Efficient Control of Epidemics over Random Networks

Marc Lelarge (INRIA-ENS)

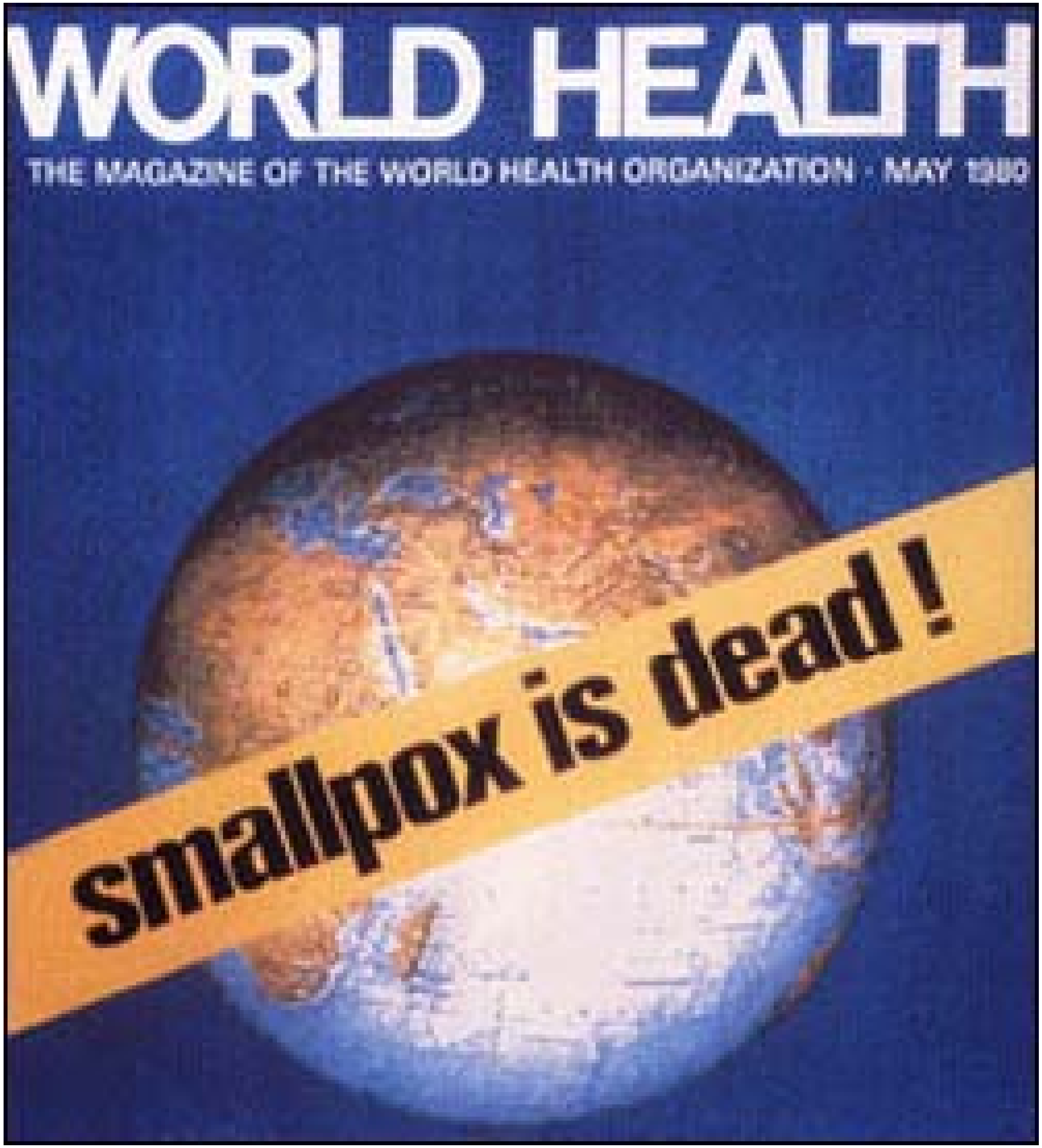
EURANDOM YEQT-III

A simple problem

- Model for an epidemic on a graph:
  - Spread of worms, email viruses…
  - Diffusion of information, gossip…
- Strategic players: Attacker / Defender
  - Defender plays first by vaccinating nodes.
  - Viral marketing.
- Information available: none for the Defender!
- Defense has to be decentralized.

A hopeless goal?

Some hope…


WORLD HEALTH
THE MAGAZINE OF THE WORLD HEALTH ORGANIZATION · MAY 1980
smallpox is dead !

