Technische Universität Graz

Institut für Mathematische Strukturtheorie

Phase Transitions for Random Walk Asymptotics on Free Products of Groups

YEP VII, Eindhoven – March, 11th 2010

Elisabetta Candellero (joint work with **Lorenz A. Gilch**)

Example of Free Product of two Finite Groups

Example of Free Product of two Finite Groups

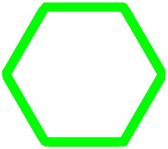


Figure: Example: $\mathbb{Z}_6 * \mathbb{Z}_3$

Example of Free Product of two Finite Groups

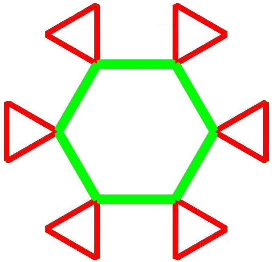


Figure: Example: $\mathbb{Z}_6 * \mathbb{Z}_3$

Example of Free Product of two Finite Groups

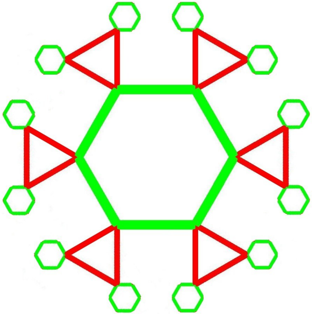


Figure: Example: $\mathbb{Z}_6 * \mathbb{Z}_3$

Example of Free Product of two Finite Groups

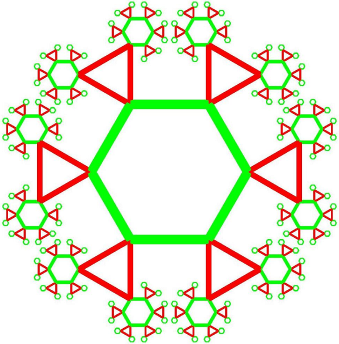


Figure: Example: $\mathbb{Z}_6 * \mathbb{Z}_3$

Example of Free Product of two Finite Groups

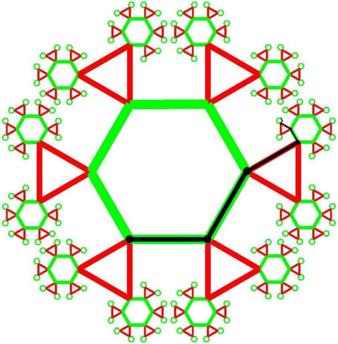


Figure: Example: $x_1 x_2 x_3 x_4 x_5$ is an element of $\mathbb{Z}_6 * \mathbb{Z}_3$

Formal Overwiev

$\Gamma :=$ group with identity e. $A :=$ set of generators of Γ; $|A| < \infty$.
$\mu :=$ probability measure on Γ: defines a RW with transition probabilities

$$\forall x, y \in \Gamma \qquad p(x,y) = \mu(x^{-1}y); \qquad (p(x,y) > 0 \text{ iff } x^{-1}y \in A).$$

$\mu^{(n)}(x^{-1}y) := p^{(n)}(x,y) =$ probability to go from x to y in n steps.
Analitically: n-th *convolution* of $p(x,y)$.

Formal Overwiev

$\Gamma :=$ group with identity e. $A :=$ set of generators of Γ; $|A| < \infty$.
$\mu :=$ probability measure on Γ: defines a RW with transition probabilities

$$\forall x, y \in \Gamma \qquad p(x, y) = \mu(x^{-1}y); \qquad (p(x, y) > 0 \text{ iff } x^{-1}y \in A).$$

$\mu^{(n)}(x^{-1}y) := p^{(n)}(x, y) =$ probability to go from x to y in n steps.
Analitically: n-th *convolution* of $p(x, y)$.

