O,DO, DO^2, Eurandom

Anton Bovier

Metastability in the random field Curie-Weiss model

Based on collaborations with:
Alessandra Bianchi and Dima Ioffe

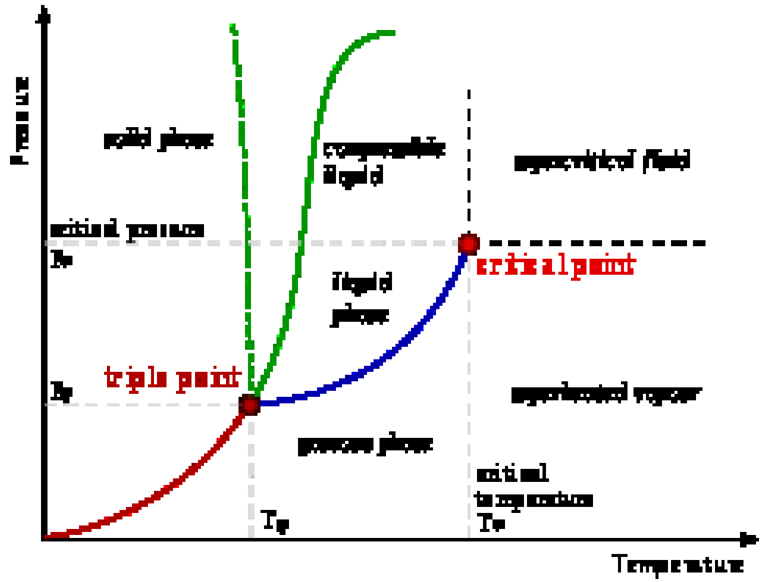
hausdorff center for mathematics | institute for applied mathematics

Plan

- ▷ Metastability
- ▷ The RFCW model
- ▷ Equilibrium properties
- ▷ Glauber dynamics
- ▷ Main theorem
- ▷ Previous work
- ▷ Elements of proof
- ▷ Conclusions and outlook

Metastability

Metastability is a common phenomenon related to the dynamics of first order phase transitions:



If the parameters of a systems are changed rapidly across the line of a first order phase transition, the system will persist for a long time in a metastable state before transiting rapidly to the new equilibrium state under the influence of random fluctuations.

Stochastic Ising models

A model context we are interested in are stochastic Ising-type models, i.e. Markov chains with

- ▷ **State space** $\mathcal{S}_\Lambda = \{-1,1\}^\Lambda$, $\Lambda \subseteq \mathbb{Z}^d$;
- ▷ **Hamiltonian** $H_\Lambda : \mathcal{S}_\Lambda \to \mathbb{R}$;
- ▷ **Gibbs measure** $\mu_{\beta,\Lambda}(\sigma) = Z_{\beta,\Lambda}^{-1} \exp(-\beta H_\Lambda(\sigma))$;
- ▷ **Order paprameter**, e.g. $m_\Lambda(\sigma) = \frac{1}{|\Lambda|} \sum_{x\in\Lambda} \sigma_x$;
- ▷ **Transition rates** $p_\beta(\sigma,\sigma')$ reversible with respect to $\mu_{\beta,\Lambda}$ and “local”, i.e. essentially single site flips only.

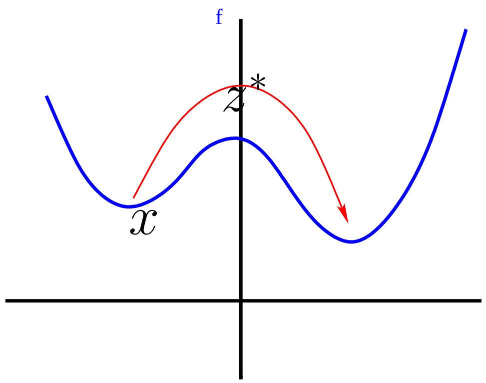
Metastability in such system can be described often in terms of the behavior of the order parameter:

If m^*_β is the equilibrium value of m_Λ, i.e. $\mu_{\beta,\Lambda}\left(m_\lambda(\sigma) \sim m^*_\beta\right) \sim 1$, there are values of m such that if at time $t = 0$, the system is prepared with $m_\Lambda(\sigma(0)) = m$, then the first time, t, such that $m_\lambda(\sigma(t)) \sim m^*_\beta$, is exceptionally large (in average).