Acquaintance vaccination

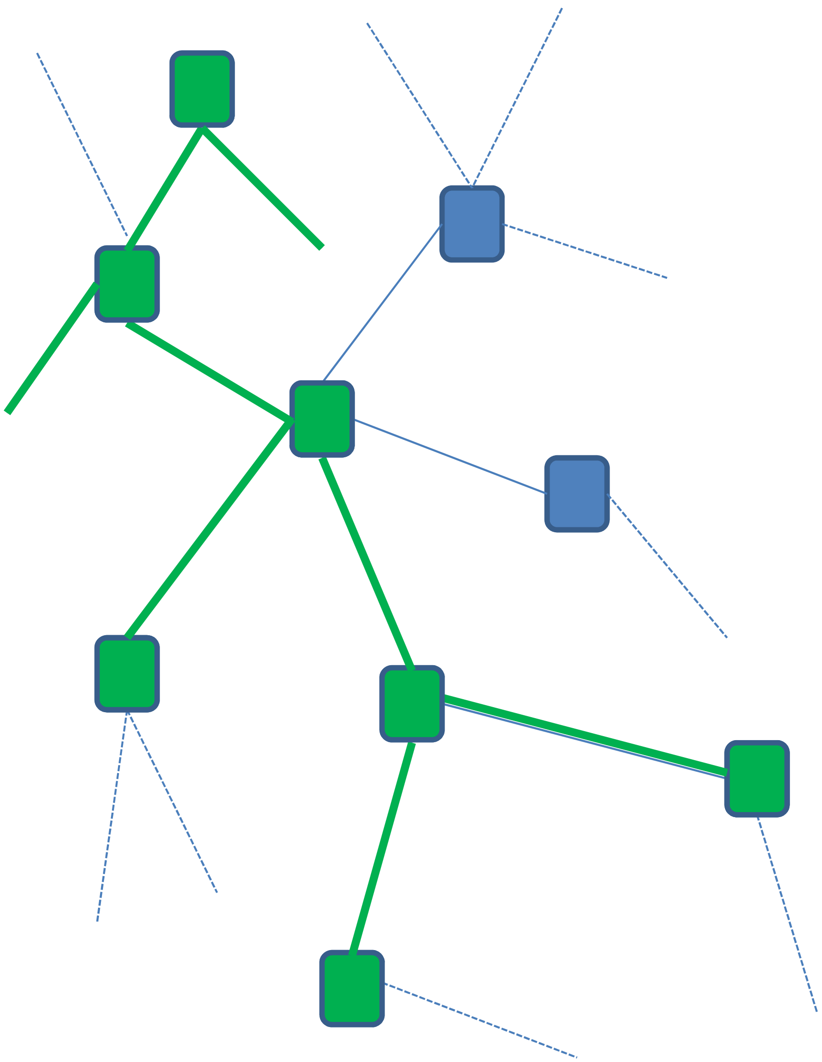
- Proposed by Cohen, Havlin and ben-Avraham in Phys.Rev.Let. 2003.
- Sample each node uniformly and inoculate a neighbor of this node taken at random.
- Why does it work?
  - Sampling with a bias toward high-degree nodes.

(1) Percolated Threshold Model

(2) Analytic Results

(3) Toolbox

(1) Percolated Threshold Model



- Bond percolation with proba. $1-\pi$
- Symmetric threshold epidemic:

$$\sum_{j\sim_\pi i} X_j \geq K_i(d_i)$$

- Seed of active nodes

(1) Versatile model for epidemics

- Null threshold = contact process
- No bond percolation = bootstrap percolation
- Some easy general results:
  - Monotonicity: only transition passive to active.
  - In a finite graph, there is only one possible final state for the epidemic.
- I will concentrate on properties of the final state, for large random graphs.

(1) Diluted Random Graphs

$(d_i)_1^n$ is a sequence of integers with $\sum_i d_i$ even and, for some probability distribution $(p_r)_{r=0}^{\infty}$ independent of n,

(i) $\#\{i : d_i = r\}/n \to p_r$ as $n \to \infty$;

(ii) $\lambda := \sum_r r p_r \in (0, \infty)$;

Molloy-Reed (95)

(iii) $\sum_i d_i/n \to \lambda$ as $n \to \infty$.

(1) Vaccination and Attack

- Perfect vaccine: remove vaccinated population from the graph (site percolation).
- Acquaintance vaccination: Sample each node uniformly and inoculate a neighbor of this node taken at random.
- Degree based attack: randomly attack a node with a probability depending on its degree.