Asymptotic Behaviour of the Return Probabilities $\mu^{(n)}(e)$

Formal Overwiev

$\Gamma :=$ group with identity e. $A :=$ set of generators of Γ; $|A| < \infty$.
$\mu :=$ probability measure on Γ: defines a RW with transition probabilities

$$\forall x, y \in \Gamma \qquad p(x,y) = \mu(x^{-1}y); \qquad (p(x,y) > 0 \text{ iff } x^{-1}y \in A).$$

$\mu^{(n)}(x^{-1}y) := p^{(n)}(x,y) =$ probability to go from x to y in n steps.
Analitically: n-th *convolution* of $p(x,y)$.

Asymptotic Behaviour of the Return Probabilities $\mu^{(n)}(e)$

In a great variety of cases:

$$\mathbf{r}^n \mu^{(n)}(e) \sim C \cdot n^{-\lambda},$$

where $1/\mathbf{r} \leq 1$ is the "spectral radius" and $\lambda > 0$ a parameter depending on the structure of Γ and on the RW.

Background and Motivation

Gerl's conjecture

Gerl [Ger81]: the n-step return probabilities of two symmetric measures on a group have the same $n^{-\lambda}$. I.e. λ is a group invariant.

Background and Motivation

Gerl's conjecture

Gerl [Ger81]: the *n*-step return probabilities of two symmetric measures on a group have the same $n^{-\lambda}$. I.e. λ is a group invariant.

Cartwright's astonishing result

Cartwright [Car89]: found a *counterexample*. There exist at least two probability measures (with same support) on $\mathbb{Z}^d * \mathbb{Z}^d$ with $d \geq 5$ with different asymptotic behaviours: $n^{-3/2}$ and $n^{-d/2}$.

Background and Motivation

Gerl's conjecture

Gerl [Ger81]: the *n*-step return probabilities of two symmetric measures on a group have the same $n^{-\lambda}$. I.e. λ is a group invariant.

Cartwright's astonishing result

Cartwright [Car89]: found a *counterexample*. There exist at least two probability measures (with same support) on $\mathbb{Z}^d * \mathbb{Z}^d$ with $d \geq 5$ with different asymptotic behaviours: $n^{-3/2}$ and $n^{-d/2}$.

Questions

- What are the motivations of Cartwright's examples? What are the possible asymptotic behaviours on $\mathbb{Z}^{d_1} * \mathbb{Z}^{d_2}$ $(d_1 \neq d_2)$?
- What happens on $\Gamma_1 * \Gamma_2$ (Γ_1 and Γ_2 finitely generated groups)?

Definitions

Starting Objects

- $\Gamma_1, \dots, \Gamma_m$: finitely generated groups with identities $\{e_i\}_{i=1}^m$;
- $\mu_1, \dots, \mu_m$: probability measures s.t. $\langle \mathrm{supp}(\mu_i) \rangle = \Gamma_i$.

Definitions

Starting Objects

- $\Gamma_1, \dots, \Gamma_m$: finitely generated groups with identities $\{e_i\}_{i=1}^m$;
- $\mu_1, \dots, \mu_m$: probability measures s.t. $\langle \mathrm{supp}(\mu_i) \rangle = \Gamma_i$.

Structure and Properties

- Free Product $\Gamma := \Gamma_1 * \dots * \Gamma_m$: the set of all finite words of the form $x_1 x_2 \dots x_n$, where $x_1, \dots, x_n$ are elements of $\bigcup_i \Gamma_i \setminus \{e_i\}$ and x_j, x_{j+1} do not belong to the same group.

Definitions

Starting Objects

- $\Gamma_1, \ldots, \Gamma_m$: finitely generated groups with identities $\{e_i\}_{i=1}^m$;
- $\mu_1, \ldots, \mu_m$: probability measures s.t. $\langle \mathrm{supp}(\mu_i) \rangle = \Gamma_i$.

