathcal{H}$ *such that*

$$Q_{ij} \overset{Law}{=} \sigma^{(i)} \cdot \sigma^{(j)} \text{ for } i \neq j$$

where $(\sigma^{(i)})_{i \in \mathbb{N}}$ *are iid* μ*-sampled vectors.*

Random Overlap Structure (*Aizenman et al*) or Descriptor (*Parisi*):
A w.e. covariance matrix with $Q_{ii} = 1$.
$\iff$ random μ on the unit ball of $\mathcal{H}$.

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Definition (∞-volume Gibbs measure)

Set of ∞-volume Gibbs measure for $\mathcal{G}_{\beta,N}$ is the convex hull of ROSt's which are limit points $\mathcal{G}_\beta$ of $\mathcal{G}_{\beta,N}$.

What is left of Gibbs ?

Properties of Spin Glass Gibbs Measures

Stochastic Stability

$$\frac{\mathcal{G}_\beta(d\sigma)\ e^{\lambda g_\sigma}}{\text{Norm.}} \stackrel{\text{Law}}{=} \mathcal{G}_\beta(d\sigma)\ \text{(up to isometry)}$$

for $g \sim \mathit{Gaussian}$ with $\mathbb{E}[g_\sigma g_{\sigma'}] = \sigma \cdot \sigma'$.

- Non-Robust holds in β-average for the limit points of $\mathcal{G}_{\beta,N}$ *Aizenman-Contucci'98, Contucci-Giardina'04*
- Robust if it holds for $\mathbb{E}[g_\sigma g'_\sigma] = (\sigma \cdot \sigma')^p\ \forall p \in \mathbb{N}$.
- Idea:

$$e^{\beta H_N(\sigma)+\lambda g_\sigma} \stackrel{\text{Law}}{=} e^{\sqrt{\beta^2+\frac{\lambda^2}{N}}H_N(\sigma)} \stackrel{\text{Law}}{\approx} e^{\beta H_N(\sigma)}$$

Properties of Spin Glass Gibbs Measures

Ghirlanda-Guerra

$$\mathbb{E}[Q_{1,s+1} f(\{Q\}_{i,j\leq s}))] = \frac{1}{s}\mathbb{E}[Q_{12}]\ \mathbb{E}[f(\{Q\}_{i,j\leq s}))] + \frac{1}{s}\sum_{l\neq 1}^{s}\mathbb{E}[Q_{1l} f(\{Q\}_{i,j\leq s})]$$

- Non-Extended holds in β-average for the limit points of $\mathcal{G}_{\beta,N}$ *Ghirlanda-Guerra '98*
- Extended if it holds for Q_{ij}^p in front of f $\forall p \in \mathbb{N}$.
- Idea: Self-averaging + Convexity +Clever

Generic Systems

Definition

Generic Systems are ROSt's which satisfy RSS and EGG.

Ultrametricity Conjecture

If a ROSt (Q_β or $\mathcal{G}_\beta$) is generic, then its support is ultrametric. Precisely it must be a Ruelle Cascade.

Question: EGG $\Longleftrightarrow$ RSS ? ($\Leftarrow$ Probably OK)

Generic Systems from Perturbations

How to get the extended and robust versions for a given system?
Perturbations

Let $\vec{\beta} = (\beta_1, ..., \beta_p, ...)$ with $\sum_{p\geq 1} \beta_{p,N}^2 < \infty$

$$\beta\tilde{H}(\sigma) = \beta_1 H(\sigma) + \sqrt{\delta}\sum_{p>1} \beta_{p,N} H^{(p)}(\sigma)$$

where $\mathbb{E}H^{(p)}(\sigma)H^{(p)}(\sigma') = N(\sigma \cdot \sigma')^p$.

- If $\delta \to 0$, the perturbation does not affect the free energy.
- However it could affect the Gibbs measure.

Theorem (Talagrand '09)
For every $\vec{\beta}$, there exists $\vec{\beta}_N \to \vec{\beta}$ such that the limit points of $\mathcal{G}_{\vec{\beta}_N,N}$ are generic.

Constructive Proof of Parisi Formula ?

