Strategic Customers Behavior in Memoryless Queues

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Eindhoven, 19.11.09

M/M/1 queue

- Poisson arrivals. Effective rate of λ.
- Exponential requirements. Mean $\frac{1}{\mu}$.
- $\rho = \lambda/\mu < 1$, server utilization.
- First-come first-served (FCFS).

Unobservable M/M/1 queue

- steady-state, mean waiting time (service inclusive)

$$W = \frac{1}{\mu}\frac{1}{1-\rho} = \frac{1}{\mu-\lambda}$$

- for a stand-by (or zero-priority) customer

$$\frac{1}{\mu}\frac{1}{(1-\rho)^2} = \frac{1}{\mu}\frac{1}{(1-\lambda/\mu)^2}$$

To join or not to join

- Λ - potential arrival rate
- C - cost per unit of waiting time
- K - reward for service completion

Assume $K > C/\mu$ (for non triviality).

Unobservable M/M/1 (Edelson and Hildebrand '75)

Nash equilibrium:

- Pure strategies: to join or not to join
- Randomization is allowed
- Join with probability p: A symmetric profile
- An equilibrium p_e:
  steady-state under $p_e \Rightarrow p_e$ is a best response.

Equilibrium

- $\mu > \Lambda$ and $C/(\mu - \Lambda) \leq K \Rightarrow p_e = 1$, dominant strategy
- $C/(\mu - \Lambda) \geq K \Rightarrow p_e$, $0 < p_e < 1$

  mixed strategy: p_e solves $C/(\mu - p\Lambda) - K = 0$

  Explanation: when all use p_e, one is indifferent between joining or not, $\Rightarrow p_e$ is one's (not uniquely) best response against p_e.

$$p_e = \frac{\mu - \frac{C}{K}}{\Lambda}$$

Stable equilibrium

p_e is evolutionarily **stable** strategy (ESS):

If $p \neq p_e$ is best against all playing p_e, then p_e is better than p against all playing p.

Equilibrium

case	λ_e	p_e	$W(\lambda_e)$
$\Lambda \leq \mu - \frac{C}{K}$	Λ	1	$\frac{1}{\mu-\Lambda}$
$\Lambda \geq \mu - \frac{C}{K}$	$\mu - \frac{C}{K}$	$\frac{\mu-\frac{C}{K}}{\Lambda}$	$\frac{K}{C}$

$f(p)$: Under p, the difference in utility between joining and not.

$$f(p) = K - C/(\mu - p\Lambda) - 0$$

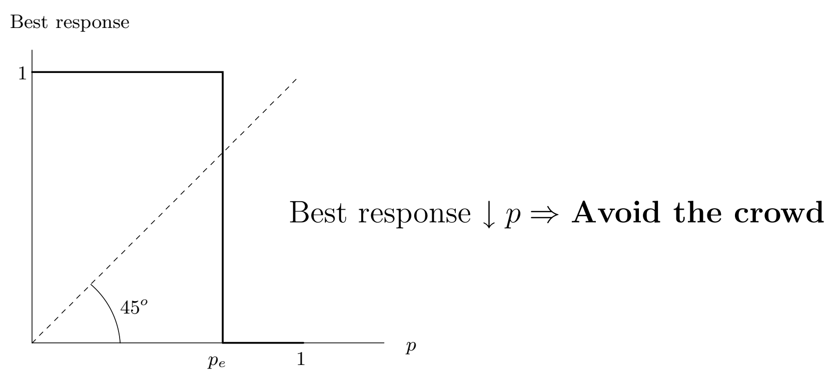
$$f(p) \downarrow p$$

The more join, the less appealing is joining.

Best response

Best response against p:

- $p \in [0, p_e] \Rightarrow 1$, i.e., join
- $p \in [p_e, 1] \Rightarrow 0$, i.e., do not join
- $p = p_e < 1 \Rightarrow$ any p



Avoid the Crowd

Best response $\downarrow p$

$$\Downarrow$$

Avoid the crowd (ATC)

Social optimization

$$p_s = \arg\max_p p\Lambda \left(K - \frac{C}{\mu - p\Lambda} \right)$$

$$p_s = \begin{cases} 1 & \Lambda < \mu - \sqrt{\frac{C\mu}{K}} \\ \frac{\mu - \sqrt{\frac{C\mu}{K}}}{\Lambda} & \text{otherwise} \end{cases}$$

When $p_s < 1$ social gain equals

$$(\sqrt{K\mu} - \sqrt{C})^2$$

not a function of Λ

Social optimization

When $p_s < 1$,

$$\frac{C}{\mu(1 - \frac{p_s \Lambda}{\mu})^2} = K$$

Under the socially optimal arrival rate, a stand-by customer is indifferent between joining and not. He imposes no externalities and under this rate, his and the society interests coincide.