Structure and Properties

- Free Product $\Gamma := \Gamma_1 * \ldots * \Gamma_m$: the set of all finite words of the form $x_1 x_2 \ldots x_n$, where $x_1, \ldots, x_n$ are elements of $\bigcup_i \Gamma_i \setminus \{e_i\}$ and x_j, x_{j+1} do not belong to the same group.
- Define on Γ the probability measure

$$\mu := \alpha_1 \mu_1 + \alpha_2 \mu_2 + \ldots + \alpha_m \mu_m,$$

s.t. $\sum_{i=1}^m \alpha_i = 1$ and $\alpha_i > 0$ for every index $i \in \{1, \ldots m\}$.
We consider a RW on Γ governed by μ.

Green Functions

Green Functions

- $G_i(z) := \sum_{n=0}^{\infty} \mu_i^{(n)}(e_i) z^n$ on the free factors Γ_i for $i = 1, \ldots, m$;
- analogously on Γ we have $G(z) := \sum_{n=0}^{\infty} \mu^{(n)}(e) z^n$.

The radii of convergence will be denoted by $\mathbf{r}_i$ and $\mathbf{r}$ respectively.
What we look for, is the asymptotic behaviour of the $\mu^{(n)}(e)$.

Green Functions

Green Functions

- $G_i(z) := \sum_{n=0}^{\infty} \mu_i^{(n)}(e_i)z^n$ on the free factors Γ_i for $i = 1, \ldots, m$;
- analogously on Γ we have $G(z) := \sum_{n=0}^{\infty} \mu^{(n)}(e)z^n$.

The radii of convergence will be denoted by $\mathbf{r}_i$ and $\mathbf{r}$ respectively.
What we look for, is the asymptotic behaviour of the $\mu^{(n)}(e)$.

Idea

We find the singular expansion of $G(z)$ near $z = \mathbf{r}$ and then apply the Darboux's Method.

Remark: It is possible to use another method known as "Singularity Analysis" (see [FS09]), but there is no advantage here.

Darboux's Method

$S(z) :=$ leading singular term of $G(z)$ near $z = \mathbf{r}$:

$$G(z) = S(z) + R(z).$$

Known: asymptotic Taylor expansion of $S(z) = \sum_{n=0}^{\infty} a_n z^n$ near $z = 0$ (when $S(z)$ has algebraic or logarithmic terms: $a_n \sim n^{-k}$, for a suitable $k > 0$).

Darboux's Method

$S(z) :=$ leading singular term of $G(z)$ near $z = \mathbf{r}$:

$$G(z) = S(z) + R(z).$$

Known: asymptotic Taylor expansion of $S(z) = \sum_{n=0}^{\infty} a_n z^n$ near $z = 0$ (when $S(z)$ has algebraic or logarithmic terms: $a_n \sim n^{-k}$, for a suitable $k > 0$).
Let us consider the following condition:

$$G(z) - S(z) \in \mathscr{C}^k \text{ for all } |z| < \mathbf{r}. \qquad (*)$$

Darboux's Method

$S(z) :=$ leading singular term of $G(z)$ near $z = \mathbf{r}$:

$$G(z) = S(z) + R(z).$$

Known: asymptotic Taylor expansion of $S(z) = \sum_{n=0}^{\infty} a_n z^n$ near $z = 0$ (when $S(z)$ has algebraic or logarithmic terms: $a_n \sim n^{-k}$, for a suitable $k > 0$).
Let us consider the following condition:

$$G(z) - S(z) \in \mathscr{C}^k \text{ for all } |z| < \mathbf{r}. \qquad (*)$$

If $(*)$ is satisfied, applying the **Riemann-Lebesgue Lemma** it follows that the coefficients of $G(z) - S(z)$ are $\mathbf{o}(n^{-k}) = \mathbf{o}(a_n)$, implying $\mu^{(n)}(e) \sim a_n$.

Darboux's Method

$S(z) :=$ leading singular term of $G(z)$ near $z = \mathbf{r}$:

$$G(z) = S(z) + R(z).$$

Known: asymptotic Taylor expansion of $S(z) = \sum_{n=0}^{\infty} a_n z^n$ near $z = 0$ (when $S(z)$ has algebraic or logarithmic terms: $a_n \sim n^{-k}$, for a suitable $k > 0$).

