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An a.s. renewal theorem for BRWs on the line

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Random walks

In what follows, let $X_1, X_2, \ldots$ denote a sequence of i.i.d. real-valued random variables and let

$$S_n = \sum_{k=1}^{n} X_k$$

denote the nth cumulative sum.

The process $(S_n)_{n \geq 0}$ will be called a *random walk*.

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If $\mu := \mathbb{E}S_1$ exists, then, by the SLLN,

$$S_n \sim n\mu \quad \text{a.s. as } n \to \infty.$$

Thus, if $\mu > 0$, one could expect that the expected number of visits of the process $(S_n)_{n \geq 0}$ of an interval $[a,b]$ is approximately $(b-a)/\mu$.

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Blackwell's renewal theorem

Theorem (Blackwell '48, Erdös, Feller, Pollard '49)

If $\mu \in (0,\infty)$, then

$$U([t,t+h]) := \mathbb{E}\sum_{n\geq 0} \mathbb{1}_{[t,t+h]}(S_n) \to \frac{h}{\mu} \quad \text{as } t \to \infty.$$

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The key renewal theorem

Theorem (Smith '54, Athreya, MacDonald, Ney '78)

If $\mu \in (0,\infty)$ and g is directly Riemann integrable, then

$$\int g(t-s)\,U(\mathrm{d}s) \;\to\; \frac{1}{\mu}\int g(s)\,\mathrm{d}s \quad \text{as } t \to \infty.$$

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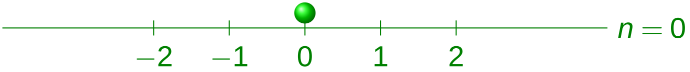
- Consider an individual, $\varnothing$, located at the origin of the real line at time $n = 0$.
- At time $n = 1$ the ancestor produces a random number Z_1 of offspring which is placed at real points according to a random point process $\mathcal{Z} = \sum_{i=1}^{Z_1} \delta_{X_i}$ on $\mathbb{R}$.
- We enumerate the ancestor's children by $1, 2, \ldots, Z_1$ where we do not exclude the case that $\mathbb{P}(Z_1 = \infty) > 0$.
- The offspring of the ancestor forms the first generation.

The population further evolves as follows.

- An individual v of the nth generation with position $S(v) \in \mathbb{R}$ produces at time $n+1$ a random number of offspring placed at random locations on $\mathbb{R}$ given by the positions of the point process $\sum_{i=1}^{Z_1(v)} \delta_{S(v)+X_i(v)}$.
- The offspring of individual v is enumerated by $v1, \ldots, vZ_1(v)$, the positions of offspring individuals are denoted by $S(vi)$, $i = 1, \ldots, Z_1(v)$.
- The point processes $\mathscr{Z}(v) = \sum_i \delta_{X_i(v)}$, $v \in \mathbb{V}$ are assumed to be i.i.d. copies of $\mathscr{Z}$.

