# Section 1

# Causal and Bayesian Networks

# The car start problem (causally)

In the morning, my car will not start. I can hear the starter turn, but nothing happens. The most probable causes are that the fuel has been stolen over night or that the spark plugs are dirty. I look at the fuel meter. It shows *half full*, so I decide to clean the spark plugs.

Events:

- Fuel?{y,n}
- Clean spark plugs?{y,n}
- Start?{y,n}
- Fuel meter{full, $\frac{1}{2}$, empty}.

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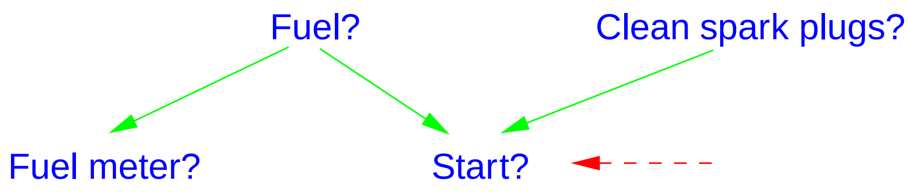
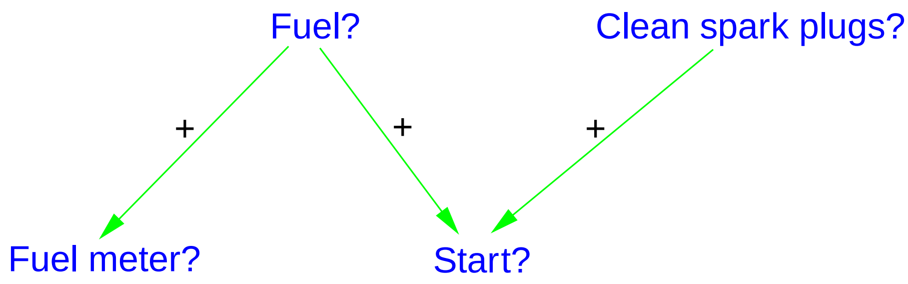
- Fuel?{y,n}
- Clean spark plugs?{y,n}
- Start?{y,n}
- Fuel meter{full, $\frac{1}{2}$,empty}.

Causal relations:




When I enter the car I have some prior belief on the various events but then start=n.

# The reasoning (explaining away)



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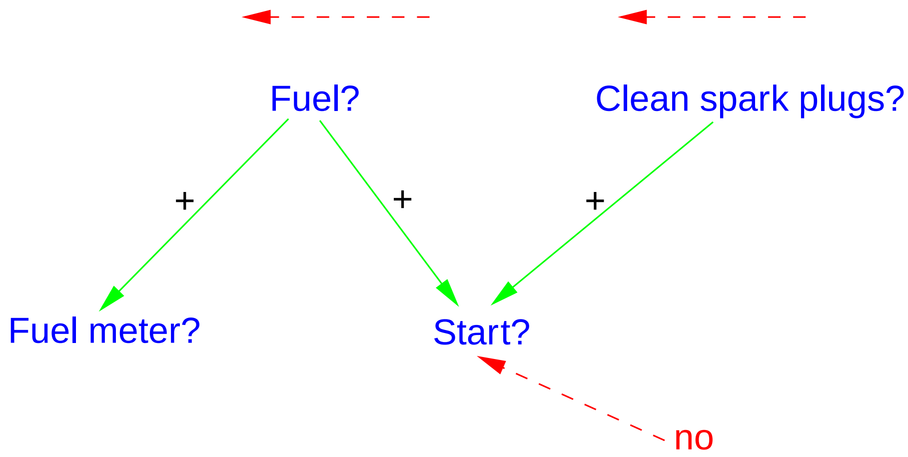
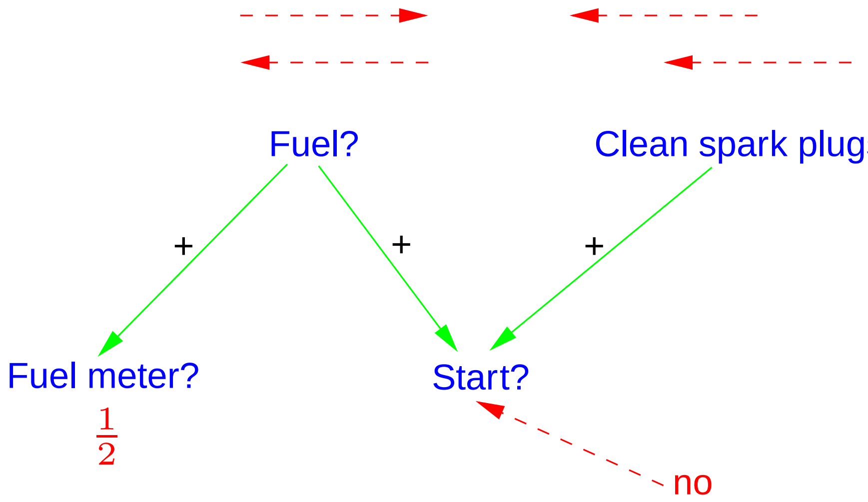
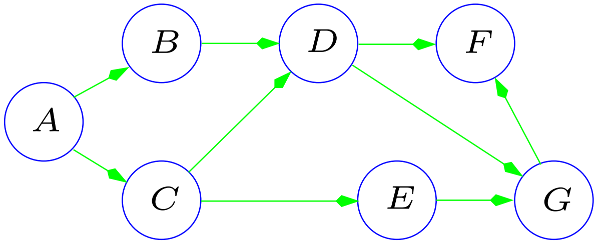


# The reasoning (explaining away)



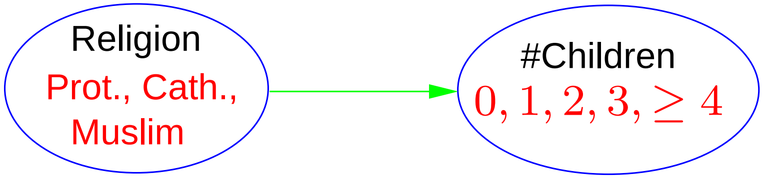
# Causal networks

A causal network is a directed acyclic graph:




- The nodes are variables with a finite set of states that are mutually exclusive and exhaustive:
  - For example {y,n}, {red, blue, green}, {0,1,2,3,42}.
- The links represent cause – effect relations.