The issue at hand is to understand in a precise way the lifetimes of such metastable states.

The heuristic theory of Kramer's and Eyring (ca. 1940) models the evolution of the order parameters by a stochastic differential equation:

$$dX_t = F'(X_t)dt + \sqrt{2\epsilon}dB_t$$



Finite state Markov chains.

If Λ is a finite set, and we consider the limit $\beta \uparrow \infty$, we have a very satisfactory theory at hand.

- Metastable states correspond to local minima of H_Λ;
- Exit from metastable states occur through minimal saddle points of H_Λ connecting one mimimum to deeper ones;
- $\mathbb{E}_x \tau_x = C \exp\left(\beta \left(H_\Lambda(saddle) - H_\Lambda(min)\right)\right)$; τ_x exp. distributed;

The simplifying feature here is that there at only “few paths”, or “nothing can beat $\exp(-\beta)$!!”

Mean field models.

If $H_\Lambda(\sigma) = E(m_\Lambda(\sigma))$, $m_\lambda(\sigma(t))$ is again Markov chain on $\{-1, -1-2/N, \dots, 1\}$;

- ▷ nearest neigbor random walk reversible with respect to measure $\exp(-\beta N F(x))$ with F free energy;
- ▷ explicitely solvable;

Thus here we essentially have exactly the situation imagined by Kramers and Eyring.

The real thing:

Whenever we are not in one of the two situations above, we have problems:

▷ There are lots of relevant paths!

▷ There is no exact reduction to a finite dimensional system!

Still, we expect an effective description of the dynamics in terms of some mesoscopic coarse grained dynamics!

In the remainder of this talk I will explain how this idea can be implemented in a simple example.

Random Hamiltonian:

$$H_N(\sigma) \equiv -\frac{N}{2}\left(\frac{1}{N}\sum_{i=1}^{N}\sigma_i\right)^2 - \sum_{i=1}^{N} h_i\sigma_i.$$

h_i, $i \in \mathbb{N}$ are (bounded) i.i.d. random variables, $\sigma \in \{-1, 1\}^N$.

Equilibrium properties: [see Amaro de Matos, Patrick, Zagrebnov (92), Külske (97)]

Gibbs measure: $\mu_{\beta,N}(\sigma) = \frac{2^{-N}e^{-\beta H_N(\sigma)}}{Z_{\beta,N}}$

Magnetization: $m_N(\sigma) \equiv \frac{1}{N}\sum_{i=1}^{N}\sigma_i$.

Induced measure: $\mathcal{Q}_{\beta,N} \equiv \mu_{\beta,N} \circ m_N^{-1}$. on the set $\Gamma_N \equiv \{-1, -1+2/N, \ldots, +1\}$.

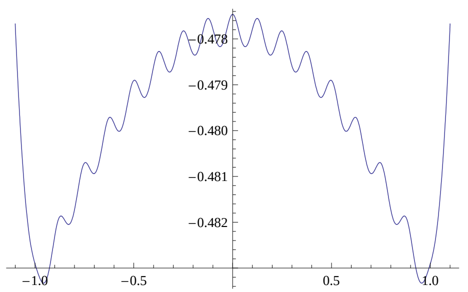
Equilibrium properties

Using sharp large deviation estimates, one gets

$$Z_{\beta,N}\mathcal{Q}_{\beta,N}(m) = \sqrt{\frac{2I_N''(m)}{N\pi}} \exp\left\{-N\beta F_N(x)\right\}(1+o(1)),$$