Optimal toll

$p_e \geq p_s$. Left to themselves, customers overcrowd the system: They ignore the negative externalities they inflict on others.

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In both schemes, the added charge coincides with the externality that one who joins inflicts on others under the social optimal joining rate.

All consumer surplus goes to the central planner.

A cab or a bus?

- Waiting for the bus: 5 minutes
- The cab leaves when the 7th arrives
- Unobservable. No regrets
- Poisson arrival, rate λ

A cab or a bus? The more use the cab, the more appealing it is

$$\Downarrow$$

Follow the Crowd (FTC)

A cab or a bus?

Symmetric strategy: Take the cab with probability p

If $\lambda < 3/5 \Rightarrow p = 0$ is dominant.

Otherwise three equilibria:

- $p = 0$ (stable)
- $p = 1$ (stable)
- $p = p_e$ (unstable)

where

$$\frac{3}{p_e \lambda} - 5 = 0$$

social optimization

1. if $\lambda < 3/5 \Rightarrow p = 0$
2. if $\lambda > 3/5 \Rightarrow p = 1$
3. if $\lambda = 3/5 \Rightarrow p = 0$ and $p = 1$

Purchasing priority (Hassin and Haviv '03)

- two priority levels. Within a class, FCFS
- high priority costs θ
- no balking or reneging (K is irrelevant)
- unobservable

Why to have priority at all?

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Why to have priority at all?

1. to overtake ordinary customers ($\Rightarrow$ ATC)

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- unobservable

Why to have priority at all?

1. to overtake ordinary customers ($\Rightarrow$ ATC)
2. to avoid being overtaken by premium customers ($\Rightarrow$ Follow the Crowd (FTC))

Priority purchasing

Strategy p: purchase priority with probability p

$f(p) =$ difference in utility between purchasing and not purchasing

$$f(p) = \frac{C\lambda}{(\mu - \lambda)(\mu - \lambda p)} - \theta$$

$f(p) \uparrow p$.

Conclusion: The more purchase priority, the higher its value (FTC).

Purchasing priority

- $f(0) \geq 0 \Rightarrow p = 1$ is dominant

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  - $p = 1$ (stable)
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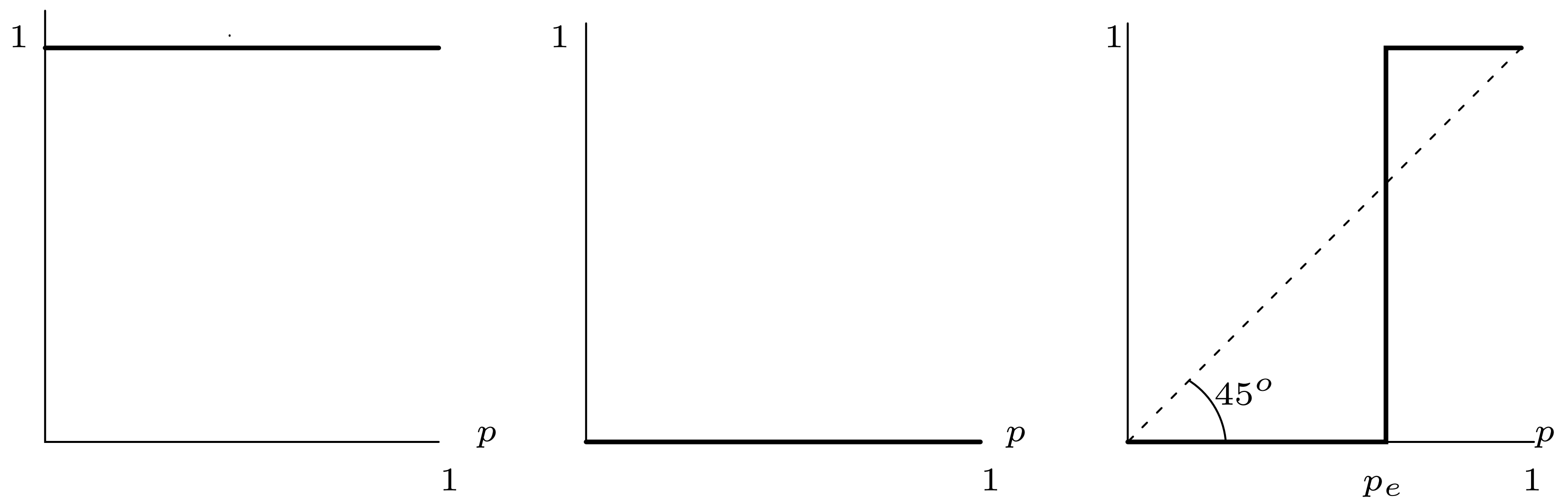
Follow the Crowd