For example:

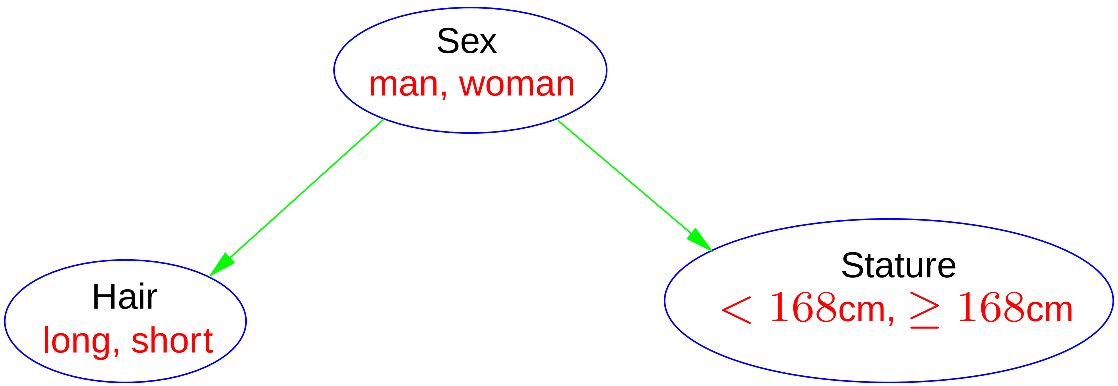



All variables are in exactly one state, but we may not know which one.

# Reasoning under uncertainty 1

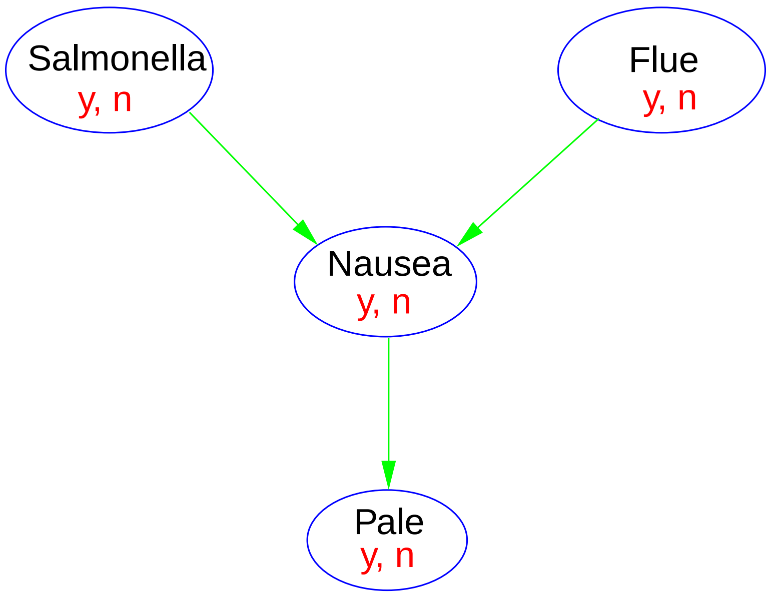

- Information on flooding may change the belief of *Rainfall*.
- If we know that the water level is high, then information on flooding will not change the belief of *Rainfall*.

# Reasoning under uncertainty 2




- Knowledge of a person's hair length may change the belief of the stature.
- If we know that it is a women, then information of hair length has no impact on the belief of her stature.

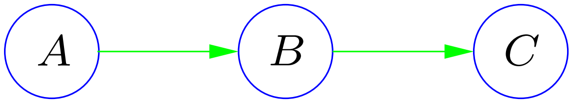
# Reasoning under uncertainty 3



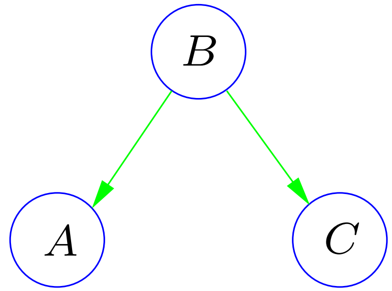

- Salmonella has nor impact on Flue?
- If a person is Pale, then knowledge of *Salmonella* may change the belief on *Flue* (explaining away).