where $F_N(x) \equiv \frac{1}{2}m^2 - \frac{1}{\beta}I_N(m)$ and $I_N(y)$ is the Legendre-Fenchel transform of

$$U_N(t) \equiv \frac{1}{N}\sum_{i\in\Lambda} \ln\cosh\left(t+\beta h_i\right)$$



Critical points: Solutions of $m^* = \frac{1}{N}\sum_{i\in\Lambda}\tanh(\beta(m^*+h_i))$.
Maxima if $\beta\mathbb{E}_h\left(1-\tanh^2(\beta(z^*+h))\right) > 1$.
Moreover, at critical points,

$$Z_{\beta,N}\mathcal{Q}_{\beta,N}(z^*) \quad = \quad \frac{\exp\left\{\beta N\left(-\frac{1}{2}\left(z^*\right)^2 + \frac{1}{\beta N}\sum_{i\in\Lambda}\ln\cosh\left(\beta(z^*+h_i)\right)\right)\right\}}{\sqrt{\frac{N\pi}{2}\left(\mathbb{E}_h\left(1-\tanh^2(\beta(z^*+h))\right)\right)}}(1+o(1))$$

We consider for definiteness discrete time Glauber dynamics with Metropolis transition probabilities

$$p_N(\sigma,\sigma') \equiv \frac{1}{N} \exp\left\{-\beta[H_N(\sigma') - H_N(\sigma)]_+\right\}$$

if σ and σ' differ on a single coordinate, and zero else.

We will be interested in transition times from a local minimum, m^*, to the set of “deeper” local minima,

$$M \equiv \{m : F_{\beta,N}(m) \leq F_{\beta,N}(m^*)\}.$$

Set $S[M] = \{\sigma \in S_N : m_N(\sigma) \in M\}$.

***Theorem* 1.** *Let m^* be a local minimum of $F_{\beta,N}$; let z^* be the critical point separating m^* from M.*

$$\begin{aligned}&\mathbb{E}_{\nu_{m^*}}\tau_{S[M]}\\&=\exp\left\{\beta N\left(F_N(z^*)-F_N(m^*)\right)\right\}\end{aligned}$$

$$\times\frac{2\pi N}{\beta|\hat{\gamma}_1|}\sqrt{\frac{\beta\mathbb{E}_h\left(1-\tanh^2\left(\beta(z^*+h)\right)\right)-1}{1-\beta\mathbb{E}_h\left(1-\tanh^2\left(\beta(m^*+h)\right)\right)}}\left(1+o(1)\right),$$

where $\hat{\gamma}_1$ is the unique negative solution of the equation

$$\mathbb{E}_h\left[\frac{1-\tanh(\beta(z^*+h))}{\left[\beta\left(1+\tanh(\beta(z^*+h))\right)\right]^{-1}-\gamma}\right]=1.$$

Note that a naive approximation by a one-dimensional chain would give the same result except the wrong constant

$$\gamma=\frac{1}{\beta\mathbb{E}_h\left(1-\tanh^2\left(\beta(z^*+h)\right)\right)}-1$$

Previous work

The model was studied in

- ▷ F. den Hollander and P. dai Pra (JSP 1996) [large deviations, logarithmic asymptotics]
- ▷ P. Mathieu and P. Picco (JSP, 1998) [binary distribution; up to polynomial errors in N]
- ▷ A.B, M. Eckhoff, V. Gayrard, M. Klein (PTRF, 2001) [discrete distribution, up to multiplicative constants]

Both MP and BEGK made heavy use of exact mapping to finite-dimensional Markov chain!

The main goal of the present work was to show that potential theoretic methods allow to get sharp estimates (i.e. precise pre-factors of exponential rates) in spin systems at finite temperature when no symmetries are present. The RFCW model is the simplest model of this kind.

