The contraction principle in two-carousel warehousing models

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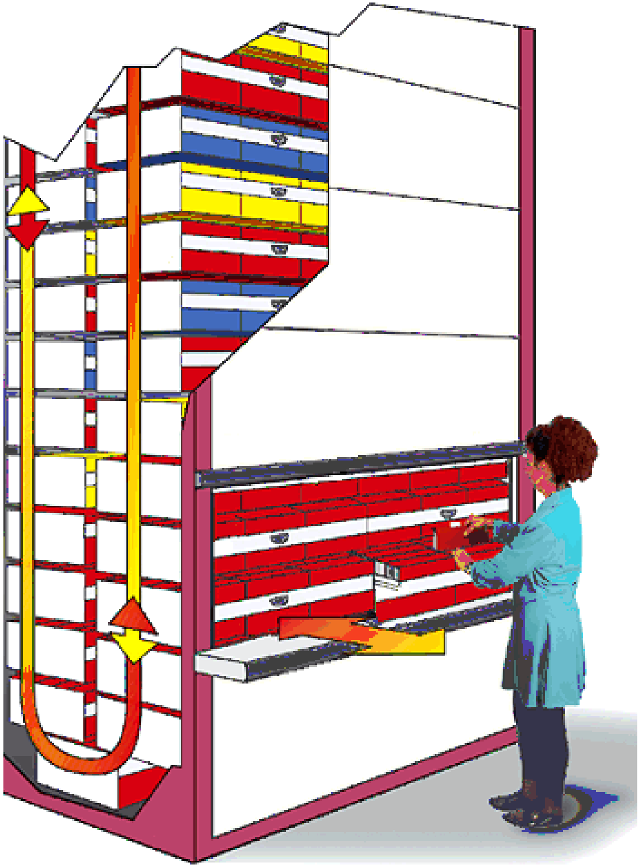
Where innovation starts

What is a carousel?



The contraction principle in two-carousel warehousing models

Research overview of carousels

- Storage and Design
  - Randomised policies: Hwang and Ha ’94, Litvak ’01
  - Two-class storage: Ha and Hwang ’94
  - Maximal adjacency principle: Stern ’86
  - Organ pipe: Lim et al. ’85, Bengü ’95, Vickson and Fujimoto ’96
  - Number of items per bin for max number of orders: Jacobs et al. ’00, Yeh ’02, Kim ’05, Li and Wan ’05
- Picking multiple orders
  FIFO: Ghosh and Wells ’92, M/G/1 analogy: Rouwenhorst et al. ’96, Rural Postman: Van den Berg ’96
- Picking a single order
  - One-carousel studies
  - Two-carousel studies
  - Multi-carousel studies

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  - Multi-carousel studies
    Emerson and Schmatz ’81

Analytic results for 2 carousels

So far:

- Realistic applications usually involve several carousels.
- Few analytic results for carousel systems.
- Koenigsberg ’86: 2 carousels is the optimal # for a single server.
- Hwang et al. ’99 reinforce this point.
- Existing results are mainly for 1 carousel.
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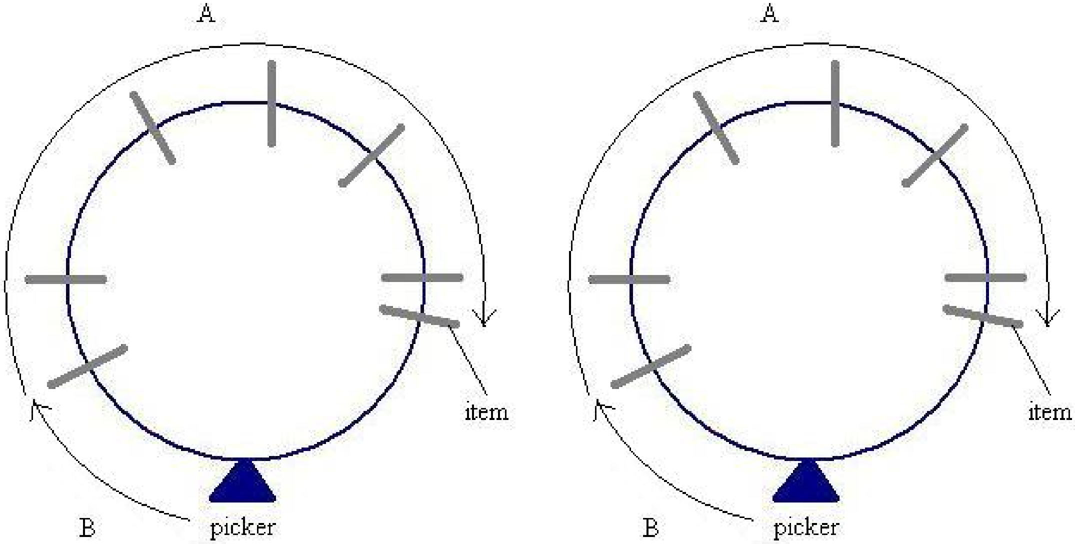
Question:
Can we get analytic results on performance for more realistic scenarios?

Contractions in carousels



In general,

$$S_{n+1} = (B_{n+1} - S_n)^+ + A_{n+1},$$

where A_{n+1} and B_{n+1} are dependent, but independent of S_n.

Contractions in carousels

In stationarity, for the distribution F_S of S we have that

$$F_S(x) = P[(B-S)^+ + A \leq x] = \int_0^x P[(B-S)^+ \leq x - y | A = y] f_A(y) dy$$
$$= \int_0^1 \int_0^x P[B \leq x - y + z | A = y] f_A(y) dy f_S(z) dz.$$

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Conjecture:
This mapping is a contraction mapping.