(1) Percolated Threshold Model

(2) Analytic Results

(3) Toolbox

(2) Cascade Condition

- Random graph with degree distribution: D

  (configuration model: Molloy-Reed 95)
- Bond percolation: π and threshold: $K(d)$.
- When can a single active node have a global impact?

$$\pi\mathbb{E}[D(D-1)\mathbb{1}(K(D)=0)] > \mathbb{E}[D]$$

- $K \equiv 0$ Epidemic contagion threshold.

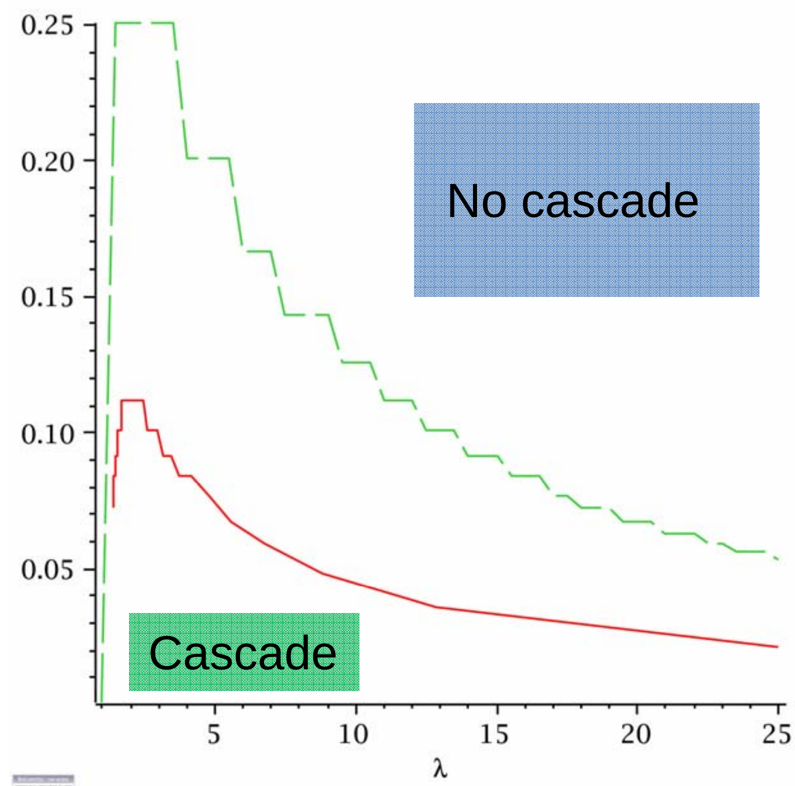
(2) Cascade Condition

- Cascade condition:

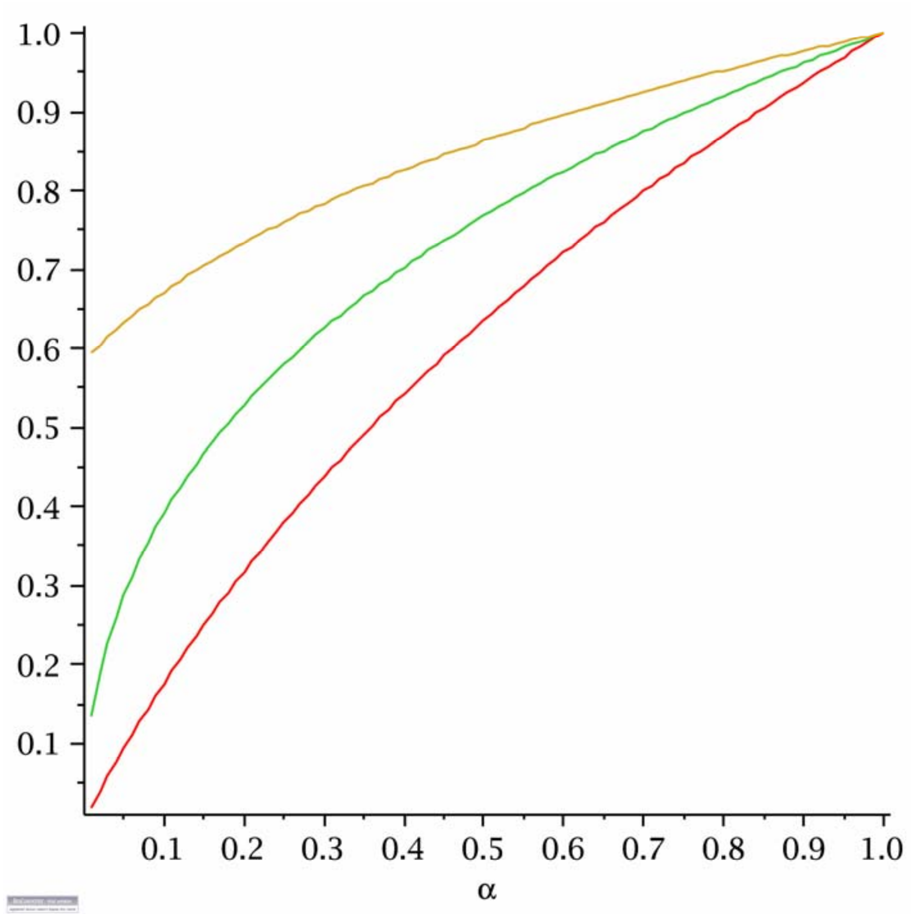
$$\pi \mathbb{E}[D(D-1)\mathbb{1}(K(D)=0)] > \mathbb{E}[D]$$