Let us consider the following condition:

$$G(z) - S(z) \in \mathscr{C}^k \text{ for all } |z| < \mathbf{r}. \qquad (*)$$

If $(*)$ is satisfied, applying the **Riemann-Lebesgue Lemma** it follows that the coefficients of $G(z) - S(z)$ are $\mathbf{o}(n^{-k}) = \mathbf{o}(a_n)$, implying $\mu^{(n)}(e) \sim a_n$.

If $(*)$ is not satisfied, we **have to** expand $G(z)$ further ($S(z)$ will contain "enough" terms), until it holds.

Full Classification of RWs on $\mathbb{Z}^{d_1} * \mathbb{Z}^{d_2}$, $\mu = \alpha_1\mu_1 + \alpha_2\mu_2$

Let us compute the singular term of the Green function for each factor $\mathbb{Z}^d, d \geq 1$:

$$S_d(z) \sim \begin{cases} (\mathbf{r}_d - z)^{(d-2)/2}, & \text{if } d \text{ is odd,} \\ (\mathbf{r}_d - z)^{(d-2)/2} \log(\mathbf{r}_d - z), & \text{if } d \text{ is even,} \end{cases}$$

Full Classification of RWs on $\mathbb{Z}^{d_1} * \mathbb{Z}^{d_2}$, $\mu = \alpha_1\mu_1 + \alpha_2\mu_2$

Let us compute the singular term of the Green function for each factor $\mathbb{Z}^d, d \geq 1$:

$$S_d(z) \sim \begin{cases} (\mathbf{r}_d - z)^{(d-2)/2}, & \text{if } d \text{ is odd,} \\ (\mathbf{r}_d - z)^{(d-2)/2} \log(\mathbf{r}_d - z), & \text{if } d \text{ is even,} \end{cases}$$

There are three possible cases:

1st possibility: first singular term of $G(z)$ (defined on $\mathbb{Z}^{d_1} * \mathbb{Z}^{d_2}$) is proportional to

$$\sqrt{\mathbf{r} - z}. \qquad \text{(1st)}$$

Full Classification of RWs on $\mathbb{Z}^{d_1} * \mathbb{Z}^{d_2}$, $\mu = \alpha_1\mu_1 + \alpha_2\mu_2$

Let us compute the singular term of the Green function for each factor $\mathbb{Z}^d, d \geq 1$:

$$S_d(z) \sim \begin{cases} (\mathbf{r}_d - z)^{(d-2)/2}, & \text{if } d \text{ is odd,} \\ (\mathbf{r}_d - z)^{(d-2)/2} \log(\mathbf{r}_d - z), & \text{if } d \text{ is even,} \end{cases}$$

There are three possible cases:

1st possibility: first singular term of $G(z)$ (defined on $\mathbb{Z}^{d_1} * \mathbb{Z}^{d_2}$) is proportional to

$$\sqrt{\mathbf{r} - z}. \tag{1st}$$

In this case:

$$\mu^{(2n)}(e) \sim C \cdot \mathbf{r}^{-2n} \cdot n^{-3/2} \quad \text{independently of the weight } \alpha_1.$$

Full Classification of RWs on $\mathbb{Z}^{d_1} * \mathbb{Z}^{d_2}$, $\mu = \alpha_1\mu_1 + \alpha_2\mu_2$

2nd possibility: first singular term of $G(z)$ is proportional to

$$(\mathbf{r} - z)^{(d_1-2)/2} \log^{\kappa}(\mathbf{r} - z) \quad \text{[leading singularity on } \mathbb{Z}^{d_1}\text{].} \qquad \text{(2nd)}$$

($\kappa = 0$ for d_1 odd, or $\kappa = 1$ for d_1 even.)