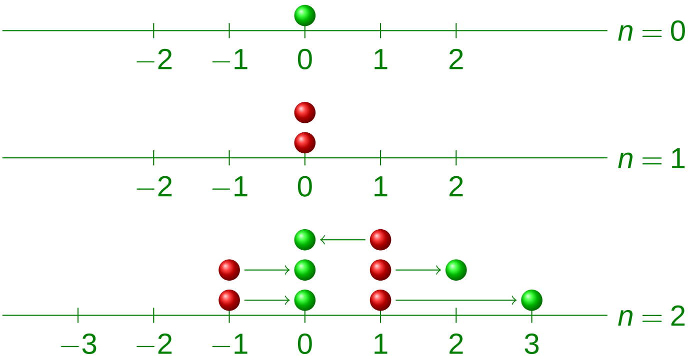
How it works

BRW



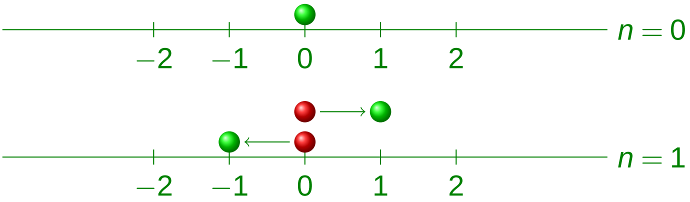
How it works

BRW



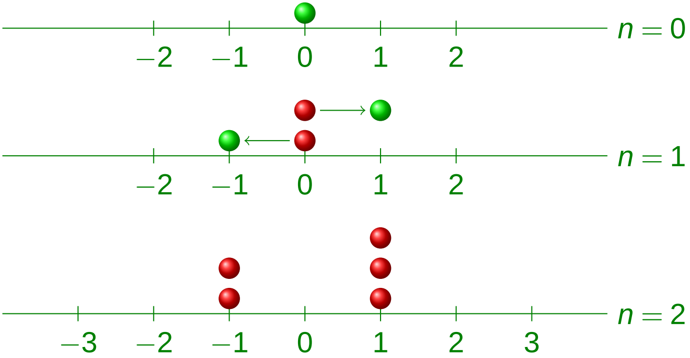
How it works

BRW



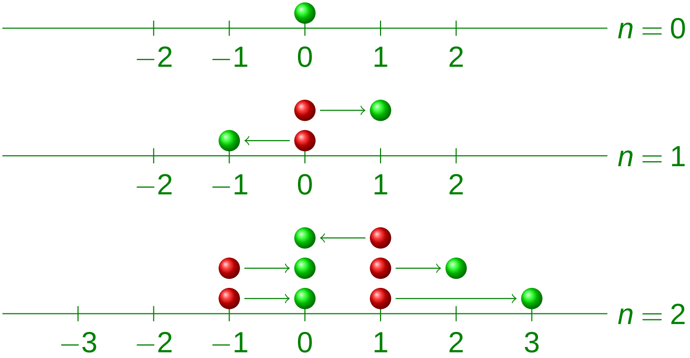
How it works

BRW



How it works

BRW



Branching Random Walks

Definition

The point process of the positions of the nth generation individuals will be denoted by $\mathscr{Z}_n$. The sequence of point processes $(\mathscr{Z}_n)_{n\geq 0}$ is then called *branching random walk*.

Assumptions

In what follows, we assume that

- The BRW is *supercritical*, that is, $\mathbb{E}Z_1 > 1$.
- The BRW is non-lattice.
- There exists a *Malthusian parameter* $\alpha > 0$, *i.e.*, an $\alpha > 0$ such that

$$m(\alpha) := \int e^{-\alpha x}\,\mathbb{E}\mathscr{Z}_1(\mathrm{d}x) = \mathbb{E}\sum_{|v|=1} e^{-\alpha S(v)} = 1.$$

Further we assume that

$$-m'(\alpha) := \mathbb{E}\sum_{|v|=1} S(v)e^{-\alpha S(v)} \in (0,\infty).$$

The limit of the martingale in the BRW

- It is well known that when the Malthusian parameter exists, the sequence

$$W_n^{(\alpha)} = \sum_{|v|=n} e^{-\alpha S(v)}, \quad n \geq 0$$

forms a non-negative martingale.

- $(W_n^{(\alpha)})_{n\geq 0}$ thus converges almost surely to a random variable $W^{(\alpha)}$.
- $\mathbb{E}W^{(\alpha)}$ equals either 0 or 1.
- In our situation (that is, when $-m'(\alpha) \in (0,\infty)$), $\mathbb{E}W^{(\alpha)} = 1$ if and only if $W_1^{(\alpha)}$ has a finite $x\log x$-moment (Biggins '77, Lyons '97).

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General branching processes

- If $\mathscr{Z}$ is concentrated on $[0, \infty)$, the BRW $(\mathscr{Z}_n)_{n \geq 0}$ will be called a *general branching process* or *CMJ branching process*.
- In this case, the position $S(v)$ can be interpreted as the birth time of the individual v.

Characteristics

Definition

Suppose that $\phi : \mathbb{R} \times \Omega \to \mathbb{R}$ is a function of the BRW. Then

$$Z_t^{\phi} := \sum_{v} [\phi]_v(t - S(v))$$

is a *general branching process counted with characteristic* ϕ.

