Metastates and short-range spin glasses
Part II(b)

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Ground states and Domain walls

Definition of ground state

Domain walls

Domain walls for ground states

Domain wall measure

Ground state domain wall geometry

Definitions

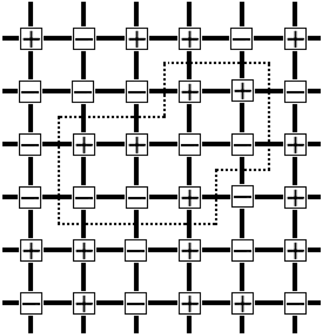
Let J be a configuration of couplings, sampled from the product measure ν.

- For any spin configuration α on $\mathbb{Z}^2$, we say that α is an *infinite-volume ground state for J* if the energy of each closed dual loop is positive.
- The *energy* of a set S of dual edges in a coupling configuration J and a spin configuration α is the sum

$$\sum_{\langle x,y\rangle \in S} J_{xy}\alpha_x\alpha_y.$$

Definitions

This coincides with the finite-volume definition of a ground state.



- In this case, the energy of a dual loop is $1/2$ of the energy needed to "flip" the spins surrounded by that loop.
- It costs positive energy to flip any finite set of spins.

Definitions

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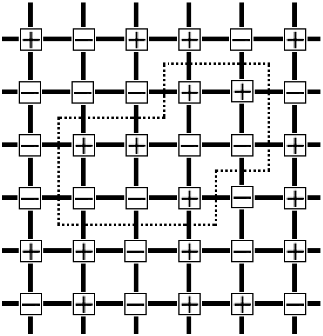


- Note that if α is a ground state then $-\alpha$ (the global flip of α) is also a ground state.
- Therefore we consider *ground state pairs* (GSP's).

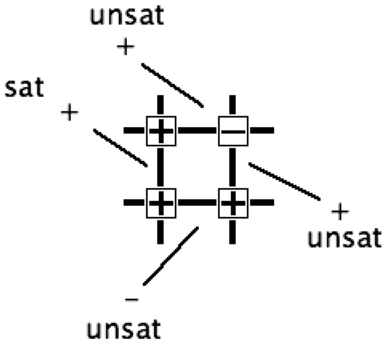
Domain walls

We begin with a metastate measure $\mathcal{K}_J$ for $\mathbb{Z}^2$ (or for the half-plane).

- This is constructed using boxes $\Lambda^{(L)}$ (or Λ_n) of the form $[-n, n] \times [-n, n]$ with periodic boundary conditions (or periodic horizontally and free vertically).
- Recall that this is a measure on infinite-volume ground states (globally flip-related spin configurations) on $\mathbb{Z}^2$ (or the half-plane).

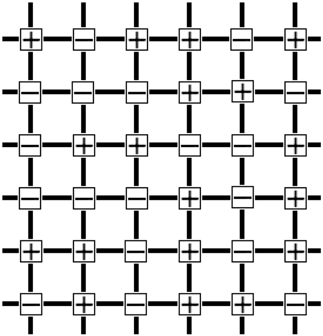
Domain walls

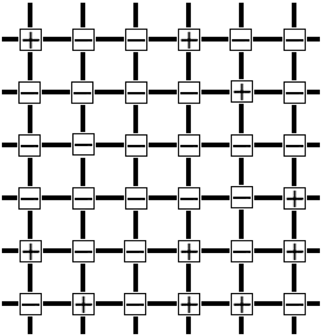
Sample two states (pairs) α and β from $\mathcal{K}_J^\sharp$. We define the *interface* $\alpha\Delta\beta$ as a subset of the dual lattice:



- a dual edge e^* is in the interface if it is satisfied in exactly one of α and β; and
- e^* is *satisfied* in α if it has positive energy in α.

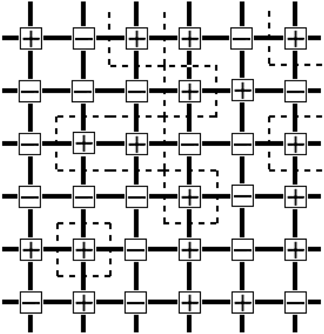
Example

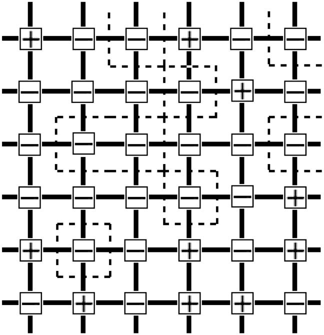




- The interface separates regions where spins agree from where spins disagree.
- Therefore the connected components (*domain walls*) form either infinite paths or closed loops (no dangling ends).

Example





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Domain walls for ground states

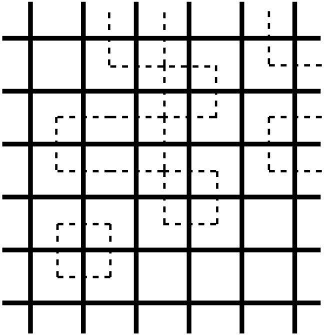
We want to study properties of domain walls between ground states. To do this:

- identify each state (spin config. pair) with its set of satisfied and unsatisfied couplings,
- choose two states from the product measure $\mathcal{K}_J \times \mathcal{K}_J$ and integrate out the couplings.