The best response function $\uparrow p$

$\Downarrow$

Follow the crowd (FTC)



When to arrive? Haviv, ’09

1. a single service will be performed at time T
2. number interested $\sim Pois(\lambda)$
3. all value it by K
4. the first to arrive gets it (ties broken randomly)
5. cost C per unit of waiting

First-price sealed-bid auction

When to arrive?

Common knowledge: each bidder thinks that the number of other bidders $\sim Pois(\lambda)$

Suits huge population of potential bidders, each becomes a bidder with a tiny probability

When to arrive? (model 1)

- losers wait too
- no restrictions on arrival time

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The equilibrium density function of arrival:

$$f(t) = \begin{cases} \frac{1}{\lambda(\frac{K}{C} - (t-a))} & a \leq t \leq T \\ 0 & \text{otherwise} \end{cases}$$

$a = T - K(1 - e^{-\lambda})/C$

$f(t)$ increases with t

Truncated beta (1,0) distribution

When to arrive? (model 2)

- losers do not wait
- no restrictions on arrival time

The equilibrium density function of arrival:

$$f(t) = \begin{cases} \frac{1}{\lambda(\frac{K}{C} - (T - t))} & a \leq t \leq T \\ 0 & \text{otherwise} \end{cases}$$

The equilibrium density function of arrival time is the mirror image of the previous density along the same support.

$f(t)$ decreases with t.

When to arrive? (model 3)

- losers wait too
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- losers wait too
- restriction: no arrivals prior to $b > a$

1. arrive at b with probability p

$$K \frac{1 - e^{-\lambda p}}{\lambda p} - C(T - b) = K e^{-\lambda}$$

2. no arrivals in (b, c), $c = T - \frac{K}{C}(e^{-\lambda p} - e^{-\lambda})$
3. arrive in $[c, T]$ with density

$$f(t) = \begin{cases} \frac{1}{\lambda(\frac{K}{C} e^{-\lambda p} - (t - c))} & c \leq t \leq T \\ 0 & \text{otherwise} \end{cases}$$

When to arrive? (model 4)

- losers do not wait
- restriction: no arrivals prior to $b > a$

1. arrive at b with probability q

$$\frac{1 - e^{-\lambda q}}{\lambda q}(K - C(T - b)) = Ke^{-\lambda}$$

2. no arrivals in (b, d), $d = T - \frac{K}{C}(e^{-\lambda q} - e^{-\lambda})$
3. arrive in $[d, T]$ with density

$$f(t) = \begin{cases} \frac{1}{\lambda(\frac{K}{C} - (T - t))} & d \leq t \leq T \\ 0 & \text{otherwise} \end{cases}$$

When to arrive?

For all versions:

- individual utility in equilibrium $Ke^{-\lambda}$
- social utility in equilibrium $\lambda K e^{-\lambda}$
- social utility in optimization $K(1-e^{-\lambda})$
- price of anarchy (PoA) $\frac{e^{\lambda}-1}{\lambda}$

Biding for priority (version 1) Glazer and Hassin, ’86

- The more one pays, the earlier one enters
- With preemption
- Ties are broken on FCFS basis
- No balking or reneging: K is irrelevant

Bidding for priority (version 1)

Equilibrium biding: mix payment continuously between $[0, a]$.