Full Classification of RWs on $\mathbb{Z}^{d_1} * \mathbb{Z}^{d_2}$, $\mu = \alpha_1\mu_1 + \alpha_2\mu_2$

2nd possibility: first singular term of $G(z)$ is proportional to

$$(\mathbf{r} - z)^{(d_1-2)/2} \log^{\kappa}(\mathbf{r} - z) \quad \text{[leading singularity on } \mathbb{Z}^{d_1}\text{].} \qquad \text{(2nd)}$$

In this case:

$$\mu^{(2n)}(e) \sim C_1 \cdot \mathbf{r}^{-2n} \cdot n^{-d_1/2}, \text{ when } d_1 \geq 5 \text{ and } \alpha_1 > \alpha_c.$$

($\kappa = 0$ for d_1 odd, or $\kappa = 1$ for d_1 even.)

Full Classification of RWs on $\mathbb{Z}^{d_1} * \mathbb{Z}^{d_2}$, $\mu = \alpha_1\mu_1 + \alpha_2\mu_2$

2nd possibility: first singular term of $G(z)$ is proportional to

$$(\mathbf{r} - z)^{(d_1-2)/2} \log^{\kappa}(\mathbf{r} - z) \quad [\text{leading singularity on } \mathbb{Z}^{d_1}]. \tag{2nd}$$

In this case:

$$\mu^{(2n)}(e) \sim C_1 \cdot \mathbf{r}^{-2n} \cdot n^{-d_1/2}, \text{ when } d_1 \geq 5 \text{ and } \alpha_1 > \alpha_c.$$

3rd possibility: first singular term of $G(z)$ is proportional to

$$(\mathbf{r} - z)^{(d_2-2)/2} \log^{\kappa}(\mathbf{r} - z) \quad [\text{leading singularity on } \mathbb{Z}^{d_2}]. \tag{3rd}$$

($\kappa = 0$ for d_1, d_2 odd, or $\kappa = 1$ for d_1, d_2 even.)

Full Classification of RWs on $\mathbb{Z}^{d_1} * \mathbb{Z}^{d_2}$, $\mu = \alpha_1\mu_1 + \alpha_2\mu_2$

2nd possibility: first singular term of $G(z)$ is proportional to

$$(\mathbf{r} - z)^{(d_1-2)/2} \log^{\kappa}(\mathbf{r} - z) \quad \text{[leading singularity on } \mathbb{Z}^{d_1}\text{].} \qquad \text{(2nd)}$$

In this case:

$$\mu^{(2n)}(e) \sim C_1 \cdot \mathbf{r}^{-2n} \cdot n^{-d_1/2}, \text{ when } d_1 \geq 5 \text{ and } \alpha_1 > \alpha_c.$$

3rd possibility: first singular term of $G(z)$ is proportional to

$$(\mathbf{r} - z)^{(d_2-2)/2} \log^{\kappa}(\mathbf{r} - z) \quad \text{[leading singularity on } \mathbb{Z}^{d_2}\text{].} \qquad \text{(3rd)}$$

In this case:

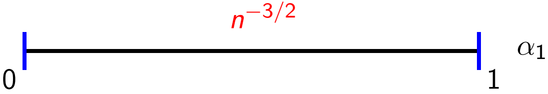
$$\mu^{(2n)}(e) \sim C_2 \cdot \mathbf{r}^{-2n} \cdot n^{-d_2/2}, \text{ when } d_2 \geq 5 \text{ and } \alpha_1 < \alpha_c.$$

($\kappa = 0$ for d_1, d_2 odd, or $\kappa = 1$ for d_1, d_2 even.)

What does it depend on?