# Rules of d-separation

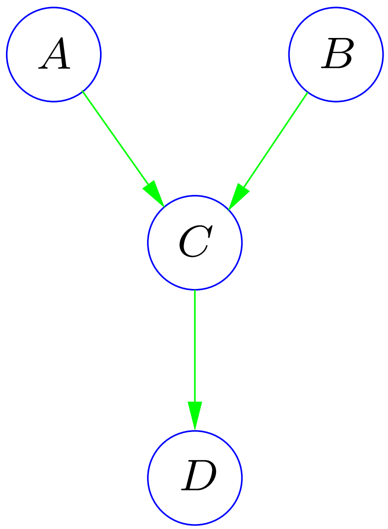
Relevance changes with evidence




Serial



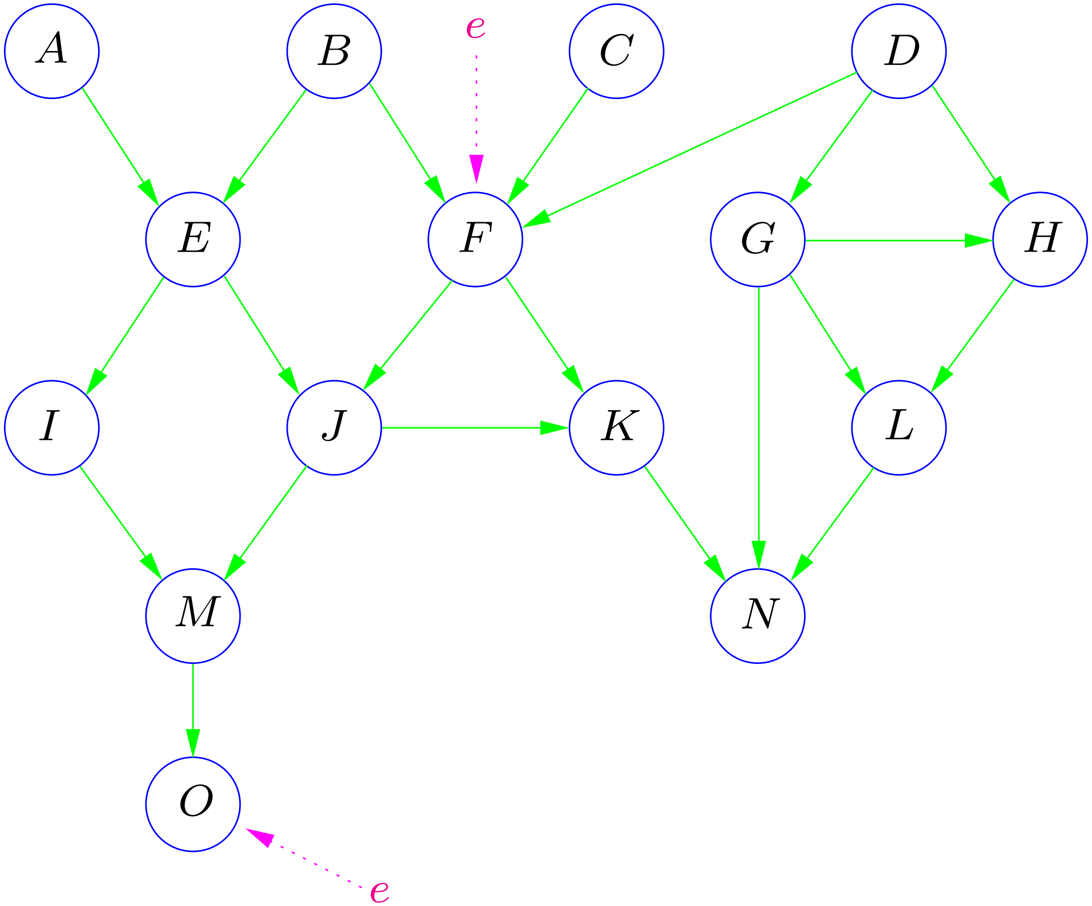

Diverging




Converging

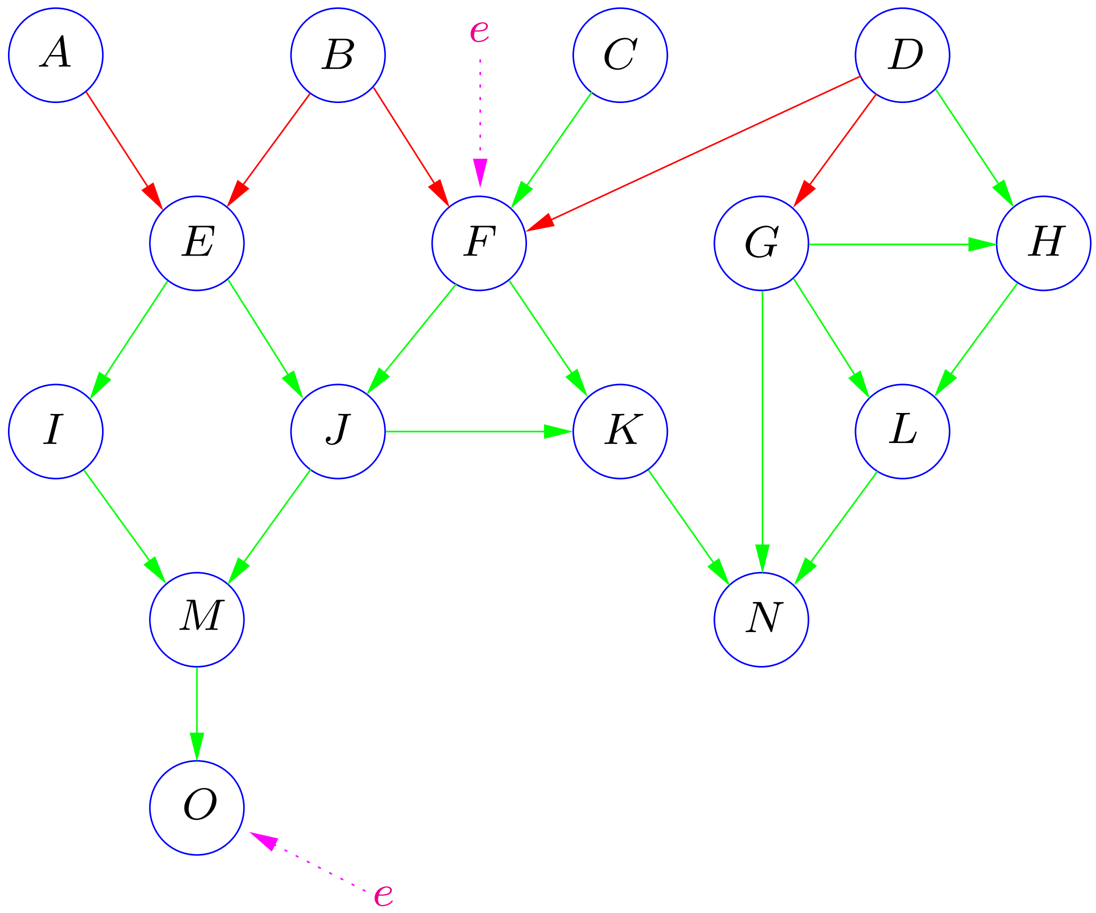
**Note:** These are general rules about reasoning under uncertainty on causal models.