$a =$ difference value between best and worst positions

$$a = \frac{C}{\mu(1-\rho)^2} - \frac{C}{\mu}$$

Equilibrium: Pay in accordance with cumulative distribution function $F(x)$, $0 \le x \le a$

Bidding for priority (version 1)

Finding $F(x)$, $0 \le x \le a$:

$$x + CW_x = 0 + \frac{C}{\mu(1-\rho)^2}, \quad 0 \le x \le a$$

$$W_x = \frac{1}{\mu(1 - \frac{\lambda(1-F(x))}{\mu})^2}, \quad 0 \le x \le a$$

$$\Rightarrow \quad F(x) = \frac{1-\rho}{\rho}\left(\sqrt{\frac{1}{1 - \frac{x\mu}{C}(1-\rho)^2}} - 1\right)$$

Biding for priority (version 2) Hassin '95

- As above but with balking
- K is back

Equilibrium joining rate as the socially optimal.

Proof: One who pays nothing imposes no externalities.

Those who join pay with some density along $[0, K - C/\mu]$.

Biding for relative priority Haviv and van der Wal, ’97

1. no balking
2. any payment $x \geq 0$
3. if payments of n in line were x_i, $1 \leq i \leq n$, customer i enters with probability

$$\frac{x_i}{\sum_{j=1}^{n} x_j}$$

How much to pay?

Biding for relative priority

Unique pure equilibrium: Pay

$$\frac{\rho C}{\mu(1-\rho)(2-\rho)}$$

Random entrance!

Not ATC or FTC

Around the equilibrium, FTC

Retrial queues Hassin and Haviv, ’96

1. M/M/1
2. λ, μ, C
3. no balking or reneging (K not relevant)
4. if server is busy, retry and retry later
5. tries cost H each

Decision problem: When to try again?

Retrial queues

The forgetful customers: Times between retrials are exponential with an identical parameter

Retrial queues

The forgetful customers: Times between retrials are exponential with an identical parameter

What is the equilibrium retrial rate?

$$\frac{C\rho + \sqrt{C^2\rho^2 + 8\mu CH(1-\rho)(2-\rho)}}{4H(1-\rho)}$$

Around the equilibrium, ATC

Socially optimal retrial rate Kulkarni, '83

$$\sqrt{\frac{C\mu}{H}}$$

Reneging

Having the option of reneging (abandonment) later changes nothing

Explanation: Under M/M/1, memoryless waiting time. If others may renege, one's future improves while waiting.

Deteriorating conditions Hassin and Haviv, ’95

Bang-bang: at time T after waiting, K drops to zero.

- to join or not to join?
- when to renege? (customers have a watch)

Unique equilibrium: Join with probability p and then renege at time T ($p = 1$ is possible).

Continuous deterioration Haviv and Ritov, ’01

$C(t)$ waiting cost per unit at time t of waiting. $C(0) = 0$.

Equilibrium reneging strategy (some technical conditions):

For some $T_1 < T_2$,

1. do not renege until T_1
2. renege with some density in $[T_1, T_2)$
3. renege with complementary probability at T_2

Observable M/M/1 queues Naor ’69

- M/M/1
- same cost/reward model (C and K)
- queue length inspected upon arrival

To join or not to join?

Equilibrium: { Join $\Leftrightarrow L \leq n_e$ }

$$n_e = \left\lfloor \frac{K\mu}{C} \right\rfloor - 1$$

Equilibrium Hassin and Haviv, '02

Multi-equilibria: For $L \leq n_e$ as above. For $L \geq n_e + 1$, anything.

{ Join $\Leftrightarrow L \leq n_e$ }. Unique subgame-perfect equilibrium (SPE).

Social optimization

$$g(n) = \frac{n(1-\rho) - \rho(1-\rho^n)}{(1-\rho)^2}$$

n_s is with

$$g(n_s - 1) \leq \frac{K}{\mu} \leq g(n_s)$$

Social optimization: Join if and only if $L < n_s$

Optimal entry fee

A right optimal one-for-all entry fee T with

$$n_s = \left\lfloor \frac{(K - T)\mu}{C} \right\rfloor$$

makes the new n_e coincide with the old n_s.

Optimal entry fee

A right optimal one-for-all entry fee T with

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makes the new n_e coincide with the old n_s.

Some surplus stays at the hands of customers: They are more informed than the central planner is.

not-FCFS and social optimization Hassin, '85

- An arrival is placed anywhere but at the last position
- Pre-emption is possible. A must if $L = 1$

Individuals' dilemma: when to renege?