Based on Guerra-Toninelli interpolation and the Aizenman-Sims-Starr scheme, one "expects" a constructive proof of the following

Theorem (Parisi Formula)

If the Ultrametricity Conjecture holds, then the free energy of the SK model is given by the Parisi formula

$$\lim_{N\to\infty}\frac{1}{N}\log\sum_{\sigma}e^{\beta H_N(\sigma)}=\min_{\mu\sim RC}\mathbb{E}\log\frac{\int\mu(d\sigma)e^{\log\cosh\beta g_\sigma}}{\int\mu(d\sigma)e^{\beta g_\sigma^{(2)}}}$$

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The proof is based on:

- the AS^2 variational principle on ROSt's.
- the restriction to systems satisfying EGG. RSS more delicate.
- the continuity of the Parisi functionals

$$\mathbb{E} \log \int_{\mathcal{H}} \mu(d\sigma) e^{\beta\psi(g_\sigma)} \ .$$

Ultrametricity Conjecture Results

The state of the conjecture is to this day:

Suppose that Q_β takes a finite number of values.

Theorem (Aizenman-A '07)

If a ROSt satisfies RSS, then its support is ultrametric.

Theorem (Panchenko '08)

If a ROSt satisfies EGG, then its support is ultrametric.

The ultrametricity actually determines the full law: Ruelle Cascade.

- The approaches are similar.
- Extension of the proof to infinite values under some restrictions by J. Miller '09.
- EGG $\Longleftrightarrow$ RSS ?

Elementary Results on RSS and EGG

Before attacking the conjecture, elementary results can be "easily deduced".

Proposition (Panchenko, Aizenman-A)

Let $\mathcal{G}$ or Q be a ROSt satisfying RSS or EGG. Then

1. *its support is contained on a sphere of $\mathcal{H}$.*
2. *its support is a point or has infinite dimension.*

With finiteness assumption:

Lemma

If Q_{ij} takes only a finite number of values, then the sampling measure $\mathcal{G}$ is supported on countable number of vectors:

$$\mathcal{G} = \sum_k p_k \delta_{\sigma_k}$$

$\sum_k p_k = 1$ *and* $\sigma_k \in \mathcal{H}$.

How to Prove Ultrametricity ? Step 1

Recall that $\mathcal{G} = \sum_k p_k \ \delta_{\sigma_k}$ where $\|\sigma_k\|^2 \equiv \|\sigma\|^2$ cste.

Note that $\sigma_i \cdot \sigma_j = \|\sigma\|^2 \iff \sigma_i = \sigma_j$.

1. Robust and Extended

If $\mathcal{G}$ satisfies the property $\forall p \in \mathbb{N}$ then also when $p \to \infty$.

- RSS:
$$\mathcal{G}(d\sigma) \stackrel{\text{Law}}{=} \frac{\mathcal{G}(d\sigma)e^{\lambda g_\sigma}}{Norm.}$$
for $\mathbb{E} g_\sigma g_{\sigma'} = \lim_{p\to\infty}(\frac{\sigma\cdot\sigma'}{\|\sigma\|^2})^p = \delta_{\sigma\sigma'}$
- EGG: An invariance also holds for $e^{\lambda\epsilon_\sigma}$ for *iid* $\epsilon_\sigma = \pm 1$ with prob $1/2$.

How to Prove Ultrametricity ? Step 2

Recall that $\mathcal{G} = \sum_k p_k \ \delta_{\sigma_k}$ where $\|\sigma_k\|^2 \equiv \|\sigma\|^2$ cste.

2. Weak Exchangeability at the Edge

- Take p_k in decreasing order.
- Define $Q'_{kl} = \sigma_k \cdot \sigma_l$. Takes same values as Q minus one.

Proposition

If $\mathcal{G}(d\sigma) \stackrel{Law}{=} \frac{\mathcal{G}(d\sigma)e^{\lambda g_\sigma}}{Norm.}$ *for* $\mathbb{E} g_\sigma g_{\sigma'} = \delta_{\sigma\sigma'}$*, then* Q' *is weakly exchangeable.*

How to Prove Ultrametricity ? Step 3

3. Dovbysh-Sudakov Representation
From weak exchangeability, there exists a measure $\mathcal{G}'$

$$Q'_{kl} = \sigma'^{(k)} \cdot \sigma'^{(l)} + (\|\sigma\|^2 - \|\sigma'^{(k)}\|^2)\delta_{kl}$$

where $(\sigma'^{(k)})_{k\in\mathbb{N}}$ iid $\mathcal{G}'$-sampled.

