# Ising models on power-law random graphs

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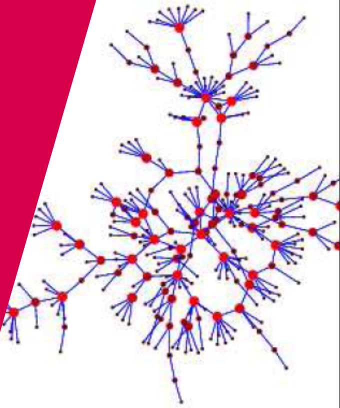

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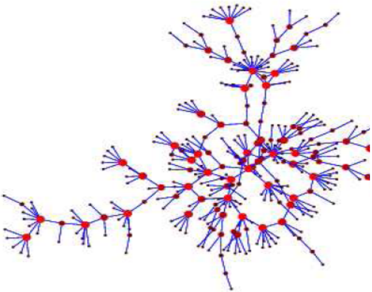
# Introduction



There are many complex real-world networks, e.g.,

- Social networks (friendships, business relationships, *sexual contacts,* ...);
- Information networks (World Wide Web, citations, ...);
- Technological networks (Internet, airline routes, ...);
- Biological networks (protein interactions, neural networks,...).



Sexual network Colorado Springs, USA (Potterat, et al., '02)



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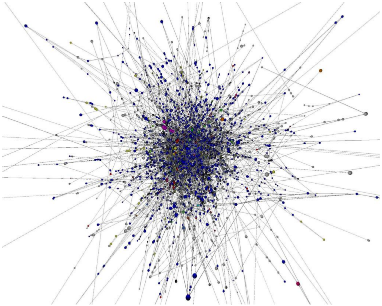

Small part of the Internet (http://www.fractalus.com/steve/stuff/ipmap/)



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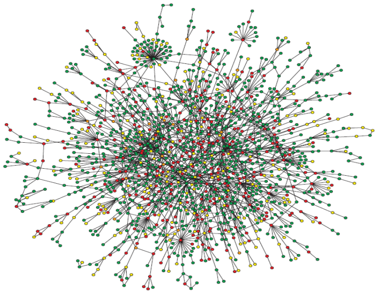


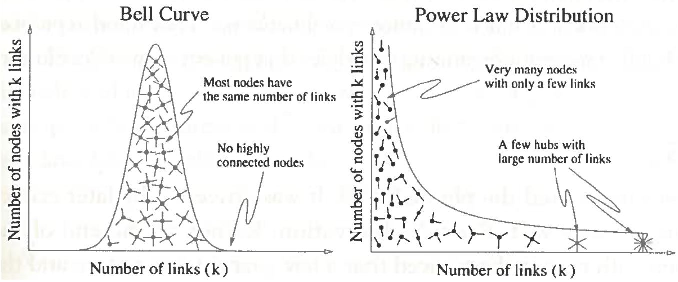
Yeast protein interaction network
(Jeong, et al., '01)

# Properties of complex networks



## *Power-law behavior*

Number of vertices with degree $k$ is proportional to $k^{-\tau}$.




Barabási, Linked, '02

## *Small worlds*

Distances in the network are small



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# Ising model



Ising model: paradigm model in statistical physics for *cooperative behavior.*

When studied on complex networks it can model for example *opinion spreading* in society.

What are effects of *structure* of complex networks on *behavior* of Ising model?

# Power-law random graphs



In the *configuration model* a graph $G_n = (V_n = [n], E_n)$ is constructed as follows.

- Let $D$ have a certain distribution (the *degree distribution*);
- *Assign* $D_i$ half-edges to each vertex $i \in [n]$, where $D_i$ are i.i.d. like $D$ (Add one half-edge to last vertex when the total number of half-edges is odd);
- *Attach* first half-edge to another half-edge *uniformly at random;*
- Continue until all half-edges are connected.

Special attention to power-law degree sequences, i.e.,

$$\mathbb{P}[D \geq k] \leq ck^{-(\tau-1)}, \qquad \tau > 1.$$



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Start from random vertex $i$ which has degree $D_i$.

Look at neighbors of vertex $i$, probability such a neighbor has degree $k+1$ is approximately,

$$\frac{(k+1)\sum_{j\in[n]} \mathbb{1}_{\{D_j=k+1\}}}{\sum_{j\in[n]} D_j}$$

# Local structure configuration model for $\tau > 2$



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$$\frac{(k+1)\sum_{j\in[n]} \mathbb{1}_{\{D_j=k+1\}}/n}{\sum_{j\in[n]} D_j/n} \longrightarrow \frac{(k+1)\mathbb{P}[D=k+1]}{\mathbb{E}[D]}, \qquad \text{for } \tau > 2.$$



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Let $K$ have distribution (the *forward degree* distribution),

$$\mathbb{P}[K=k] = \frac{(k+1)\mathbb{P}[D=k+1]}{\mathbb{E}[D]}.$$

*Locally tree-like* structure: a branching process with offspring $D$ in first generation and $K$ in further generations. Also, *uniformly sparse.*