Answer: renege when n_s customers are ahead: The one at the back inflicts no externalities. His and the society's interests coincide.

Queue-length dependent entry fee

A queue-length dependent fee leaves them with zero surplus:

$$T_n = \begin{cases} K - \frac{C(n+1)}{\mu} & 0 \le n \le n_s - 1 \\ \infty & \text{otherwise} \end{cases}$$

Purchasing priority: Observable case H. and H.,'9

High priority costs θ.

An arrival observes the two queue lengths.

Only the number of regular customers matters.
$\Rightarrow$ Assume the number of premium customers is zero.

Purchasing priority: Observable case

A pure threshold equilibrium n: do not pay iff the number of regular customers is below n.

$W(n)$=mean queueing time of the worst regular customers when all use strategy n.

Result: n is an equilibrium iff

$$\theta - CB \leq CW(n) \leq \theta$$

where $B = 1/(\mu - \lambda)$ (mean busy period)

Purchasing priority: Observable case

1. at least one pure equilibrium exists
2. consecutive multiple equilibria are possible. At most $\lfloor 1/(1-\rho) \rfloor$ pure equilibria
3. both bounds are attainable
4. between two pure equilibria, (usually) one mixed.

Inferring quality from long queues, Debo et. al '09

- $M/M/1$, $C = 0$, FCFS, observable
- homogeneous service value: $low\ < 0$ or $high\ > 0$
  P$(high) = p$
- private independent signals: good or bed
  P$(good|high) = P(bad|low) = q$.

To queue or not to queue?

Inferring quality from long queues

- Customers with good signals $\Rightarrow$ ATC
- Customers with bad signals $\Rightarrow$ FTC

Equilibrium:

- **good signal**: join with prob. $\alpha > 0$ when $n = 0$ ($\alpha = 1$ possible). Join when $n \geq 1$.
- **bad signal**: up to $n_e \geq 1$ (exclusive) do not join. At n_e join with prob. β, $0 \leq \beta \leq 1$. Join when $n > n_e$.

Multiple equilibria

Strategic Customers Behavior in the M/G/1 Queue

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Joint with Yoav Kerner

Eindhoven, 20.11.09

M/G/1 queue

- Single server
- Poisson arrival process (λ)
- G - Service distribution

  $\overline{x}$ - mean service time

  $\overline{x^2}$ - 2^{nd} moment of service time

  $G^*(s) = \int_{x=0}^{\infty} e^{-sx} dG(x)$ - LST of service time

M/G/1: unobservable case

Conceptually as in M/M/1

- - service value, K
  - queueing costs, C per unit of time
- The pure strategies: to join or not to join
- Mixed strategy: Randomize between joining and not

Unobservable system

All join with probability $p \Rightarrow W_q = \frac{\lambda p \overline{x^2}}{2(1-\lambda p \overline{x})}$ (K-P)

Symmetric Nash equilibrium:

$$p_e \in \arg \max_{0 \le p \le 1} p \left(K - \frac{C \lambda p_e \overline{x^2}}{2(1 - \lambda p_e \overline{x})} \right) + (1 - p) \cdot 0$$

$$p_e = \min \left\{ \frac{2K}{\lambda(2K\overline{x} + C\overline{x^2})}, 1 \right\}$$

Unobservable system

Properties of Nash equilibrium

- Unique
- ESS Evolutionarily Stable Strategy
- ATC Avoid the Crowd

M/G/1 and residual service times

The unconditional mean residual service time

$$= \frac{\lambda \overline{x^2}}{2}$$

But:

- the residual service time and the queue length are **not** independent
- when balking is with queue dependent probabilities, the distribution of the residual service time (given the queue length) is a function of (early) balking probabilities

Partially observable systems Haviv and Kerner, '07

- Same cost/reward model (assume no waiting costs during service)
- Information upon arrival: $L = 0$, $L = 1$ or, $L =$ many

All join when informed of $L = 0$.