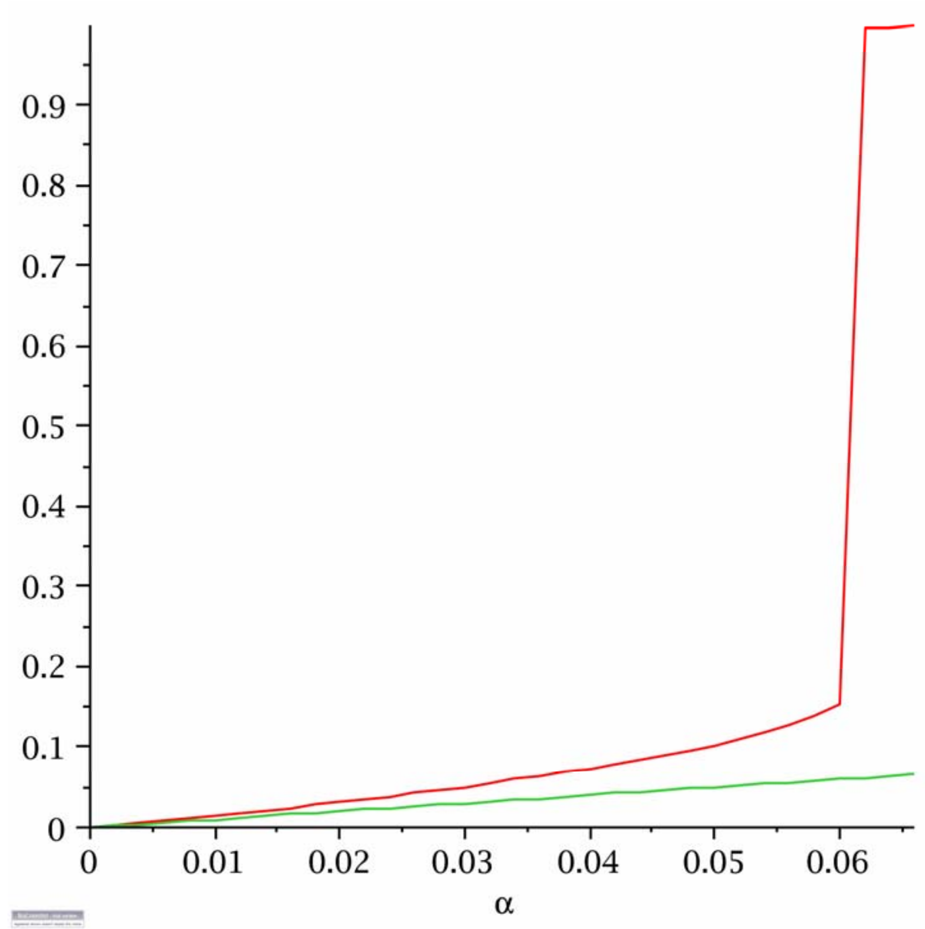
- Contagion threshold

K(d)=qd (Watts 02)