# Transmission of evidence



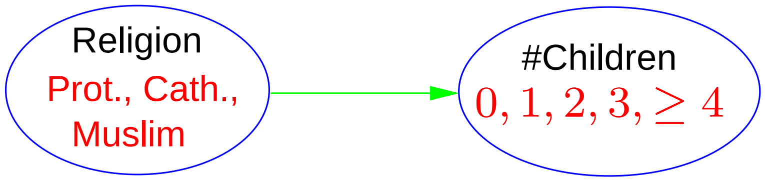

Can knowledge of $A$ have an impact on our knowledge of $G$?

# Transmission of evidence




Can knowledge of $A$ have an impact on our knowledge of $G$? yes!

# Quantification of causal networks



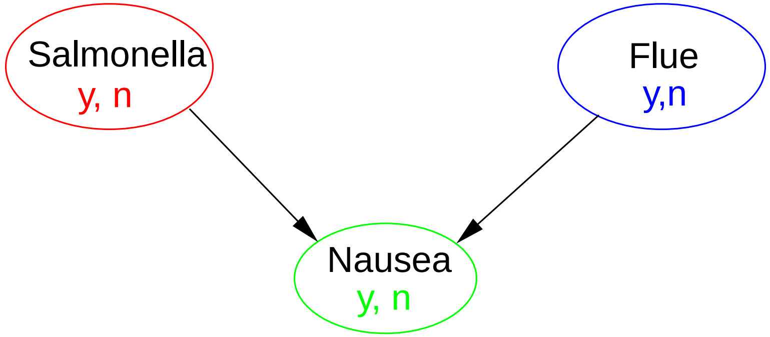

The <u>strength</u> of the link is represented by probabilities:

$$\begin{array}{ccc} P(0|p) & P(0|c) & P(0|m) \\ P(1|p) & P(1|c) & P(1|m) \\ P(2|p) & P(2|c) & P(2|m) \\ P(3|p) & P(3|c) & P(3|m) \\ P(\geq 4|p) & P(\geq 4|c) & P(\geq 4|m) \end{array} \quad \Rightarrow$$

| | | Religion | | |
|---|---|---|---|---|
| | | p | c | m |
| #Children | $0$ | $0.15$ | $0.05$ | $0.05$ |
| | $1$ | $0.2$ | $0.1$ | $0.1$ |
| | $2$ | $0.4$ | $0.2$ | $0.1$ |
| | $3$ | $0.2$ | $0.4$ | $0.1$ |
| | $\geq 4$ | $0.05$ | $0.25$ | $0.35$ |

$P$(#Children|Religion)

# Several parents




$$P(y|y,y) \quad P(y|y,n)$$
$$P(n|y,y) \quad P(n|y,n)$$
$$P(y|n,y) \quad P(y|n,n)$$
$$P(n|n,y) \quad P(n|n,n)$$

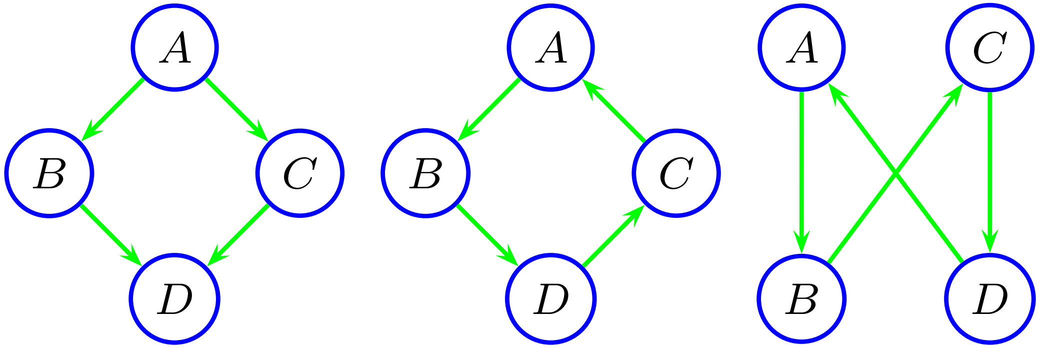
$$\Longrightarrow$$

| | | Salmonella y | Salmonella n |
|---|---|---|---|
| Flue | y | $(0.9, 0.1)$ | $(0.6, 0.4)$ |
| Flue | n | $(0.8, 0.2)$ | $(0.1, 0.9)$ |

$P$(Nausea|Salmonella,Flue)

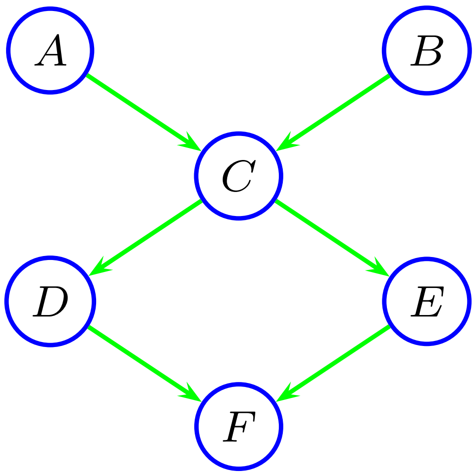
# Bayesian networks

A causal network without directed cycles:



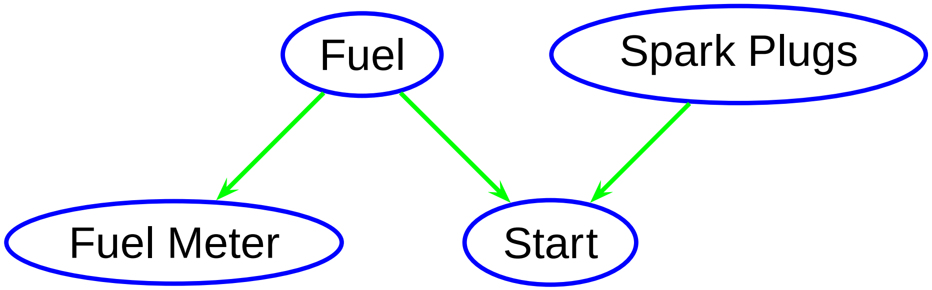

OK ¬OK ¬OK

For each variable $A$ with parents $B_1, \ldots, B_n$ there is a conditional probability table $P(A|B_1, \ldots, B_n)$.