# Definition of the Ising model



On a graph $G_n$, the *ferromagnetic Ising model* is given by the following Boltzmann distributions over $\sigma \in \{-1, +1\}^n$,

$$\mu(\sigma) = \frac{1}{Z_n(\beta, B)} \exp\left\{ \beta \sum_{(i,j)\in E_n} \sigma_i\sigma_j + B \sum_{i\in[n]} \sigma_i \right\},$$

where

- $\beta \geq 0$ is the inverse temperature;
- $B$ is the external magnetic field;
- $Z_n(\beta, B)$ is a normalization factor (the *partition function*), i.e.,

$$Z_n(\beta, B) = \sum_{\sigma\in\{-1,1\}^n} \exp\left\{ \beta \sum_{(i,j)\in E_n} \sigma_i\sigma_j + B \sum_{i\in[n]} \sigma_i \right\}.$$

# Critical temperature



Define the *magnetization* on $G_n$ as

$$m_n(\beta, B) = \frac{1}{n} \sum_{i=1}^{n} \langle \sigma_i \rangle_\mu .$$

Then, the *spontaneous magnetization,*

$$m(\beta, 0+) = \lim_{B \downarrow 0} \lim_{n \to \infty} m_n(\beta, B) \begin{cases} = 0, & \beta < \beta_c; \\ > 0, & \beta > \beta_c. \end{cases}$$

The *critical inverse temperature* $\beta_c$ is given by

$$\mathbb{E}[K](\tanh \beta_c) = 1.$$

Note that, for $\tau \in (2, 3)$, we have $\mathbb{E}[K] = \infty$, so that $\beta_c = 0$.



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# Pressure in thermodynamic limit ($\mathbb{E}[K] < \infty$)



**Theorem (Dembo, Montanari, '08)**

*For a* locally tree-like *and* uniformly sparse *graph sequence* $\{G_n\}_{n\geq 1}$ *with* $\mathbb{E}[K] < \infty$, *the pressure per particle,*

$$\psi_n(\beta, B) = \frac{1}{n} \log Z_n(\beta, B),$$

*converges, for* $n \to \infty$, *to*

$$\varphi_h(\beta, B) \equiv \frac{\mathbb{E}[D]}{2} \log\cosh(\beta) - \frac{\mathbb{E}[D]}{2} \mathbb{E}[\log(1 + \tanh(\beta)\tanh(h_1)\tanh(h_2))] + \mathbb{E}\left[\log\left(e^B \prod_{i=1}^{D}\{1+\tanh(\beta)\tanh(h_i)\} + e^{-B} \prod_{i=1}^{D}\{1-\tanh(\beta)\tanh(h_i)\}\right)\right].$$



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# Pressure in thermodynamic limit ($\mathbb{E}[D] < \infty$)

**Theorem (DGvdH, '10)**

*Let $\tau > 2$. Then, in the configuration model, the pressure per particle,*

$$\psi_n(\beta, B) = \frac{1}{n} \log Z_n(\beta, B),$$

*converges almost surely, for $n \to \infty$, to*

$$\varphi_h(\beta, B) \equiv \frac{\mathbb{E}[D]}{2} \log\cosh(\beta) - \frac{\mathbb{E}[D]}{2} \mathbb{E}[\log(1 + \tanh(\beta)\tanh(h_1)\tanh(h_2))]$$
$$+ \mathbb{E}\left[\log\left(e^B \prod_{i=1}^{D}\{1 + \tanh(\beta)\tanh(h_i)\}\right.\right.$$
$$\left.\left. + e^{-B} \prod_{i=1}^{D}\{1 - \tanh(\beta)\tanh(h_i)\}\right)\right].$$



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# Tree recursion



## Proposition

*Let* $K_t$ *be i.i.d. like* $K$ *and* $B > 0$. *Then, the recursion*

$$h^{(t+1)} \stackrel{d}{=} B + \sum_{i=1}^{K_t} \operatorname{atanh}(\tanh(\beta)\tanh(h_i^{(t)})),$$

*has a* unique fixed point $h^*_\beta$.

Interpretation: the *effective field* of a vertex in a *tree* expressed in that of its neighbors.

Uniqueness shown by showing that effect of *boundary conditions* on generation $t$ vanishes for $t \to \infty$.

This is done using *monotonicity* in $\beta$ and $B$ and *concavity* in $B$ of the magnetization in the ferromagnetic Ising model.