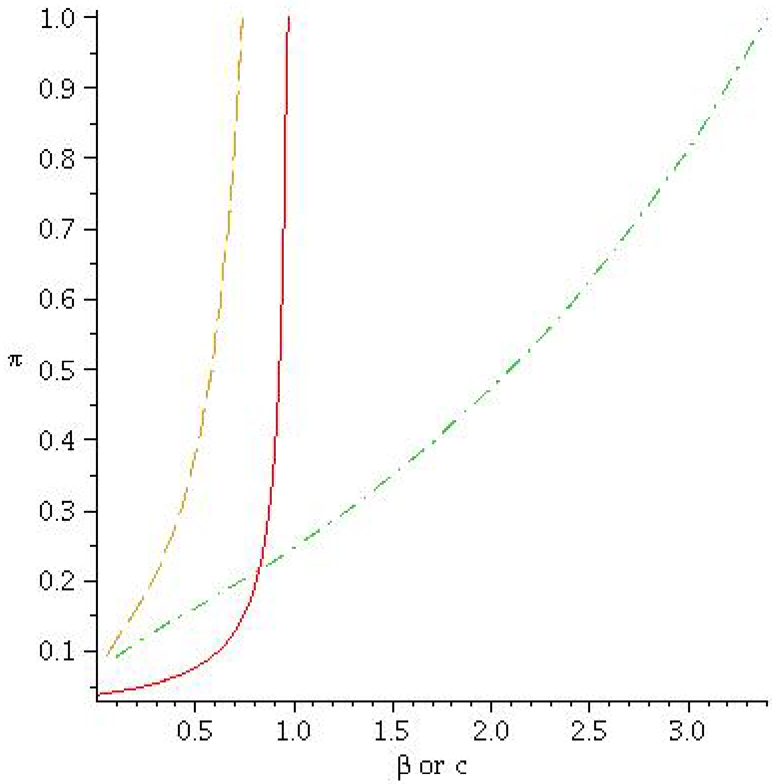


(2) Phase transition



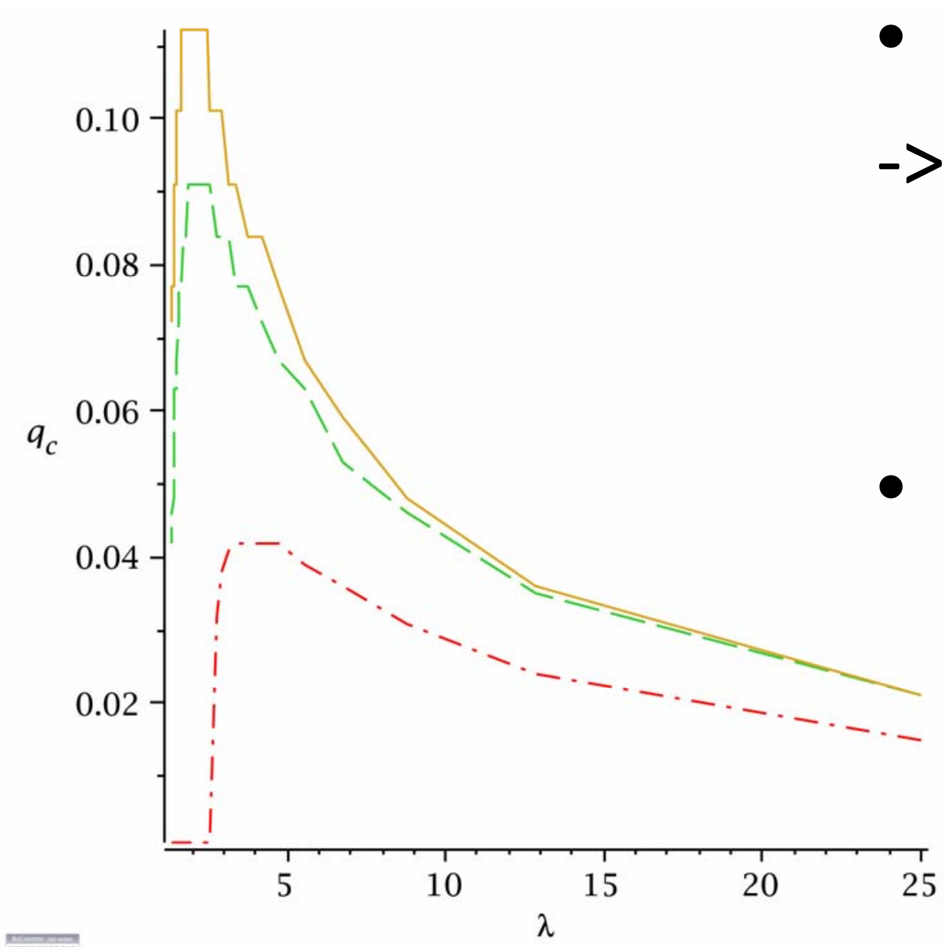


(2) Vaccination for the contact process



- Epidemic threshold as a function of vaccinated population.
- If $\mathbb{E}[D^2] = \infty$, uniform vaccination is useless. Acquaintance vaccination can stop epidemic!