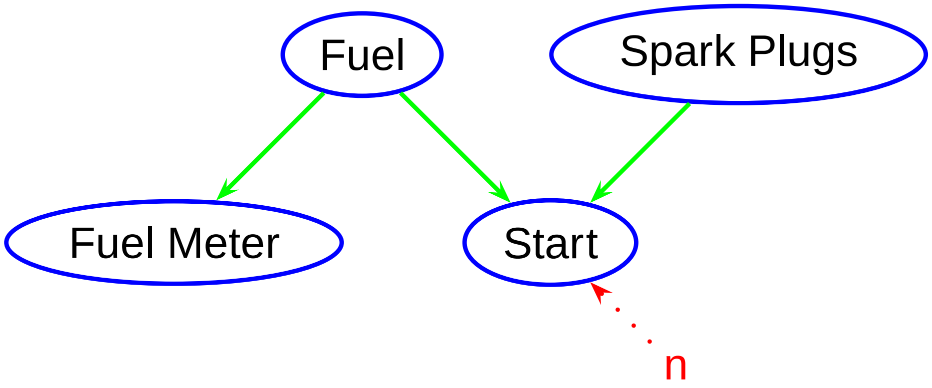



Note: Nodes without parents receive a prior distribution.

# Bayesian networks as a tool for reasoning



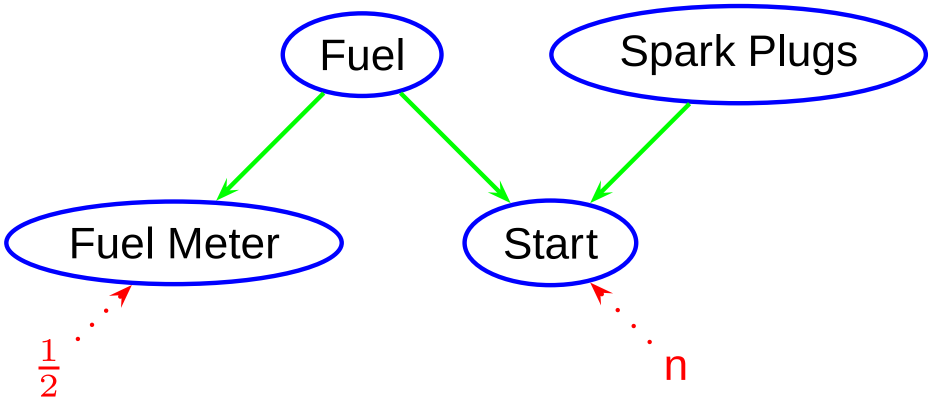
# Bayesian networks as a tool for reasoning




Consider evidence $e_1$ =(Start=n) and find:

- $P($Spark Plugs$|e_1) = ??$
- $P($Fuel$|e_1) = ??$
- $P($Fuel Meter$|e_1) = ??$

# Bayesian networks as a tool for reasoning



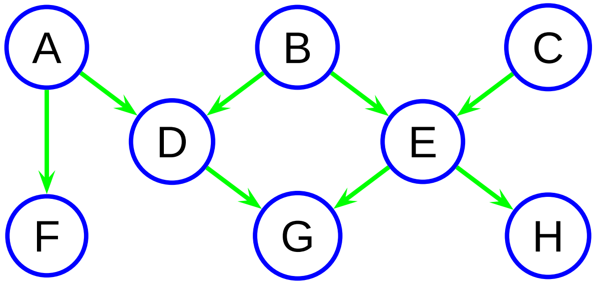

Consider evidence $e_1$ =(Start=n) and find:

- $P($Spark Plugs$|e_1) = ??$
- $P($Fuel$|e_1) = ??$
- $P($Fuel Meter$|e_1) = ??$

If we also have evidence $e_2$=(Fuel Meter $= \frac{1}{2}$) what is:

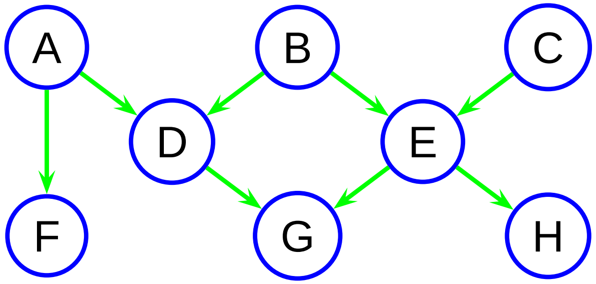
- $P($Spark Plugs$|e_1, e_2) = ??$
- $P($Fuel$|e_1, e_2) = ??$

# Bayesian belief updating




Find $P(B|a, f, g, h)$

# Bayesian belief updating




Find $P(B|a, f, g, h)$

We can if we have access to $P(a, B, C, D, E, f, g, h)$:

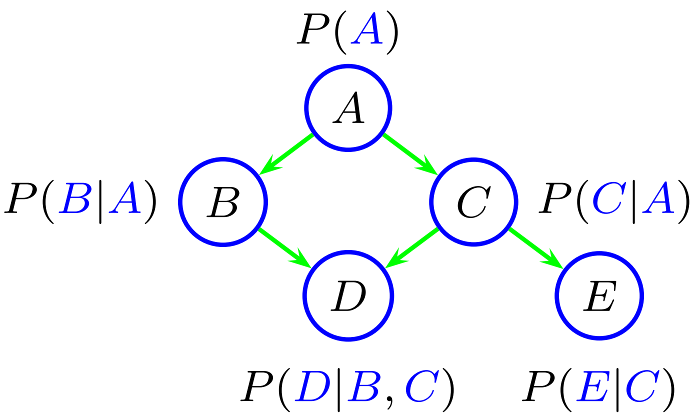
$$P(B, a, f, g, h) = \sum_{C,D,E} P(a, B, C, D, E, f, g, h)$$

$$P(B|a, f, g, h) = \frac{P(B, a, f, g, h)}{P(a, f, g, h)},$$

where

$$P(a, f, g, h) = \sum_{B} P(B, a, f, g, h)$$

# Joint probabilities




Calculate $P(A, B, C, D, E)$

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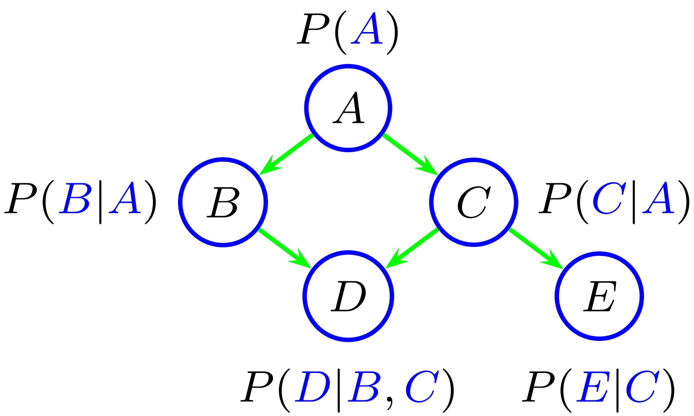


Calculate $P(A, B, C, D, E)$

$$P(A, B, C, D, E) = P(E|A, B, C, D)P(A, B, C, D)$$

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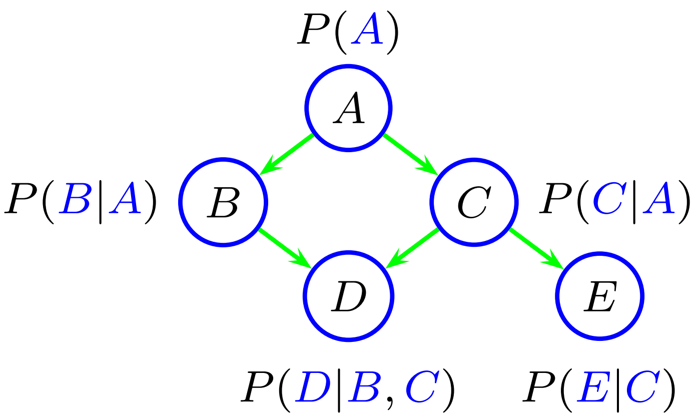


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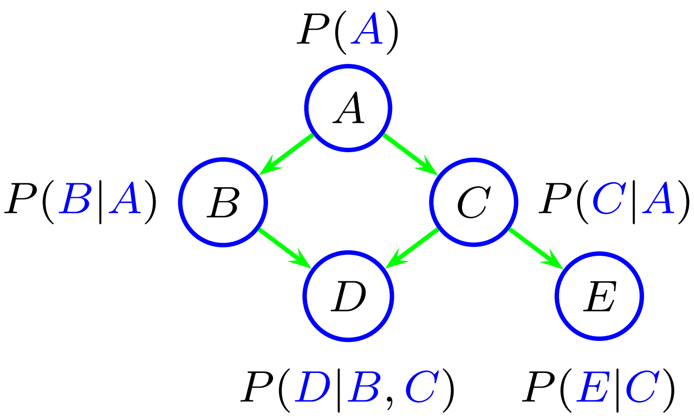


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$$

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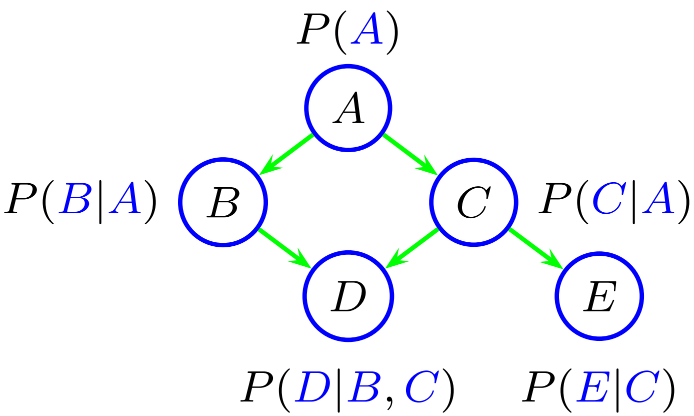


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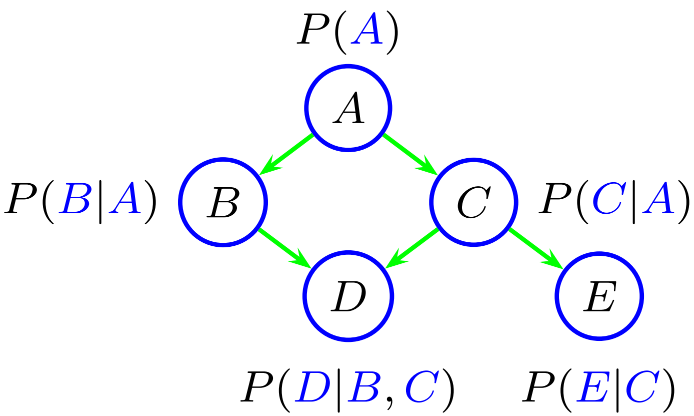