Advantages:
Accurate (to any level) and fast (geometric convergence) performance estimates (e.g. throughput), without the need of simulation (i.e. irrespective of the underlying distributions).

Special cases (1)

One-directional carousels

Observe that

$$f_A(y) = \begin{cases} n(n-1)(1-y)y^{n-2} & 0 < y < 1, \\ 0 & \text{else}, \end{cases}$$

and that

$$P[B \leq x | A = y] = \frac{x}{1-y}, \qquad 0 \leq x \leq 1-y.$$

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Filling these into the differential equation for F_S yields:

$$F(x) = nx^{n-1} - (n-1)x^n - nx^{n-1} \int_0^{1-x} F(z)dz.$$

Let Ω represent this mapping.

Special cases (1)

Ω is a contraction mapping. I.e. for $|F_1(x) - F_2(x)| \leq \delta$, and all $x \in [0,1]$ and $n \geq 2$

$$\begin{aligned} |\Omega(F_1(x)) - \Omega(F_2(x))| &\leq nx^{n-1} \int_0^{1-x} |F_1(z) - F_2(z)| dz \\ &\leq nx^{n-1} \int_0^{1-x} \delta dz = \delta n x^{n-1}(1-x) < \delta. \end{aligned}$$

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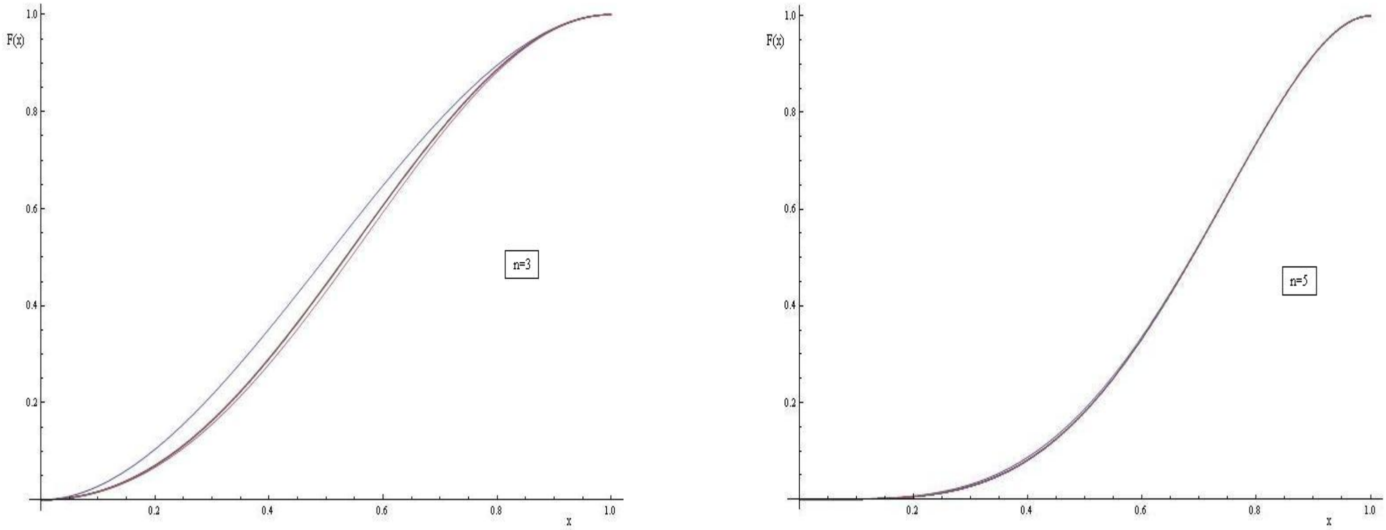
Moreover, all moments are recursively found by:

$$\begin{aligned}a_k &= \int_0^1 x^k F(x) dx = \frac{n}{n+k} - \frac{n-1}{n+k+1} - \frac{n}{n+k} b_{k+n} \\ b_k &= \int_0^1 (1-x)^k F(x) dx \\ &= \sum_{i=0}^{n-1} \binom{n-1}{i} (-1)^{n-1-i} \left(c_{n,k} + d_{n,k}(a_{n-i+k} - a_0)\right)\end{aligned}$$

with $c_{n,k} = \frac{1+n(n-i+k)}{(n-i+k)(n+1-i+k)}$, $d_{n,k} = \frac{n}{n-i+k}$.

Convergence



Starting with $F_1(x) = nx^{n-1} - (n-1)x^n$, we iteratively approximate F_S for $n = 3$ and $n = 5$.

Special cases (2)

Two-directional carousels

A few strategies:

1. go to closest item, continue in the same direction
2. avoid the biggest gap
3. go to closest item, continue in one direction giving the shortest path

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Major difference: F_A and F_B are modified. E.g. for strategy 3:

$$P[B \leq x] = 1 - (1-2x)^n, \qquad 0 \leq x \leq \tfrac{1}{2},$$

and for $0 \leq y \leq 1-2x$:

$$P[A \leq y | B = x] = \begin{cases} \left(\frac{y}{1-2x}\right)^{n-1} + \left(\frac{(y-2x)^+}{1-2x}\right)^{n-1} - \left(\frac{(2y-1)^+}{1-2x}\right)^{n-1}, & 0 \leq x \leq \frac{1}{4}, \\ \left(\frac{y}{1-2x}\right)^{n-1}, & \frac{1}{4} \leq x \leq \frac{1}{2}. \end{cases}$$

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- This remains the case even if orders have a variable number of items
- We conjecture that the generic stochastic recursion for the sojourn time is a contraction
- All theoretical results agree with simulation experiments
- We can use these results for accurate approximations of performance