This gives a measure $\mathcal{D}$ on domain wall configurations. It is translation-invariant because each $\mathcal{K}_J$ was built with periodic boundary conditions.

Domain walls for ground states

If the $\mathcal{K}_J$ were not supported on GSP's, we could sample something like this from $\mathcal{D}$ (the configuration in the last example):



But interfaces between GSP's cannot contain closed loops (reasons soon).

Domain walls for ground states

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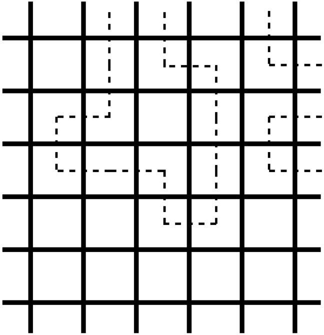


But interfaces between GSP's cannot contain closed loops (reasons soon). The config. at left is now admissible.

Newman-Stein Theorem

For the rest of the talk we will discuss the geometry of ground state domain walls.

Theorem (Newman-Stein)

Let α and β be two GSP's on $\mathbb{Z}^2$ (or on the half-plane).

1. *The interface $\alpha\Delta\beta$ contains no loops or dangling ends.*
2. *If α and β are sampled from $\mathcal{K}_J$, (on $\mathbb{Z}^2$) then for a.e. J, $\alpha\Delta\beta$ contains no "branching points." In other words, the domain walls are double-infinite paths in which each (dual) vertex has degree two.*

Here "dangling end" means a (dual) vertex of degree one in the interface.

Proof idea

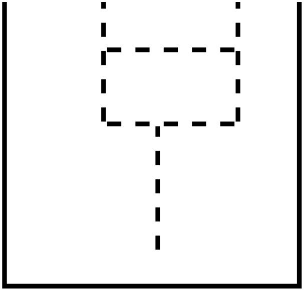
We saw that an interface between *any* two states cannot have dangling ends. (Interfaces form boundaries of sets of spins.) We must show:

- The interface between GSP's cannot have loops; and
- any interface sampled from $\mathcal{D}$ (a.s.) cannot have branching points.

The first statement concerns *all* GSP interfaces. The second refers only to those constructed through the metastate (with p.b.c.)

Proof idea

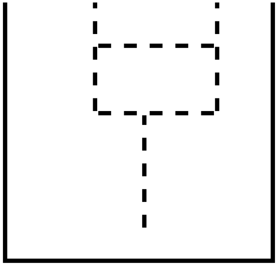
In this figure, the interface has (a) a loop, (b) a dangling end, and (c) a branching point.



Loops

Loops

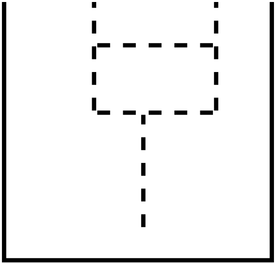
Why can't a GSP interface have loops?



- Recall the definition of a GSP: each dual loop has positive energy.
- Therefore the loop would have positive energy in both states α and β.
- But the satisfaction status of each coupling on the loop is different in each state.

Loops

Why can't a GSP interface have loops?



We use the fact that an edge in the interface has opposite energy in both states:

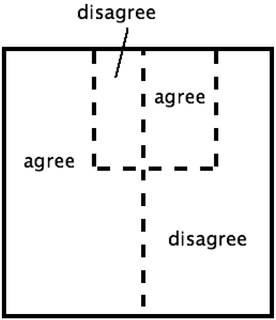
$$E_\alpha(L) = -E_\beta(L).$$

These can only both be non-negative if they are zero.

Branching points

Branching points

Now we must deal with branching points.



- Notice that since there are no loops, branching points must connect distinct paths to infinity.
- Also they must connect an even number (4) of paths.

Branching points

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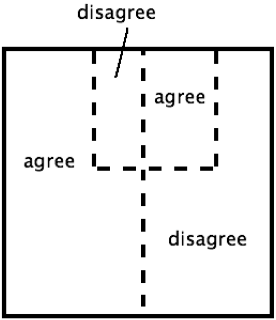
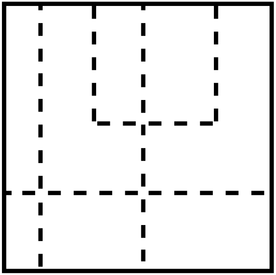


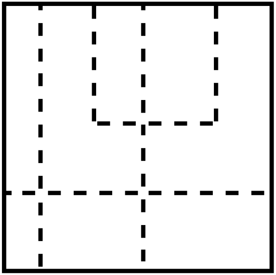
To rule out branching points (dual vertices of degree ≥ 3) we use the standard Burton-Keane argument.

Branching points – Burton-Keane



- We can associate with each branching point a unique point on the boundary. (Adding a branching point adds at least one new boundary point.)

Branching points – Burton-Keane

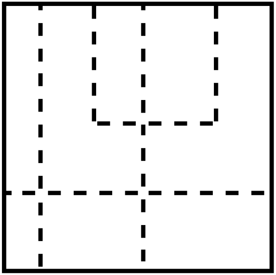


- This gives

  number of branching points in box

  $\leq$ surface area.

Branching points – Burton-Keane



- But by ergodicity, in expectation, number of branching points in box $\geq c$ (volume).
- This is a contradiction.

Final picture

Because of the theorem, if interfaces exist, they consist of disjoint doubly-infinite paths.