How to Prove Ultrametricity ? Step 4

4. Induction

The property is lifted: $\mathcal{G}'$ satisfies RSS ($\sim$ hard) and EGG (easy)

Therefore

- it is supported on a sphere of radius $\|\sigma'\| < \|\sigma\|$

$$Q'_{kl} = \sigma'^{(k)} \cdot \sigma'^{(l)} + (\|\sigma\|^2 - \|\sigma'\|^2)\delta_{ij}$$

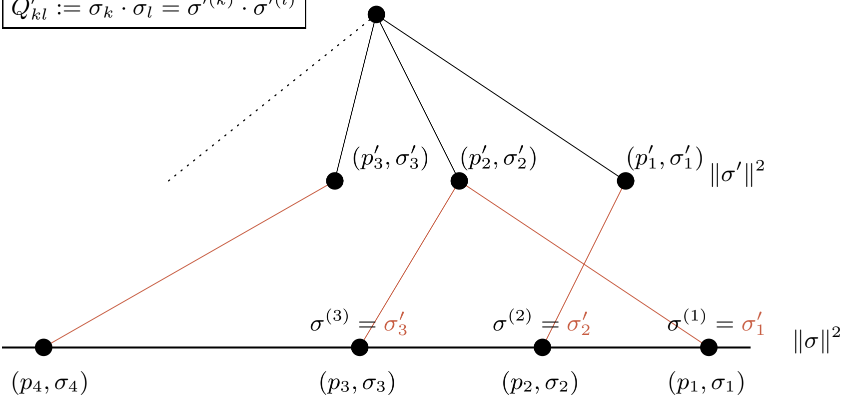
- Since Q takes a finite number of values,

$$\mathcal{G}' = \sum_m p'_m \delta_{\sigma'_m}$$

Redo the argument with $\mathcal{G}'$.
If Q' takes only one off-diagonal value, $\mathcal{G}' = \delta_{\sigma'}$.

Pictorially

$$Q'_{kl} := \sigma_k \cdot \sigma_l = \sigma'^{(k)} \cdot \sigma'^{(l)}$$



Beyond EGG and RSS

The ultrametric conjecture is not true if only the non-robust or non-extended version holds.

Example (Franz-Parisi-Virasoro '91 and Bolthausen-Kistler '06)

$$H_N(\sigma) = \sqrt{a} X^1_{\sigma^1} + \sqrt{1-a} X^2_{\sigma^2}$$

where X^1 *and* X^2 *iid Gaussian*$(0, N)$

$$\mathbb{E} H_N(\sigma) H_N(\sigma') = N(a\delta_{\sigma^1 = \sigma'^1} + (1-a)\delta_{\sigma^2 = \sigma'^2})$$

$$X^1_{\sigma^1} \qquad X^{12}_{\sigma^1\sigma^2} \qquad X^2_{\sigma^2}$$

$$+ - + - + + + - + - - + \;\Big|\; - + + - - + + + + - - -$$

$$\sigma^1 \in \{\pm 1\}^{N/2} \qquad \sigma^2 \in \{\pm 1\}^{N/2}$$

Non-Hierarchical GREM's

- It does satisfy GG and Stoch. Stability (in β-average)
- The Gibbs measure of the system is a product measure: NOT ultrametric

Question (Bolthausen-Franz)

Can ultrametricity be retrieved by adding a small perturbation ?
If so, how small can it be ? Small = Same Free Energy

YES:

- Take a Generic System with the same free energy.
- Apply Ultrametricity Theorems

- Perturbation of order $N^{7/8}$ needed to get generic systems in general.
- For this system, $\log N$ is actually sufficient (A-Kistler '08).

Open Questions

- EGG and/or RSS $\Rightarrow$ Ultrametricity (infinite # of values)?
- EGG $\Leftrightarrow$ RSS ?
- Parisi formula ?
- Beyond perturbation, what can we say about the Gibbs measure ?
  non-extended GG, non-robust Stoch. Stab.
- Is $\log N$ the smallest perturbation that modifies the Gibbs measure ?