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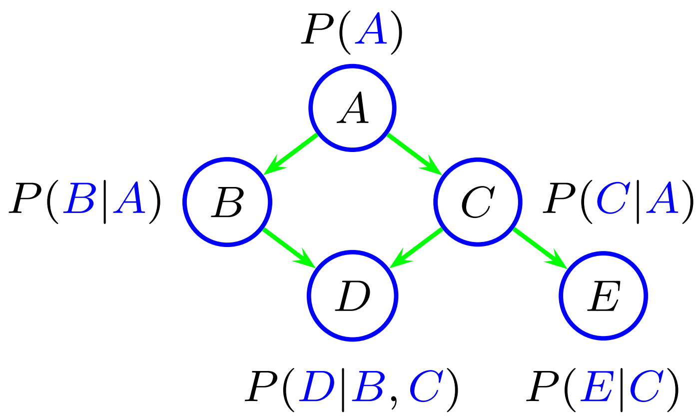
# Joint probabilities

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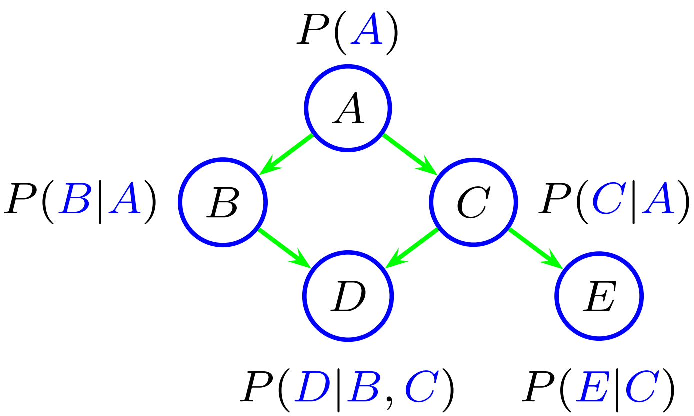


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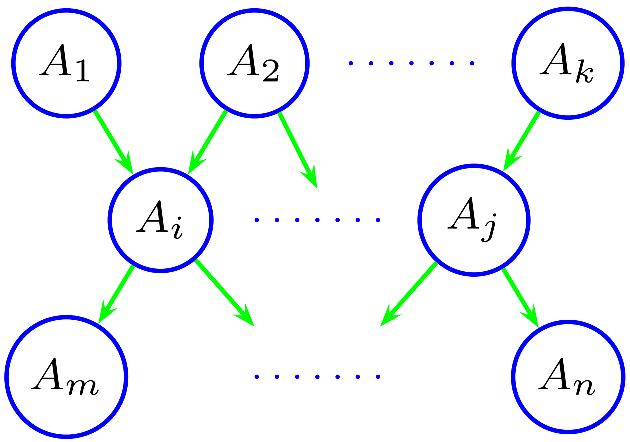
&= P(E|C)P(D|B,C)P(C|A)P(B|A)P(A)
\end{aligned}
$$

# The chain rule



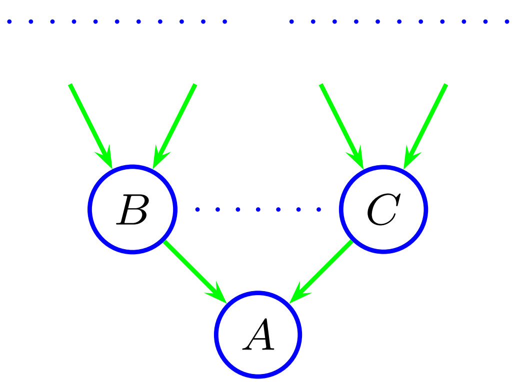

Let BN be a Bayesian network over $\mathcal{U} = \{A_1, \ldots, A_n\}$

Then:

$$P(\mathcal{U}) = \prod_i P(A_i|\text{Pa}(A_i)),$$

where Pa$(A_i)$ are the parents of $A_i$.

- $P(\mathcal{U})$ is the product of the potentials specified in BN.
- BN is a compact representation of $P(\mathcal{U})$.




$$P(\mathcal{U}) = P(A|\mathcal{U} \setminus \{A\}) P(\mathcal{U} \setminus \{A\})$$

$$= P(A|B, \ldots, C) \prod_{X \in \mathcal{U} \setminus \{A\}} P(X|\text{Pa}(X))$$

# Evidence I

Consider a variable $A$ with five states $a_1, a_2, a_3, a_4, a_5$ and with probability:

$$P(A) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}, \quad \sum_{i=1}^{5} x_i = 1$$

Assume that we get the evidence $e$: "$A$ is either in state $a_2$ or $a_4$". Then:

$$P(A, e) = \begin{pmatrix} 0 \\ x_2 \\ 0 \\ x_4 \\ 0 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Thus, $e$ can be represented by a potential $\bar{e} = (0, 1, 0, 1, 0)^T$ and:

$$P(A, e) = P(A) \cdot \bar{e}$$

# Evidence II

Definition: Let $A$ be a variable with $n$ states. A finding on $A$ is an $n$-dimensional table with $0$s and $1$s.

Semantics: The states marked with a $0$ are impossible.