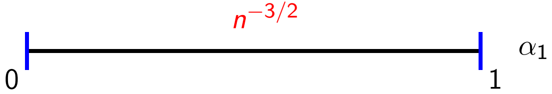
Complete Picture from the Point of View of α_1

Case (1st): suitable μ_1 and μ_2

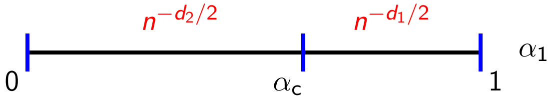


Complete Picture from the Point of View of α_1

Case (1st): suitable μ_1 and μ_2



Cases (2nd) and (3rd): e.g. μ_1 and μ_2 Simple RWs

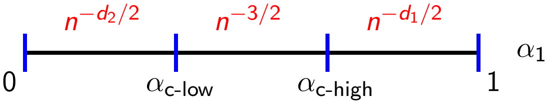


Here there is a value α_c which determines a *phase transition*.

Complete Picture from the Point of View of α_1

For some μ_1 and μ_2 it is possible to obtain all three behaviours, just depending on the value of the parameter α_1.

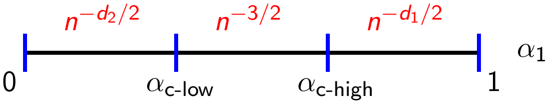
Keep μ_1, μ_2 fixed, α_1 varies: all possible behaviours



Complete Picture from the Point of View of α_1

For some μ_1 and μ_2 it is possible to obtain all three behaviours, just depending on the value of the parameter α_1.

Keep μ_1, μ_2 fixed, α_1 varies: all possible behaviours



Possible Combinations

It is possible to have one or two sub-intervals collapsing, depending on the properties of a Functional Equation concerning $G(z)$.

The Functional Equation

The trick to understand what happens, is to consider a functional equation (concerning $G(z)$), seen as a function of the paramenter α_1.

It behaves approximately like a truncated parabola, in particular it can have $0, 1$ or 2 zeros, according to its characteristics.

The Functional Equation

The trick to understand what happens, is to consider a functional equation (concerning $G(z)$), seen as a function of the paramenter α_1.

It behaves approximately like a truncated parabola, in particular it can have $0, 1$ or 2 zeros, according to its characteristics.

- For its **positive** values, the asymptotic behaviour obeys one of the $n^{-d_i/2}$-laws ($i = 1, 2$): which one? It depends on α_1.

The Functional Equation

The trick to understand what happens, is to consider a functional equation (concerning $G(z)$), seen as a function of the paramenter α_1.

It behaves approximately like a truncated parabola, in particular it can have $0, 1$ or 2 zeros, according to its characteristics.

- For its **positive** values, the asymptotic behaviour obeys one of the $n^{-d_i/2}$-laws ($i = 1, 2$): which one? It depends on α_1.
- Otherwise, the asymptotic behaviour obeys the $n^{-3/2}$-law.

The Functional Equation

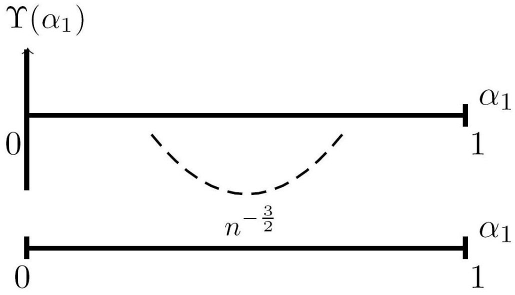


Figure: (1st) Case.

The Functional Equation

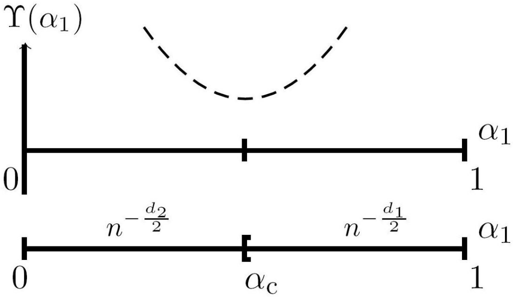


Figure: (2nd) Case, e.g if μ_1 and μ_2 are Simple RW.