Example

If $\phi(t) := \mathbb{1}_{[0,\infty)}(t)$, then

$$Z_t^{\phi} = \sum_{v} \mathbb{1}_{\{S(v) \leq t\}} = \#\{v : v \text{ is born up to time } t\}.$$

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The associated random walk

- The assumption that $m(\alpha) = \mathbb{E}\sum_{|v|=1} e^{-\alpha S(v)} = 1$ (Malthusian parameter) provides us with the possibility to make a change of measure.
- $m(\alpha) = 1$ implies that $\mu_\alpha := \mathbb{E}\sum_{|v|=1} e^{-\alpha S(v)} \delta_{S(v)}$ is a probability distribution.
- If $(S_n)_{n\geq 0}$ denotes a random walk with increment distribution μ_α, then

$$\mathbb{E}f(S_n) = \mathbb{E}\sum_{|v|=n} e^{-\alpha S(v)} f(S(v)).$$

- In particular, $\mathbb{E}S_1 = \mathbb{E}\sum_{|v|=1} e^{-\alpha S(v)} S(v) = -m'(\alpha)$.

Asymptotic behaviour of $\mathbb{E}Z_t^\phi$

$$
\begin{aligned}
e^{-\alpha t}\mathbb{E}Z_t^\phi &= \mathbb{E}\sum_v e^{-\alpha S(v)} e^{-\alpha(t-S(v))}[\phi]_v(t-S(v)) \\
&= \sum_{n\geq 0}\mathbb{E}\sum_{|v|=n} e^{-\alpha S(v)} e^{-\alpha(t-S(v))}[\phi]_v(t-S(v)) \\
&= \sum_{n\geq 0}\mathbb{E}g(t-S_n) = \int g(t-s)\,U(\mathrm{d}s) \\
&\to \frac{1}{-m'(\alpha)}\int_0^\infty e^{-\alpha s}\mathbb{E}\phi(s)\,s.
\end{aligned}
$$

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A.s. renewal theorem for CMJ processes

Theorem (Nerman '81)

Suppose that $\mathcal{Z}$ *is a.s. concentrated on* $[0,\infty)$ *and further suppose that* $\phi(t) = 0$ *for all* $t < 0$ *and that for some decreasing integrable function* $h : [0,\infty) \to (0,\infty)$

$$\sup_{t \geq 0} \frac{e^{-\alpha t}\phi(t)}{h(t)} \in \mathcal{L}^1.$$

Then, under a moment assumption concerning $\mathcal{Z}$*,*

$$e^{-\alpha t} Z_t^{\phi} \to \frac{W^{(\alpha)}}{-m'(\alpha)} \int_0^{\infty} e^{-\alpha s} \mathbb{E}\phi(s)\, \mathrm{d}s \quad \text{a.s. as } t \to \infty,$$

where $W^{(\alpha)}$ *denotes the limit of the martingale in the BRW.*

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Asymptotic number of particles $\leq t$

Corollary

Under the moment assumption concerning the $\mathscr{Z}$*,*

$$\#\{v : v \text{ is born up to time } t\} \sim e^{\alpha t} \frac{W^{(\alpha)}}{\alpha(-m'(\alpha))}$$

almost surely on $\{W^{(\alpha)} > 0\}$.

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Theorem

Suppose that $\phi(t)$ has càdlàg paths and that with $h(t) = 1 \wedge (t^{-1} \log^{-(1+\varepsilon)} t)$

$$\sup_{t \in \mathbb{R}} \frac{e^{-\alpha t} \phi(t)}{h(t)} \in \mathscr{L}^1 \quad \textit{and}$$

$$\mathbb{E} \sum_{|v|=1} e^{-\alpha S(v)} (S(v)^-)^2 \log^{1+\varepsilon}(S(v)^-) < \infty.$$

Then, under a moment assumption concerning the positive points generated by $\mathscr{Z}$,

$$e^{-\alpha t} Z_t^{\phi} \to \frac{W^{(\alpha)}}{-m'(\alpha)} \int_{-\infty}^{\infty} e^{-\alpha s} \mathbb{E}\phi(s) \, ds \quad \textit{a.s. as } t \to \infty.$$

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The main idea

- The main idea is to generalize the strategy of the proof in the case of ordinary random walks.
- To this end, we need a stopping concept for BRWs to extract an embedded BRW with positive steps (CMJ branching process).