(2) Vaccination for threshold model



- Threshold $K(d) = qd$

-> become active when fraction of active neighbors $\geq q$

- Contagion threshold as a function of mean degree.

(1) Percolated Threshold Model

(2) Analytic Results

(3) Toolbox

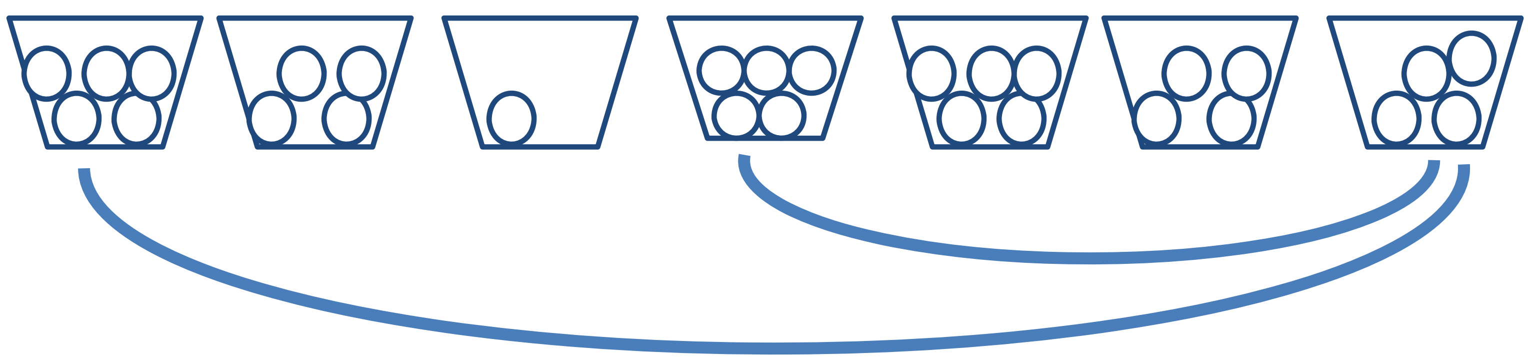
(3) General methodology

- (i) Theorem for the epidemic spread for a general graph with degree sequence D and degree based attack.
- (ii) Proposition: how vaccination modifies the distribution of D
- (iii) Show that we can apply the first Theorem on the graph obtained after vaccination.
- The steps (i) and (ii) are independent and can be used in different context…

(3) Configuration Model

- Vertices = bins and half-edges = balls

Bollobás (80)



(3) Percolated Threshold Model

- Bond percolation: randomly delete each edge with probability 1-π.
- Bootstrap percolation with threshold K(d): Seed of active nodes, $S = \{i,\ \sigma_i = 1\}$
  Deterministic dynamic: set $X_i = 1$ if