The Functional Equation

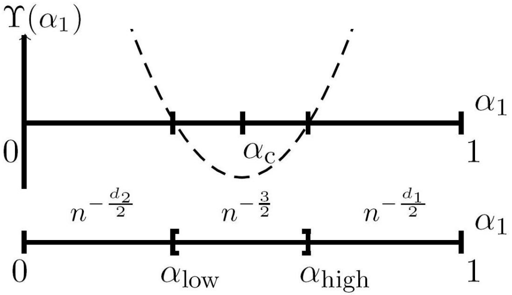


Figure: General Case (2 phase transitions).

Asymptotics

What is the meaning of our result?

According to the properties of the functional equation and to the value of α_1, the RW on Γ inherits its (non-exponential) behaviour either from the RW defined on $\mathbb{Z}^{d_1}$ or from the RW defined on $\mathbb{Z}^{d_2}$.

If those properties are not satisfied, we have the $n^{-3/2}$-behaviour.

More general Groups

On $\Gamma := \Gamma_1 * \Gamma_2$

Assume the $G_i(z)$ have *algebraic* or *logarithmic* singular expansion. Then *up to* 3 different asymptotic behaviours are possible for $\mu^{(n)}(e)$:

$$C \cdot \mathbf{r}^{-n} n^{-3/2}, \qquad C_1 \cdot \mathbf{r}^{-n} n^{-\lambda_1} \log^{\kappa_1} n, \qquad C_2 \cdot \mathbf{r}^{-n} n^{-\lambda_2} \log^{\kappa_2} n.$$

($\lambda_1, \lambda_2 > 0$, $\kappa_1, \kappa_2 \geq 0$ are parameters related to the singular expansions of $G_1(z)$ and $G_2(z)$ respectively).

More general Groups

On $\Gamma := \Gamma_1 * \Gamma_2$

Assume the $G_i(z)$ have *algebraic* or *logarithmic* singular expansion. Then *up to* 3 different asymptotic behaviours are possible for $\mu^{(n)}(e)$:

$$C \cdot \mathbf{r}^{-n} n^{-3/2}, \qquad C_1 \cdot \mathbf{r}^{-n} n^{-\lambda_1} \log^{\kappa_1} n, \qquad C_2 \cdot \mathbf{r}^{-n} n^{-\lambda_2} \log^{\kappa_2} n.$$

($\lambda_1, \lambda_2 > 0$, $\kappa_1, \kappa_2 \geq 0$ are parameters related to the singular expansions of $G_1(z)$ and $G_2(z)$ respectively).

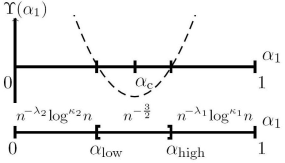


Figure: General Case

Résumé

Theorem: General Result (C. and Gilch, '09)

Define $\Gamma := \Gamma_1 * \ldots * \Gamma_m$, by induction we find:

The asymptotic behaviour of a (transient) RW on a Free Product of m (finitely generated) Groups obeys one of the following laws:

$$\mu^{(n)}(e) \sim \begin{cases} C_i \mathbf{r}^{-n} n^{-\lambda_i} \log^{\kappa_i} n & \text{for one } i \in \{1, \ldots, m\} \\ C_0 \mathbf{r}^{-n} n^{-3/2} & \end{cases}$$

According to the properties of the functional equation and to the values of the α_i, the RW on Γ inherits its (non-exponential) behaviour from the RW defined on one of the Γ_i.

If those properties are not satisfied, we have the $n^{-3/2}$-behaviour.

Skip the Proof: go to the Open Questions

Remarks

Idea of the used Method

- Define new functions $\xi_i(z)$ (where $i = 1, \dots, m$) s.t.

$$\alpha_i z G(z) = \xi_i(z) G_i(\xi_i(z)).$$

- Find the singular expansion for $\xi_i(z)$ near $z = \mathbf{r}$: either it has the same form of the expansion of one of the $G_j(z')$ near $z' = \mathbf{r}_j$, or it has a square root singular term.
- Find the singular expansion of $G(z)$ near $z = \mathbf{r}$: the same as $\xi_i(z)$.
- Apply method of Darboux.