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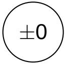
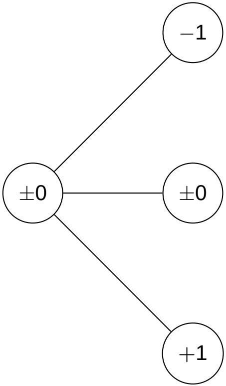
A strictly ascending ladder line



A strictly ascending ladder line



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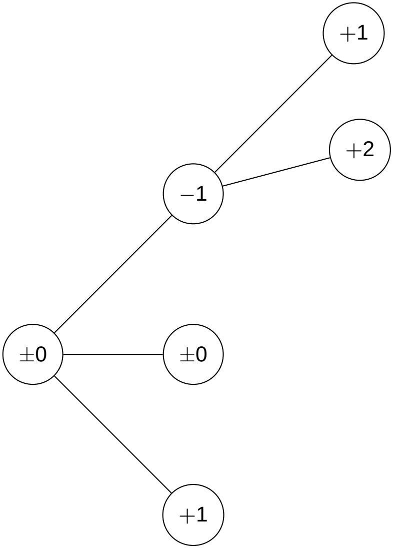

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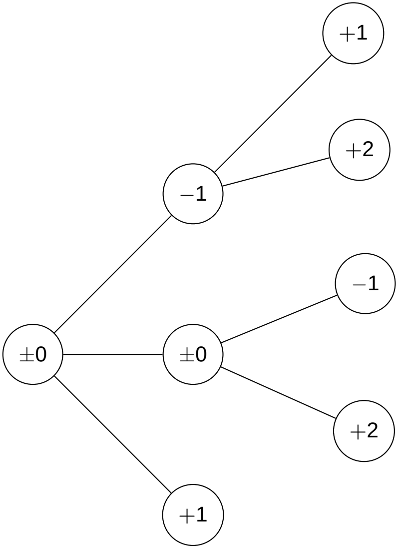

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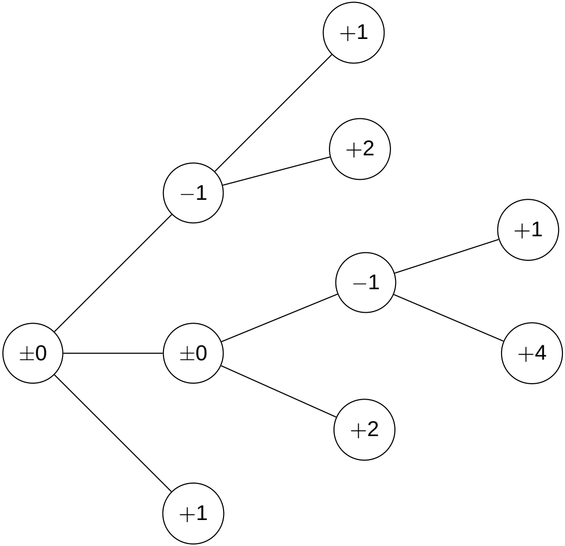

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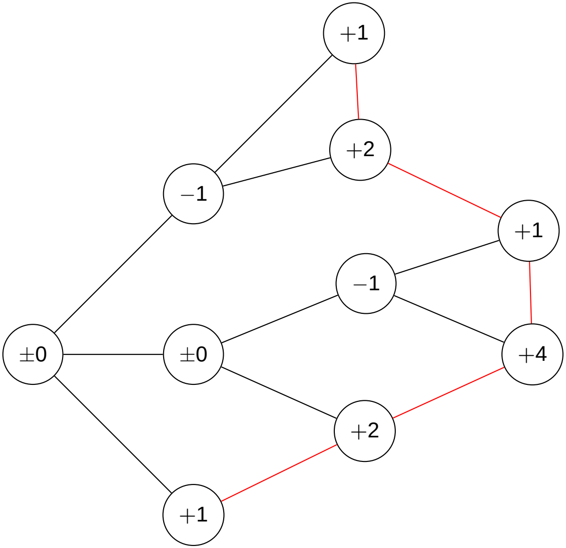

The generations of the embedded process

- Denote by $\mathscr{G} \subseteq \mathbb{V}$ the family tree of the process. We define the first strictly ascending ladder line by

$$\mathscr{G}_1^> := \{v \in \mathscr{G} : S(v) > 0 \text{ and } S(u) \leq 0 \text{ for all } u \prec v\}.$$

  Here, $u \prec v$ means that u is a strict ancestor of v.