Theorem: Let BN be a Bayesian network over the universe $\mathcal{U} = \{A_1, \ldots, A_n\}$, and let $\bar{e}_1$, $\bar{e}_2$,. . . , $\bar{e}_m$ be findings. Then:

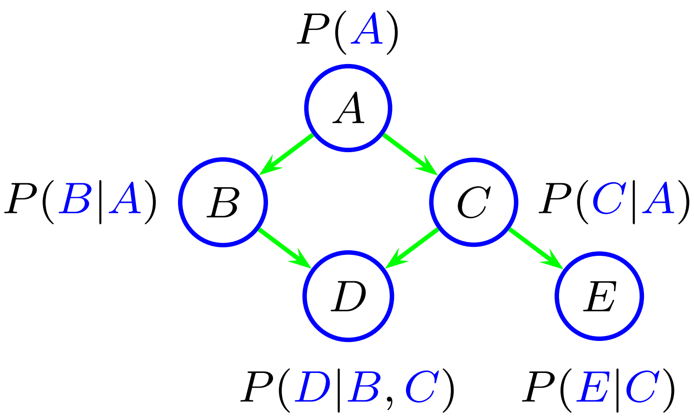
$$P(\mathcal{U}, e) = P(\mathcal{U}) \cdot \prod_{i=1}^{m} \bar{e}_i$$

$$= \prod_{i=1}^{n} P(A_i | \text{Pa}(A_i)) \prod_{j=1}^{m} \bar{e}_j .$$

Hence, to find $P(A|e)$ we use:

$$P(A|e) = \frac{\sum_{\mathcal{U} \setminus \{A\}} P(\mathcal{U}, e)}{P(e)}.$$

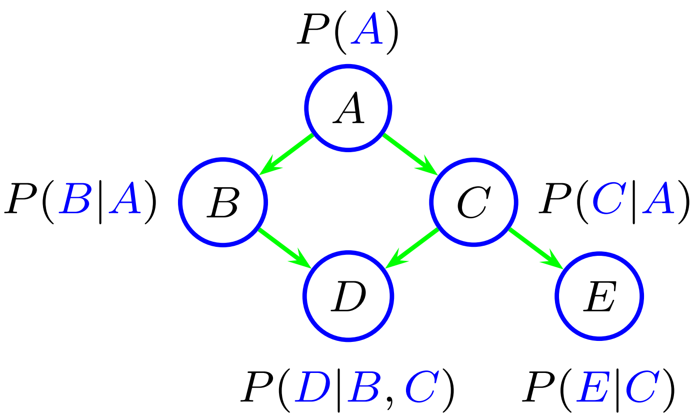
# Variable elimination




Do we need $P(\mathcal{U}) = P(A, B, C, D, E)$ in order to calculate $P(A|c, e)$?

Note: $P(A|c,e) = \frac{\sum_B \sum_D P(A,B,c,D,e)}{P(c,e)}$.

# Variable elimination

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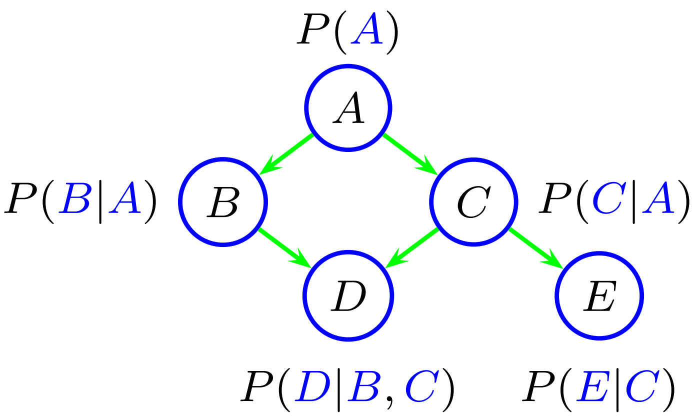
$$\sum_B \sum_D P(A, B, c, D, e) = \sum_B \sum_D P(e|c)P(c|A)P(D|c, B)P(A)P(B|A)$$

# Variable elimination

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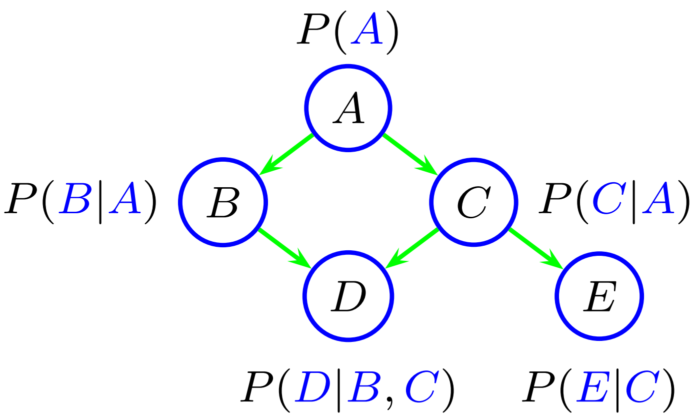


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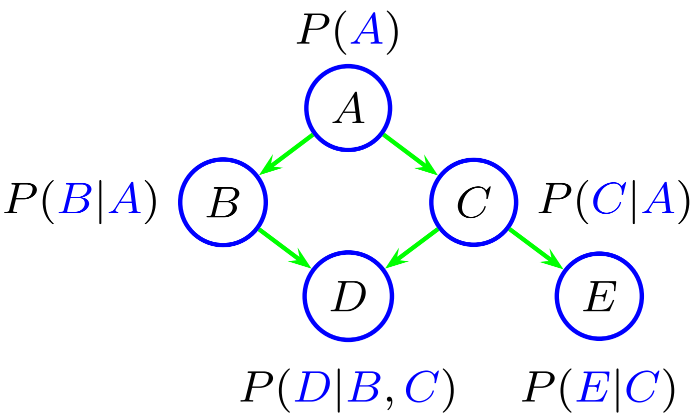
$$= P(e|c)P(c|A)P(A) \sum_B P(B|A) \sum_D P(D|c,B)$$

# Variable elimination

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Note: $P(A|c, e) = \frac{\sum_B \sum_D P(A,B,c,D,e)}{P(c,e)}$.

$$\sum_B \sum_D P(A, B, c, D, e) = \sum_B \sum_D P(e|c)P(c|A)P(D|c, B)P(A)P(B|A)$$

$$= P(e|c)P(c|A)P(A) \sum_B \sum_D P(D|c, B)P(B|A)$$

$$= P(e|c)P(c|A)P(A) \sum_B P(B|A) \sum_D P(D|c, B)$$

$$= P(e|c)P(c|A)P(A)$$

So instead of constructing a table with $2^5$ entries we only need $2$ numbers!