Balking strategies

Pure strategies: When $\{L = 1\}$ or $\{L = \text{many}\}$, join or not

Mixed strategies: When $\{L = 1\}$ (or $\{L = \text{many}\}$), join with probability p (or q), for some p and q

Equilibrium strategies

(p, q), when selected by all, is also an individual's best response

Qualitative findings:

- $q > p$ is possible
- $\{L = 1\} \Rightarrow$ Both 'avoid the crowd' (ATC) and 'follow the crowd' (FTC) (or none) are possible (it depends on G)
- $\{L > 1\} \Rightarrow$ ATC

Expected residual service, $L = 1$

Under steady-state, conditioning upon $L = 1$, Mandelbaum and Yechiali, '79:

$$\mathsf{E}(R|L=1) = \begin{cases} \frac{\overline{x}}{1-G^*(\lambda p)} - \frac{1}{\lambda p} & p > 0 \\ \\ \frac{\overline{x^2}}{2\overline{x}} & p = 0 \end{cases}$$

Example 1: Zero-one service Altman and Hassin '02

$$\mathsf{E}(R|L=1) = \frac{1}{1-e^{-\lambda p}} - \frac{1}{\lambda p}$$

$$\Downarrow$$

$$\mathsf{E}(R|L=1) \uparrow p$$

$$\Downarrow$$

$$ATC$$

Example 2: G=Exp(μ), $\mu \sim$ U(1,2)

$$\mathsf{E}(R|L=1) = \frac{\log 2}{\lambda p(\log(\lambda p + 2) - \log(\lambda p + 1))} - \frac{1}{\lambda p}$$

$$\Downarrow$$

$$\mathsf{E}(R|L=1) \downarrow p$$

$$\Downarrow$$

$$FTC$$

Example 3: non-monotone residual

$x = 0.5$ w.p. 0.8 and $x = 3$ w.p. 0.2.

$$\Downarrow$$

$$\mathsf{E}(R|L = 1) = \frac{1}{1 - .2e^{-3\lambda p} - .8e^{-.5\lambda p}} - \frac{1}{\lambda p}$$

Not monotone with p

Nash Equilibrium p_e

- if $\forall p$, $C\mathsf{E}(R(p)|L = 1) \leq K \Rightarrow p_e = 1$, 'dominant'

- if $\forall p$, $C\mathsf{E}(R(p)|L = 1) \geq K \Rightarrow p_e = 0$, 'dominant'

-
  - if $C\mathsf{E}(R(1)|L = 1) \leq K \Rightarrow p_e = 1$
  - if $C\mathsf{E}(R(0)|L = 1) \geq K \Rightarrow p_e = 0$
  - if $C\mathsf{E}(R(p)|L = 1) = K \Rightarrow p_e = p$

Increasing service residuals, L=1

Increasing failure rate (IFR) service distribution

$$\Downarrow$$

$$\mathsf{E}(R(p)|L = 1) \uparrow p$$

$$\Downarrow$$

ATC

$$\Downarrow$$

p_e unique and ESS

Decreasing service residuals, L=1

Decreasing failure rate (DFR) service distribution

$$\Downarrow$$

$$\mathsf{E}(R(p)|L=1) \downarrow p$$

$$\Downarrow$$

FTC

If no dominance $\Rightarrow$ three equilibria

$$\Downarrow$$

$p_e = 0$ and $p_e = 1$ are ESS

$0 < p_e < 1$ is not ESS

Example 3: non-monotone residual

$x = 0.5$ w.p. 0.8 and $x = 3$ w.p. 0.2.

$$\Downarrow$$

$$\mathsf{E}(R|L=1) = \frac{1}{1 - .2e^{-3\lambda p} - .8e^{-.5\lambda p}} - \frac{1}{\lambda p}$$

Not monotone with p

If no dominance $\Rightarrow$ multiple equilibria

Some of the equilibria are ESS but some are not

Expected queueing times at arrival times

In terms of:

- Decision variables p,q
- First and second moments of service $\overline{x}, \overline{x^2}$
- The potential arrival rate λ
- LST of service time at a single value, $G^*(\lambda p)$

Mean waiting when $L \geq 2$

Mean queueing time conditioning on $L \geq 2$:

$$\overline{x} + \frac{\lambda q \overline{x^2}}{2(1 - \lambda q \overline{x})} + \frac{\lambda p \overline{x^2}}{2(\lambda p \overline{x} + G^*(\lambda p) - 1)} - \frac{1}{\lambda p}$$

- Separability in p and q
- Monotone increasing in q

Nash Equilibrium q_e

For any equilibrium $p_e > 0$ there exists a unique q_e such that (p_e, q_e) is an equilibrium