Equilibrium potential for $A \cap B = \emptyset$, $-L = P - 1$ generator, solution of

$$(Lh_{B,A})(\sigma) = 0, \quad \sigma \notin A \cup B,$$

with boundary conditions

$$h_{B,A}(\sigma) = \begin{cases} 1, & \text{if } \sigma \in B \\ 0, & \text{if } \sigma \in A \end{cases}.$$

Equilibrium measure $e_{B,A}(\sigma) \equiv -(Lh_{B,A})(\sigma)$.

Capacity: $\sum_{\sigma \in B} \mu(\sigma) e_{B,A}(\sigma) \equiv \text{cap}(B, A)$.

Dirichlet form $\Phi_N(f) \equiv \frac{1}{2} \sum_{\sigma,\sigma' \in S_N} \mu(\sigma) p_N(\sigma, \sigma')[f(\sigma) - f(\sigma')]^2$.

Dirichlet principle: $\text{cap}(B, A) = \Phi(h_{B,A}) = \inf_{h \in \mathcal{H}_{B,A}} \Phi_N(h)$.

Probabilistic interpretation:

$$\mathbb{P}_\sigma[\tau_B < \tau_A] = \begin{cases} h_{B,A}(\sigma), & \text{if } \sigma \notin A \cup B \\ e_{B,A}(\sigma), & \text{if } \sigma \in A. \end{cases}$$

Elements of the proof: 1. Potential theory

Equilibrium potentials and equilibrium measures also determine the Green's function:

$$h_{B,A}(\sigma) = \sum_{\sigma' \in B} G_{S_N \setminus A}(\sigma, \sigma') e_{A,B}(\sigma')$$

Mean hitting times:

$$\sum_{\sigma \in B} \mu(\sigma) e_{A,B}(\sigma) \mathbb{E}_\sigma \tau_A = \sum_{\sigma' \in S_N} \mu(\sigma') h_{A,B}(\sigma'),$$

or

$$\mathbb{E}_{\nu_{B,A}} \tau_A = \frac{1}{\text{cap}(B,A)} \sum_{\sigma' \in S_N} \mu(\sigma') h_{B,A}(\sigma').$$

where

$$\nu_{A,B}(\sigma) = \frac{\mu_{\beta,N}(\sigma) e_{B,A}(\sigma)}{\text{cap}(B,A)}.$$

Thus we need

- ▷ precise control of capacities and some
- ▷ rough control of equilibrium potential.

Remark on starting measure

The discusion above explains why it is natural in our formalism to get results for hitting times of the process started in the special measure $\nu_{m^*,\mathcal{S}[M]}$.

Of course one would expect that in most cases, the same results hold uniformly pointwise within a suitable set of in itial configurations.

In our case, we can show this to be true using a rather elaborate coupling argument.

All this would be much simpler if we had a reasonably qunatitative version of elliptic Harnack-inequalties for such processes.

Elements of Proof 2: Coarse graining

$I_\ell, \ell \in \{1,\ldots,n\}$: partition of the support of the distribution of the random field.

Random partition of the set $\Lambda \equiv \{1,\ldots,N\}$

$$\Lambda_k \equiv \{i \in \Lambda : h_i \in I_k\}$$

Order parameters

$$\boldsymbol{m}_k(\sigma) \equiv \frac{1}{N} \sum_{i \in \Lambda_k} \sigma_i$$

$$H_N(\sigma) = -NE(\boldsymbol{m}(\sigma)) + \sum_{\ell=1}^{n} \sum_{i \in I_\ell} \sigma_i \tilde{h}_i$$

where $\tilde{h}_i = h_i - \bar{h}_\ell$, $i \in \Lambda_\ell$. Note $|\tilde{h}_i| \leq c/n$;

$$E(\boldsymbol{x}) \equiv \frac{1}{2} \left(\sum_{\ell=1}^{n} \boldsymbol{x}_\ell \right)^2 + \sum_{\ell=1}^{n} \bar{h}_\ell \boldsymbol{x}_\ell$$