$$\sum_{j \sim_\pi i} X_j > K_i(d_i)$$

(3) Algorithm

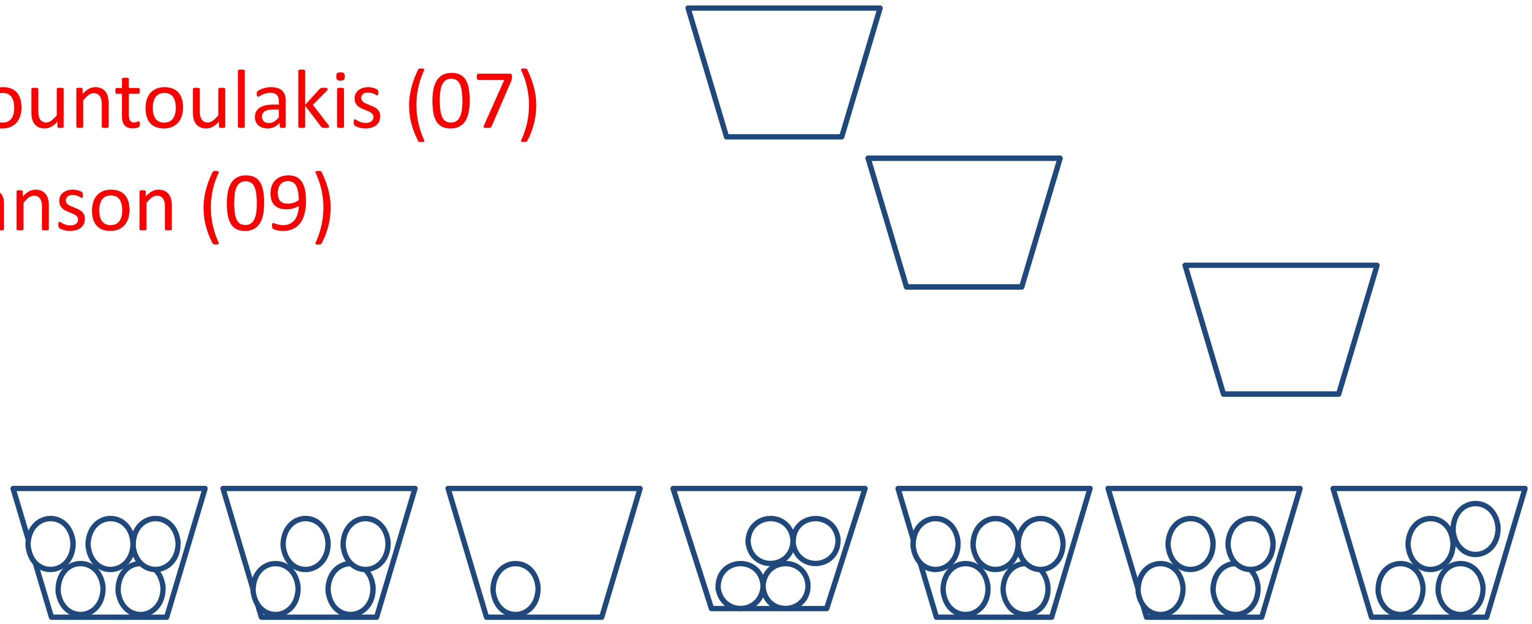
- Remove vertices S from graph G
- Recursively remove vertices i such that:

$$d_i^A \leq d_i - K_i(d_i)$$

- All removed vertices are active and all vertices left are inactive.
- Variations:
  – remove edges instead of vertices.
  – remove half-edges of type B.

(3) Site percolation

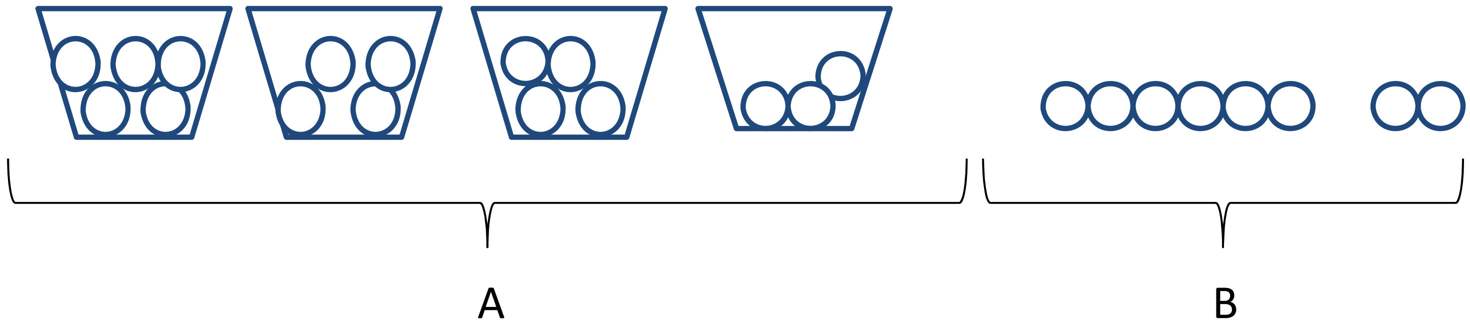
Fountoulakis (07)
Janson (09)