Future Development and Open Questions

The idea is to develop this topic further, extending it to the study of -transient- *Branching Random Walks* (BRWs) on Free Products.

Future Development and Open Questions

The idea is to develop this topic further, extending it to the study of -transient- *Branching Random Walks* (BRWs) on Free Products.

Helpful literature: Hueter and Lalley (see [HL00]) study the asymptotic behaviour of BRWs on homogeneous trees.

Future Development and Open Questions

The idea is to develop this topic further, extending it to the study of -transient- *Branching Random Walks* (BRWs) on Free Products.

Helpful literature: Hueter and Lalley (see [HL00]) study the asymptotic behaviour of BRWs on homogeneous trees.

Remark: free products of groups are tree-like structures $\Rightarrow$ using this property, we get a first generalization of the results in [HL00] to free products of *finite* groups.

Future Development and Open Questions

Open Problems

Let us consider just the case $\Gamma := \Gamma_1 * \Gamma_2$:

Future Development and Open Questions

Open Problems

Let us consider just the case $\Gamma := \Gamma_1 * \Gamma_2$:

1st Open Problem: if at least one of the free factors is infinite, the previous reasoning fails: we have troubles in finding a suitable metric on Γ, in order to "measure" its limit set (boundary).

Future Development and Open Questions

Open Problems

Let us consider just the case $\Gamma := \Gamma_1 * \Gamma_2$:

1st Open Problem: if at least one of the free factors is infinite, the previous reasoning fails: we have troubles in finding a suitable metric on Γ, in order to "measure" its limit set (boundary).

2nd Open Problem: to compute "how big" the limit set of the BRW is, in relation to the boundary of Γ.

Future Development and Open Questions

Open Problems

Let us consider just the case $\Gamma := \Gamma_1 * \Gamma_2$:

1st Open Problem: if at least one of the free factors is infinite, the previous reasoning fails: we have troubles in finding a suitable metric on Γ, in order to "measure" its limit set (boundary).

2nd Open Problem: to compute "how big" the limit set of the BRW is, in relation to the boundary of Γ.

3rd Open Problem: are there any Phase Transitions (with respect to α_1) for the dimension of this limit set?

Thank you for your Attention!

Bibliography I

- Donald I. Cartwright, *Some examples of random walks on free products of discrete groups*, Ann. Mat. Pura Appl. (4) **151** (1988), 1–15. MR MR964500 (90f:60018)
- ______, *On the asymptotic behaviour of convolution powers of probabilities on discrete groups*, Monatsh. Math. **107** (1989), no. 4, 287–290. MR MR1012460 (91a:60024)
- Donald I. Cartwright and P. M. Soardi, *Random walks on free products, quotients and amalgams*, Nagoya Math. J. **102** (1986), 163–180. MR MR846137 (88i:60120a)
- Philippe Flajolet and Robert Sedgewick, *Analytic combinatorics*, Cambridge University Press, Cambridge, 2009. MR MR2483235

Bibliography II

- Peter Gerl, *A local central limit theorem on some groups*, The First Pannonian Symposium on Mathematical Statistics (Bad Tatzmannsdorf, 1979), Lecture Notes in Statist., vol. 8, Springer, New York, 1981, pp. 73–82. MR MR621143 (82h:60022)
- Irene Hueter and Steven P. Lalley, *Anisotropic branching random walks on homogeneous trees*, Probab. Theory Related Fields **116** (2000), no. 1, 57–88. MR MR1736590 (2001f:60094)
- Wolfgang Woess, *Nearest neighbour random walks on free products of discrete groups*, Boll. Un. Mat. Ital. B (6) **5** (1986), no. 3, 961–982. MR MR871708 (88i:60120b)
- ______, *Random walks on infinite graphs and groups*, Cambridge Tracts in Mathematics, vol. 138, Cambridge University Press, Cambridge, 2000. MR MR1743100 (2001k:60006)