Equilibrium distribution of the variables $\boldsymbol{m}[\sigma]$

$$\mu_{\beta,N}(\boldsymbol{m}(\sigma) = \boldsymbol{x}) \equiv \mathcal{Q}_{\beta,N}(\boldsymbol{x})$$

Coarse grained Dirichlet form:

$$\widehat{\Phi}(g) \equiv \sum_{\boldsymbol{x},\boldsymbol{x}' \in \Gamma_N} \mathcal{Q}_{\beta,N}[\omega](\boldsymbol{x}) r_N(\boldsymbol{x},\boldsymbol{x}') \left[g(\boldsymbol{x}) - g(\boldsymbol{x}')\right]^2$$

with

$$r_N(\boldsymbol{x},\boldsymbol{x}') \equiv \frac{1}{\mathcal{Q}_{\beta,N}[\omega](\boldsymbol{x})} \sum_{\sigma:\boldsymbol{m}(\sigma)=\boldsymbol{x}} \mu_{\beta,N}[\omega](\sigma) \sum_{\sigma':\boldsymbol{m}(\sigma)=\boldsymbol{x}'} p(\sigma,\sigma').$$

The key step in the proof of both upper and lower bounds is to find a function that is almost harmonic in a small neighborhood of the relevant saddle point. This will be given by

$$h(\sigma) = g(\boldsymbol{m}(\sigma)) = f((\boldsymbol{v}, (\boldsymbol{z}^* - \boldsymbol{m}(\sigma))))$$

for suitable vector $\boldsymbol{v} \in \mathbb{R}^n$ and $f : \mathbb{R} \to \mathbb{R}_+$

$$f(a) = \sqrt{\frac{\beta N \hat{\boldsymbol{\gamma}}_1^{(n)}}{2\pi}} \int_{-\infty}^{a} e^{-\beta N |\hat{\gamma}_1| u^2/2} du.$$

This yields a straightforward upper bound for capacities which will turn out to be the correct answer, as $n \uparrow \infty$!

Lower bounds use a variational principle from Berman and Konsowa [1990]:

Let $f : \mathcal{E} \to \mathbb{R}_+$ be a non-negative unit flow from $A \to B$, i.e. a function on edges such that

- ▷ $\sum_{a\in A} \sum_b f(a,b) = 1$
- ▷ for any a, $\sum_b f(b,a) = \sum_b f(a,b)$ (Kirchhoff's law).

Set $q^f(a,b) \equiv \frac{f(a,b)}{\sum_b f(a,b)}$, and let the initial distribution for $a \in A$ be $F(a) \equiv \sum_b f(a,b)$.

This defines a Markov chain on paths $\mathcal{X} : A \to B$, with law $\mathbb{P}^f$.

Theorem 2. *For any non-negative unit flow, f, one has that, for $\mathcal{X} = (a_0, a_1, \ldots, a_{|\mathcal{X}|})$,*

$$\textit{cap}(A, B) \geq \mathbb{E}^f_{\mathcal{X}} \left[\sum_{\ell=0}^{|\mathcal{X}|-1} \frac{f(a_\ell, a_{\ell+1})}{\mu(a_\ell) p(a_\ell, a_{\ell+1})} \right]^{-1}$$

Note: the variational principle is sharp, as equality is reached for the harmonic flow

$$f(a, b) = \frac{1}{\text{cap}(A, B)} \mu(a) p(a, b) \left[h^*(b) - h^*(a) \right]_+$$

Elements of proof: Construction of flow

Again, care has to be taken in the construction of the flow only near the saddle point.

Two scale construction:

- ▷ Construct mesoscopic flow on variables $\boldsymbol{m}$ from approximate harmonic function used in upper bound. This gives good lower bound in the mesoscopic Dirichlet form.
- ▷ Construct microscopic flow for each mesoscopic path.
- ▷ Use the magnetic field is almost constant and averaging that conductance of most mesoscopic paths give the same values as in mesoscopic Dirichlet function.