(3) Coupling

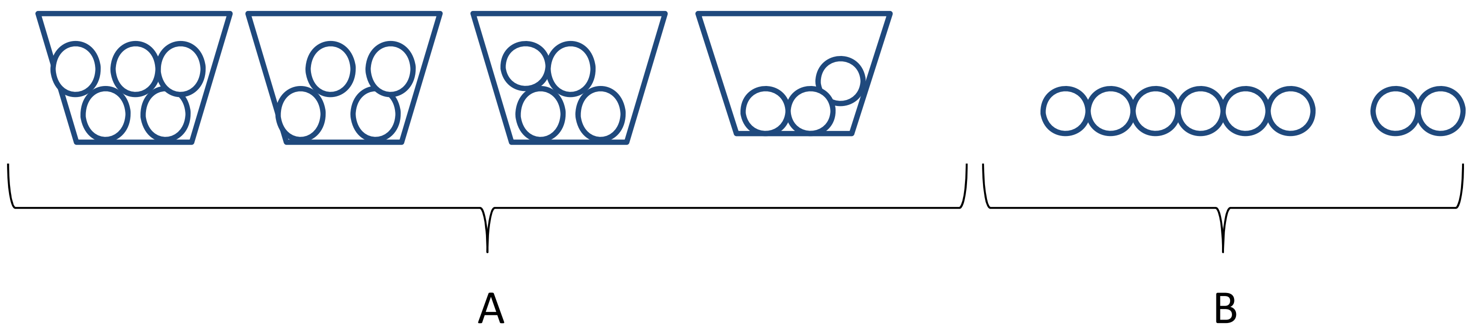
- Type A if $d_i^A \geq d_i - K_i$

Janson-Luczak (07)



(3) Deletion in continuous time

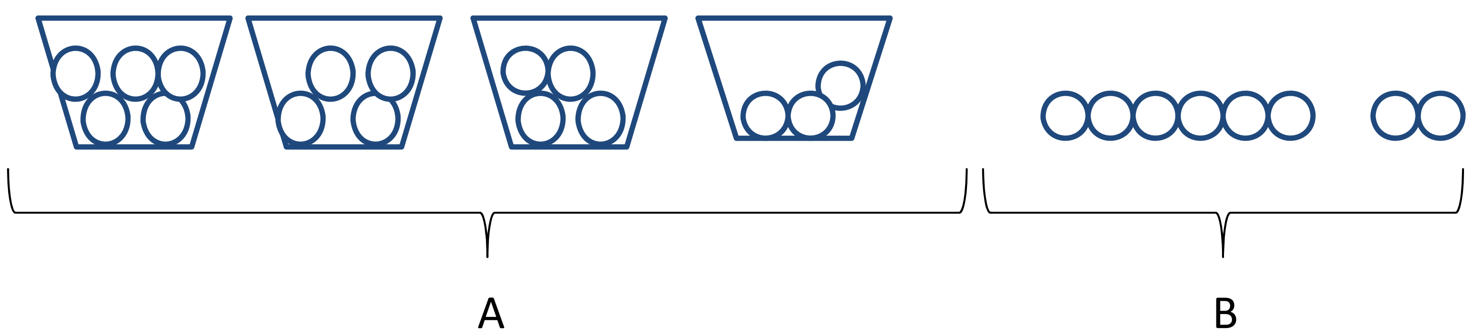
- Each white ball has an exponential life time.



(3) Percolated threshold model

- Bond percolation: immortal balls

A



(3) Death processes

- Rate 1 death process (Glivenko-Cantelli):

$$\sup_{t \geq 0} |N^{(n)}(t)/n - e^{-t}| \xrightarrow{p} 0$$

- Death process with immortal balls:

$$\frac{U_{s\ell,r}(t)}{n} \sim p_{s\ell} b_{sr} (1 - \pi + \pi e^{-t})$$

(3) Death Processes for white balls

- For the white A and B balls:

$$\frac{A(t)+B(t)}{n} \sim \lambda e^{-t}(1-\pi+\pi e^{-t}).$$

- For the white A balls:

$$\frac{A(t)}{n} \sim \sum_{s,r\geq s-\ell} r(1-\alpha_s)p_{s\ell}b_{sr}(1-\pi+\pi e^{-t}).$$

(3)Branching Process Approximation

- Local structure of G = random tree
- Recursive Distributional Equation:

$$Y_i = 1 - (1 - \sigma_i)\mathbb{1}\left(\sum_{\ell \to i} B_{\ell i} Y_\ell \leq K(d_i)\right)$$

(3) Solving the RDE

$$z = \mathbb{P}(Y = 0)$$

$$\lambda z(1 - \pi + \pi z) = h(z)$$

$$h(z) = \sum_{s, r \geq s - \ell} r(1 - \alpha_s) p_{s\ell} b_{sr} (1 - \pi + \pi z)$$

To conclude

- Acquaintance vaccination is impressively effective!
- Economics of epidemics: incentives for vaccination.

  Economics of Malware: Epidemic Risks Model, Network Externalities and Incentives, M.L., Allerton 09
- Applications to viral marketing, gossip:

  Duncan Watts \ Kempe, Kleinberg, Tardos

  Diffusion of innovations on random networks: Understanding the chasm, M.L., WINE 08.
- Technical details : www.di.ens.fr/~lelarge/