- if $\forall q$, E$(W_Q(p_e, q)|L > 1) \leq \frac{K}{C} \Rightarrow q_e = 1$, 'dominant'
- if $\forall q$, E$(W_Q(p_e, q)|L > 1) \geq \frac{K}{C} \Rightarrow q_e = 0$, 'dominant'
- Otherwise, q_e solves:

  E$(W_Q(p_e, q)|L > 1) = K/C$

Nash Equilibrium q_e

$$\forall p \quad \mathsf{E}(W_Q(p,q) | L > 1) \quad \uparrow q$$

$$\Downarrow$$

ATC

$$\Downarrow$$

q_e unique and ESS

The fully observable case Kerner, ’08, ’09

A decision model

- service value, K
- waiting costs, C

Decision: to join or not to join

p_n: Joining probability given $L = n$, $\lambda_n = \lambda p_n$

Problem: the distribution of

$$W|_{L=n} = \sum_{i=1}^{n} X_i + R_n$$

Fully observable M/G/1 queue

A typical profile: $\underline{p} = (p_1, p_2, \ldots)$

Equilibrium strategy: $\underline{p}^e = (p_1^e, p_2^e, \ldots)$, one's best response when all use it (under steady-state)

p_1^e as in the partially observable case

p_n^e are derived recursively:

$$p_n^e \in \arg\max_{0 \leq p \leq 1} \left\{ p \left(K - C\mathsf{E}(W_n(p_1^e, \ldots, p_{n-1}^e, p_n^e)) \right) \right\}$$

Of course, $n \geq K/(C\overline{x}) \Rightarrow p_n = 0$.

The $M_n/G/1$ queue

Join with $p_n \Rightarrow$ when n, is the arrival rate $\lambda_n = \lambda p_n$
The queueing model for analysis:

- Arrival rate when n, $n \geq 0$, customers, λ_n
- $X \sim G$, Service distribution

  $\overline{x}$: mean service time

  $\overline{x^2}$: 2^{nd} moment of service time

  $G^*(s) = \int_{x=0}^{\infty} e^{-sx} dG(x)$ - LST of service time
- Goal: R_n: residual service time (given n)

Recursion in $M/G/1$ queues

π_i: limit probability of queue length i, $i \geq 0$

- a recursion on the limit probabilities is well-known
- $\pi_0 = 1 - \lambda\overline{x}$

$$\Downarrow$$

π_i, $i \geq 0$, are computable

Recursion in $M_n/G/1$ queues

- we developed a recursion on the limit probabilities
- but π_0 is a function of λ_i, $i \geq 0$

$$\Downarrow$$

No finite way to compute π_i, $i \geq 0$

But things are better when inspecting the residuals!

Recursion on R_n in $M_n/G/1$

- the case $n = 1$ was dealt with above
- an arrival who sees $n \geq 2$ upon arrival
  - with prob. $1 - G^*(\lambda_n)$: is first during the current service $\Rightarrow R_1$ with λ_n
  - with prob. $G^*(\lambda_n)$: faces the residual of the residual R_{n-1} with $\lambda_1, \ldots, \lambda_{n-1}$.

$$\Downarrow$$

R_n, $n \geq 1$, can be solved recursively

Recursion on R_n

$R_n^*(s)$=LST of the conditional residual R_n

$$R_1^*(s) = \frac{\lambda_1}{\lambda_1 - s} \frac{G^*(s) - G^*(\lambda_1)}{1 - G^*(\lambda_1)}$$

$$R_n^*(s) = \frac{\lambda_n}{s - \lambda_n} \left(G^*(\lambda_n) \frac{1 - R_{n-1}^*(s)}{1 - R_{n-1}^*(\lambda_n)} - G^*(s) \right), n \geq 2$$

Recursion on $\mathsf{E}(R_n)$

$$\mathsf{E}(R_1) = \frac{\overline{x}}{1 - G^*(\lambda_1)} - \frac{1}{\lambda_1}$$

$$\mathsf{E}(R_n) = \frac{G^*(\lambda_n)}{1 - R^*_{n-1}(\lambda_n)} \mathsf{E}(R_{n-1}) - \frac{1}{\lambda_n} + \overline{x}, \qquad n \geq 2$$

Some properties

- $$\pi_n = \frac{\lambda_0 \pi_0}{\lambda_n} \prod_{i=0}^{n-1} \frac{1 - R_i^*(\lambda_{i+1})}{G^*(\lambda_{i+1})}, \quad n \geq 0$$

- An arrival who finds $n \geq 1$ upon arrival, is the first to arrive during the current service with probability $1 - G^*(\lambda_n)$.

$$\Downarrow$$

In M/G/1, the event of being the first to arrive during the current service period and the number in the system then, are independent.

