Historic behaviour

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We consider deterministic dynamical systems given by a continuous map $\varphi : X \to X$

an orbit $\{x, \varphi(x), \varphi^2(x), \dots\}$ has historic behaviour if for some continuous $f: X \to \mathbb{R}$ the average

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{i=0}^{n} f(\varphi^i(x))$$

does not exist

main question: is historic behaviour exceptional? i.e. is it not persistent?

the same question is relevant for dynamical systems with continuous time, i.e. given by a differential equation one <u>expects</u> the answer to the main question to be '<u>yes</u>': e.g. a change of the 'average temperature' on the earth is not expected without a change in the underlying dynamical system

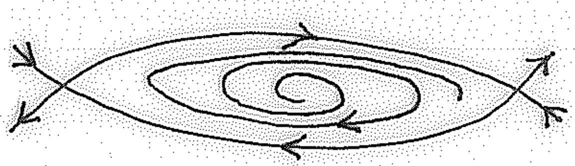
<u>some restrictions</u>: X, or at least the closure of the orbit, should be <u>compact</u> (an orbit whose closure is not compact has historic behaviour)

also, we assume φ to be <u>smooth</u>

<u>remark</u>: the notion 'historic behaviour' is due to D. Ruelle: Historic behaviour in smooth dynamical systems, in: Global Analysis of Dynamical Systems, ed. H.W. Broer et al, IoP, 2001,

who relates this with historic behaviour of spin glasses examples

1. For a classical dynamical system with compact energy levels, the initial points of orbits with historic behaviour have measure zero (Birkhoff's individual ergodic theorem)
2. Hofbauer & Keller (1990) provided a quadratic inteval map with historic behaviour (for a full measure set of initial positions) showing that in $C^3(I)$ there is a codimension 1 manifold with historic behaviour
3. Bowen (?) continuous time



heteroclinic loop with dominating attraction

Gaunersdorfer (1992)
Takens (1994)

Persistent versions of Bowens example

3^a for dynamical systems (continuous time) in $\mathbb{R}^3$, restricted to

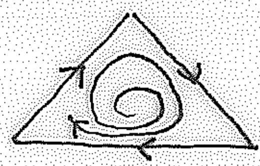
$$\{(x,y,z) \mid x,y,z \geq 0\}$$

(population dynamics)

one can have 'persistently'

$\Delta = \{x+y+z=1\}$ invariant and attracting

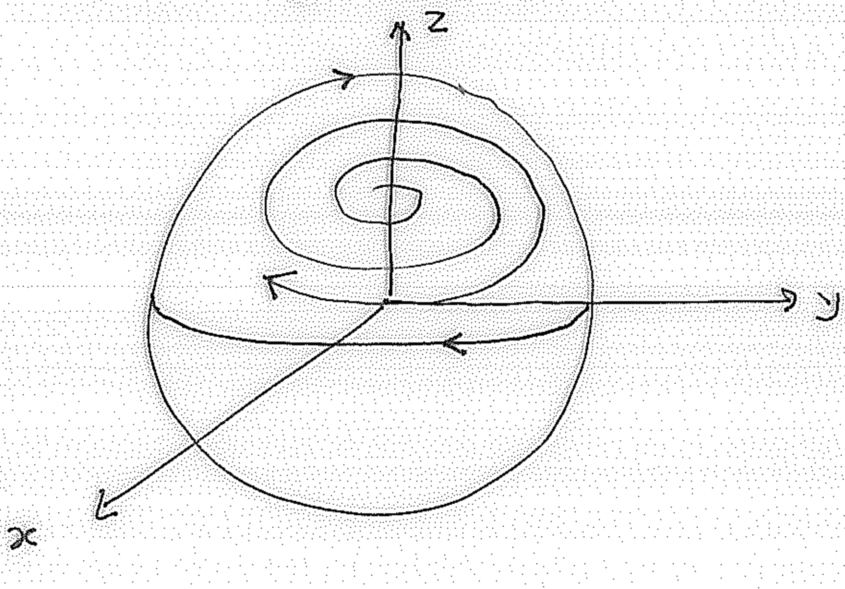
with heterodinic cycle with dominating attraction:



3^b for a dynamical system in $\mathbb{R}^3$ with symmetries

$$(x, y, z) \longmapsto (\pm x, y, \pm z)$$

the unit sphere can be made 'persistently' attracting and invariant with dynamics:



Historic behaviour in hyperbolic systems

As prototype of hyperbolic system consider $X = S^1 = \mathbb{R}/1$ and $\varphi(x) = 2x \mod 1$ (or any perturbation with $\varphi' > 1$ everywhere)

we will see that for 'generic' $x \in X$, the orbit $\{x, \varphi(x), \varphi^2(x), \ldots\}$ has historic behaviour

'for generic $x \ldots$' means that there is a residual subset $R \subset X$ such that for $x \in R$ the conclusion holds

a subset is residual in X if it contains a countable intersection of open and dense subsets of X

residual

open and dense subsets are big subsets (containing 'almost all' points) in the topological sense

For technical reasons also countable intersections of these are considered as big – note that in a complete metric space residual sets are dense

so

residual = almost all in the topological sense

still residual is very different from 'full Lebesgue measure', e.g. for the doubling map ($x \mapsto 2x \bmod 1$ on S^1) the set of initial points of orbits without historic behaviour has full Lebesgue measure (Birkhoff)

<u>proposition</u> (Dowker (1953))

let $\varphi : X \to X$ and $f : X \to \mathbb{R}$ be continuous

if φ has a dense orbit $\{\bar{x}, \varphi(\bar{x}), \dots\}$ for which $\lim\limits_{n\to\infty} \frac{1}{n+1} \sum\limits_{i=0}^{n} f(\varphi^i(\bar{x}))$ does not exist, then the set of $x \in X$, for whose orbit this limit does not exist, is residual in X

<u>proof</u>

let

$$\liminf_{n\to\infty} \frac{1}{n+1} \sum_{i=0}^{n} f(\varphi^i(\bar{x})) < \alpha < \beta < \limsup_{n\to\infty} \frac{1}{n+1} \sum_{i=0}^{n} f(\varphi^i(\bar{x}))$$

define $\mathcal{U}_h = \Big\{ x \in X \;\Big|\; \exists\ j \geq h,\ k \geq h$ such that

$$\frac{1}{j+1} \sum_{i=0}^{j} f(\varphi^i(x)) < \alpha \quad \text{and} \quad \frac{1}{k+1} \sum_{i=0}^{k+1} f(\varphi^i(x)) > \beta \Big\}$$

$\mathcal{U}_h$ is dense (it contains the $\bar{x}$ orbit) and open

$\bigcap\limits_{h=0}^{\infty} \mathcal{U}_h$ is residual

symbolic dynamics for the doubling map

$I_0 = [0, 1/2) \quad I_1 = [1/2, 1) \subset S^1$

each orbit $\{\varphi^i(x)\}$ defines a symbolic sequence

$$s^i(x) = \begin{cases} 0 & \text{if } \varphi^i(x) \in I_0 \\ 1 & \text{if } \varphi^i(x) \in I_1 \end{cases}$$

1-1 correspondence

$S^1 \ni x \longleftrightarrow$ symbolic sequences without 'tail of 1's'

construction of symbolic sequence with historic behaviour

consider two orbits

$\{\varphi^i(x')\}$ and $\{\varphi^i(x'')\}$

with symbolic sequences

$\{s^i(x')\}$ and $\{s^i(x'')\}$

such that

— $\{\varphi^i(x')\}$ is dense in S^1

— there is a continuous $f: S^1 \to \mathbb{R}$ with

$$\alpha = \lim_{n\to\infty} \frac{1}{n+1} \sum_{i=0}^{n} f(\varphi^i(x')) \neq \lim_{n\to\infty} \frac{1}{n+1} \sum_{i=0}^{n} f(\varphi^i(x'')) = \beta$$

<u>mixing the two sequences</u>

$\{s_0, \dots, s_{N_1-1}, s_{N_1}, \dots, s_{N_2-1}, s_{N_2} \dots \}$ ~ point x'''
(with $N_0 = 0$)

with

$$\{s_{N_j}, \dots, s_{N_{j+1}-1}\}$$

a (long) segment of $\{s^i(x')\}$ resp. $\{s^i(x'')\}$ if j = even resp. odd; so long that, independent of the continuation beyond $N_{j+1}-1$,

$$\frac{1}{N_{j+1}} \sum_{i=0}^{N_{j+1}-1} f(\varphi^i(x''')) \in 2^{-j}\text{-neighbourhood}$$

of α resp. β