# Outline of the proof



$$\lim_{n\to\infty} \psi_n(\beta, B)$$

$$= \lim_{\varepsilon\downarrow 0} \lim_{n\to\infty} \left[ \psi_n(0, B) + \int_0^{\varepsilon} \frac{\partial}{\partial\beta'} \psi_n(\beta', B) \mathrm{d}\beta' + \int_{\varepsilon}^{\beta} \frac{\partial}{\partial\beta'} \psi_n(\beta', B) \mathrm{d}\beta' \right]$$

$$= \varphi_h(0, B) + 0 + \lim_{\varepsilon\downarrow 0} \int_{\varepsilon}^{\beta} \frac{\partial}{\partial\beta'} \varphi(\beta', B) \mathrm{d}\beta'$$

$$= \varphi_h(\beta, B).$$

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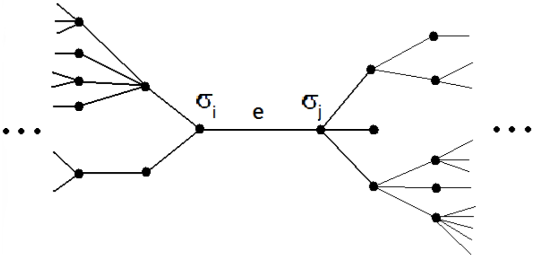
# Internal energy



$$\frac{\partial}{\partial \beta}\psi_n(\beta, B) = \frac{1}{n}\sum_{(i,j)\in E_n}\left\langle \sigma_i\sigma_j\right\rangle_\mu = \frac{|E_n|}{n}\frac{\sum_{(i,j)\in E_n}\left\langle \sigma_i\sigma_j\right\rangle_\mu}{|E_n|}$$

$$\longrightarrow \frac{\mathbb{E}[D]}{2}\mathbb{E}\left[\left\langle \sigma_i\sigma_j\right\rangle_\mu\right]$$



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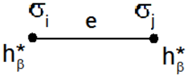



$$\frac{\mathbb{E}[D]}{2}\mathbb{E}\left[\left\langle \sigma_i\sigma_j\right\rangle_\mu\right] \longrightarrow \frac{\mathbb{E}[D]}{2}\mathbb{E}\left[\left\langle \sigma_i\sigma_j\right\rangle_e\right]$$

# Derivative of $\varphi$



$$\frac{\partial}{\partial\beta}\varphi_{h^*_\beta}(\beta, B) = \frac{\mathbb{E}[D]}{2}\mathbb{E}\left[\left\langle\sigma_i\sigma_j\right\rangle_e\right].$$

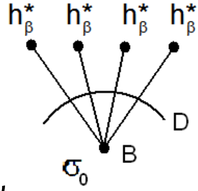
$$\varphi_h(\beta, B) = \frac{\mathbb{E}[D]}{2}\log\cosh(\beta) - \frac{\mathbb{E}[D]}{2}\mathbb{E}[\log(1+\tanh(\beta)\tanh(h_1)\tanh(h_2))]$$
$$+\mathbb{E}\left[\log\left(e^B\prod_{i=1}^{D}\{1+\tanh(\beta)\tanh(h_i)\} + e^{-B}\prod_{i=1}^{D}\{1-\tanh(\beta)\tanh(h_i)\}\right)\right]$$

- Show that we can ignore dependence of $h^*_\beta$ on $\beta$;
  (*Interpolation* techniques. Split analysis into two parts, one for *small degrees* and one for *large degrees*)
- Compute the derivative with assuming $\beta$ fixed in $h^*_\beta$.

## Corollary

*Let $\tau > 2$. Then, in the configuration model, a.s.:*
*The magnetization is given by*

$$m(\beta, B) \equiv \lim_{n\to\infty} m_n(\beta, B) = \frac{\partial}{\partial B}\varphi_{h^*}(\beta, B) = \mathbb{E}\left[\left\langle \sigma_0 \right\rangle_{\nu_{L+1}}\right].$$




*The susceptibility is given by*

$$\chi(\beta, B) \equiv \lim_{n\to\infty} \frac{\partial M_n(\beta, B)}{\partial B} = \frac{\partial^2}{\partial B^2}\varphi_{h^*}(\beta, B).$$



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# Distances in power-law random graphs



Let $H_n$ be the graph distance between two *uniformly chosen connected* vertices in the configuration model. Then:

- For $\tau > 3$ and $\mathbb{E}[K] > 1$ (vdH, Hooghiemstra, Van Mieghem, '05),

$$H_n \sim \log n,$$

- For $\tau \in (2, 3)$ (vdH, Hooghiemstra, Znamenski, '07),

$$H_n \sim \log \log n;$$

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For $\tau > 3$ and $\tau \in (2, 3)$ similar results hold for the diameter of linear preferential attachment models (D, vdH, Hooghiemstra, '10).



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# Critical exponents

Predictions by physicists (e.g. Leone, Vázquez, Vespignani, Zecchina, '02).

Critical behavior of *magnetization* $m$, and *susceptibility* $\chi$.

| | $m(\beta, 0^+), \beta \downarrow \beta_c$ | $m(\beta_c, B), B \downarrow 0$ | $\chi(\beta, 0^+), \beta \downarrow \beta_c$ |
|---|---|---|---|
| $\tau > 5$ | $\sim (\beta - \beta_c)^{1/2}$ | $\sim B^{1/3}$ | $\sim (\beta - \beta_c)^{-1}$ |
| $\tau \in (3, 5)$ | $\sim (\beta - \beta_c)^{1/(\tau-3)}$ | $\sim B^{1/(\tau-2)}$ | |
| $\tau \in (2, 3)$ | $\sim (\beta - \beta_c)^{1/(3-\tau)}$ | $\sim B^{1}$ | $\sim (\beta - \beta_c)^{1}$ |