Back to decision making

$$\lambda_n \longrightarrow \lambda p_n$$

$$\mathsf{E}(R_n) \longrightarrow \mathsf{E}(R_n(p_1, \ldots, p_n))$$

Equilibrium:

$$p_n^e \in \arg \min_{0 \leq p \leq 1} p(K - C(\mathsf{E}(R_n(p_1^e, \ldots, p_n^e)) + (n-1)\overline{x}))$$

Equilibrium joining probabilities

For $n \geq 1$,

$$(n-1)\overline{x} + \mathsf{E}(R_n(p_1^e, \ldots, p_{n-1}^e, 1)) \leq \frac{K}{C} \Rightarrow p_n^e = 1$$

$$(n-1)\overline{x} + \mathsf{E}(R_n(p_1^e, \ldots, p_{n-1}^e, 0)) \geq \frac{K}{C} \Rightarrow p_n^e = 0$$

$$(n-1)\overline{x} + \mathsf{E}(R_n(p_1^e, \ldots, p_{n-1}^e, p)) = \frac{K}{C} \Rightarrow p_n^e = p$$

Stop when $p_n^e = 0$

Example 1 (cont.)

$$G(x) = \epsilon 1_{x \geq 1} + (1-\epsilon) 1_{x \geq 0}$$

$$C = 1, K = 0.7$$

- $\lambda \leq 2.51 \Rightarrow p_1^e = p_2^e = 1$
- $2.51 < \lambda < 2.59 \Rightarrow 0 < p_1^e < 1$, $p_2^e = 1$
- $\lambda > 2.59 \Rightarrow 0 < p_1^e < p_2^e < 1$

Uniqueness issues

IFR $\Rightarrow$ ATC, unique threshold equilibrium, p_n^e, $n \geq 1$.

DFR $\Rightarrow$ FTC, non-unique equilibrium, p_n^e, $n \geq 1$

Note: $p_n^e < p_{n+1}^e$ is possible

The IFR case

IFR $\Rightarrow$ ATC, unique threshold equilibrium

Initialize with $p_0^e = 1$,

$$p_n^e = \begin{cases} 1 & (n-1)\overline{x} + \mathsf{E}(R_n(p_1^e, \ldots, p_{n-1}^e, 1)) \leq \frac{K}{C} \\ 0 & (n-1)\overline{x} + \mathsf{E}(R_n(p_1^e, \ldots, p_{n-1}^e, 0)) \geq \frac{K}{C} \\ p & (n-1)\overline{x} + \mathsf{E}(R_n(p_1^e, \ldots, p_{n-1}^e, p)) = \frac{K}{C} \end{cases}$$

as long as $p_{n-1}^e > 0$.

The DFR case

DFR $\Rightarrow$ FTC

$$(n-1)\overline{x} + \mathsf{E}(R_n(p_1^e, \ldots, p_{n-1}^e, 1)) \leq \frac{K}{C} \Rightarrow p_n^e = 0, \quad \text{'dominant'}$$

$$(n-1)\overline{x} + \mathsf{E}(R_n(p_1^e, \ldots, p_{n-1}^e, 0)) \geq \frac{K}{C} \Rightarrow p_n^e = 1, \quad \text{'dominant'}$$

$$(n-1)\overline{x} + \mathsf{E}(R_n(p_1^e, \ldots, p_{n-1}^e, p)) = \frac{K}{C} \Rightarrow p_n^e = 0, p_n^e = p, p_n^e = 1$$

Example 2 (cont.) DFR

$$G(x) = 1 - \frac{e^{-x} - e^{-2x}}{x}$$

$$\lambda = 1, K = 2.81, C = 1$$

$$\Downarrow$$

$p_1^e = p_2^e = 1,$ unique

$$p_3^e = 0, 1, 0.654$$

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THANK YOU