This yields upper lower bound that differs from upper bound only by factor $1 + O(1/n))$.

If $A = \{\sigma : \boldsymbol{m}_N(\sigma) = \boldsymbol{m}_1\}$, $B = \{\sigma : \boldsymbol{m}_N(\sigma) = \boldsymbol{m}_2\}$, and $\boldsymbol{z}^*$ is the essential saddle point connecting them, then

$$\text{cap}(A,B) = \mathcal{Q}_{\beta,N}(\boldsymbol{z}^*)\frac{\beta|\hat{\gamma}_1|}{2\pi N}\left(\prod_{\ell=1}^{n}\sqrt{r_\ell}\right)\left(\frac{\pi N}{2\beta}\right)^{n/2}\frac{1}{\sqrt{\prod_{j=1}^{n}|\hat{\gamma}_j|}}(1+O(\epsilon))$$

This can be re-written as:

***Theorem* 3.**

$$\begin{aligned} &Z_{\beta,N}\textit{cap}(A,B) \\ &= \frac{\beta|\hat{\gamma}_1^{(n)}|}{2\pi N}\frac{\exp\left\{\beta N\left(-\frac{1}{2}\left(z^*\right)^2+\frac{1}{\beta N}\sum_{i\in\Lambda}\ln\cosh\left(\beta(z^*+h_i)\right)\right)\right\}(1+o(\epsilon))}{\sqrt{\beta\mathbb{E}_h\left(1-\tanh^2(\beta(z^*+h))\right)-1}}. \end{aligned}$$

Elements of proof: control of harmonic function

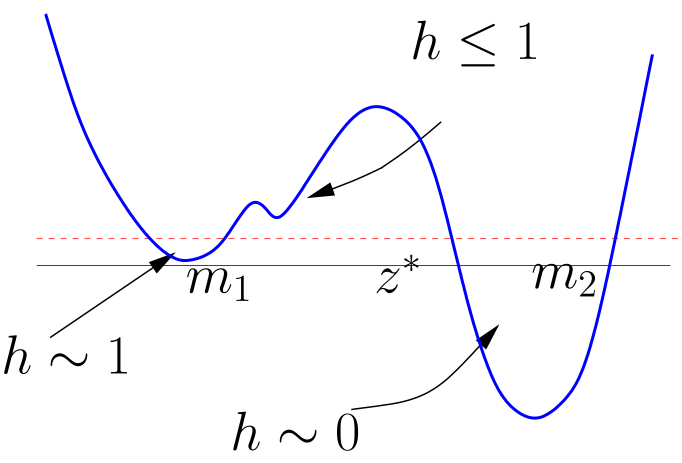
Final step in control of mean hitting times:
Compute

$$\sum_\sigma \mu_{\beta,N}(\sigma) h_{A,B}(\sigma) \sim \mathcal{Q}_{\beta,N}([\rho+m_1, m_1-\rho])$$

This requires to show that: $h_{A,B}(\sigma) \sim 1$, if σ near A, and

$$h_{A,B}(\sigma) \leq \exp\left\{-\mathbb{N}(F_{\beta,N}(z^*) - F_{\beta,N}(m_N(\sigma)) - \delta)\right\}$$

if $F_{\beta,N}(m_N(\sigma) \leq F_{b,N}(m_1)$.



Can be done using super-harmonic barrier function.

Conclusions

Nice features:

▷ We have obtained sharp estimates on exit times in a model without symmetry when entropy is relevant.

▷ Avoided use of renewal estimates for harmonic functions.

Future challenges:

▷ Control of small eigenvalues!

▷ Beyond mean field models: Kac model should be next candidate.

▷ Full scale Glauber or Kawasaki dynamics for lattice Ising!

Work on all this is in progress with Alessandra Bianchi, Frank den Hollander, Dima Ioffe, and Cristian Spitoni

Thank you for your attention!