- The nth strictly ascending ladder line is

$$\mathscr{G}_n^> := \{vw : v \in \mathscr{G}_{n-1}^>, w \in [\mathscr{G}_1^>]_w\},$$

  where $[\cdot]_v$ is the shift into the subtree rooted at v.

The embedded CMJ process

The embedded ladder line process $(\mathscr{Z}_n^{>})_{n\geq 0}$ is defined by

$$\mathscr{Z}_n^{>} := \sum_{v\in\mathscr{G}_n^{>}} \delta_{S(v)}, \qquad n\geq 0.$$

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The embedded CMJ process

The embedded ladder line process $(\mathscr{Z}_n^>)_{n\geq 0}$ is defined by

$$\mathscr{Z}_n^> := \sum_{v \in \mathscr{G}_n^>} \delta_{S(v)}, \qquad n \geq 0.$$

Fact (Jagers '89)

$(\mathscr{Z}_n^>)_{n\geq 0}$ is a CMJ process.

Decomposition of $\mathscr{Z}_t^{\phi}$

- For any characteristic ψ, we define

$$Z^{>,\psi}(t) := \sum_{v \in \mathscr{G}^{>}} [\psi]_v(t - S(v)) \qquad (t \in \mathbb{R}).$$

Decomposition of $\mathscr{Z}_t^{\phi}$

- For any characteristic ψ, we define

$$Z^{>,\psi}(t) := \sum_{v \in \mathscr{G}^{>}} [\psi]_v(t - S(v)) \qquad (t \in \mathbb{R}).$$

-

$$\phi^{>}(t) := \sum_{v \prec \mathscr{G}_1^{>}} [\phi]_v(t - S(v)), \qquad t \in \mathbb{R}.$$

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Decomposition of $\mathcal{Z}_t^{\phi}$

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$$\phi^{>}(t) := \sum_{v \prec \mathcal{G}_1^{>}} [\phi]_v(t - S(v)), \qquad t \in \mathbb{R}.$$

-

$$Z_t^{\phi} = \sum_{v \in \mathcal{G}} [\phi]_v(t - S(v))$$

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Decomposition of $\mathscr{Z}_t^{\phi}$

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$$Z^{>,\psi}(t) := \sum_{v \in \mathscr{G}^>} [\psi]_v(t - S(v)) \qquad (t \in \mathbb{R}).$$

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$$\phi^>(t) := \sum_{v \prec \mathscr{G}_1^>} [\phi]_v(t - S(v)), \qquad t \in \mathbb{R}.$$

-

$$\begin{aligned} Z_t^{\phi} &= \sum_{v \in \mathscr{G}} [\phi]_v(t - S(v)) \\ &= \sum_{v \in \mathscr{G}^>} \sum_{w \prec [\mathscr{G}_1^>]_v} [\phi]_{vw}(t - S(vw)) \end{aligned}$$

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Decomposition of $\mathcal{Z}_t^{\phi}$

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-

$$\begin{aligned} Z_t^{\phi} &= \sum_{v \in \mathcal{G}} [\phi]_v(t - S(v)) \\ &= \sum_{v \in \mathcal{G}^>} \sum_{w \prec [\mathcal{G}_1^>]_v} [\phi]_{vw}(t - S(vw)) \\ &= \sum_{v \in \mathcal{G}^>} [\phi^>]_v(t - S(v)) \ = \ Z_t^{>,\phi^>}, \qquad t \in \mathbb{R}. \end{aligned}$$

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What remains to do?

1. Show that $(\mathscr{Z}_n^{>})_{n \geq 0}$ satisfies the assumptions of Nerman's theorem.

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What remains to do?

1. Show that $(\mathscr{Z}_n^{>})_{n \geq 0}$ satisfies the assumptions of Nerman's theorem.

   This is a commonplace in BRW theory.

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What remains to do?

1. Show that $(\mathscr{Z}_n^{>})_{n\geq 0}$ satisfies the assumptions of Nerman's theorem.

   This is a commonplace in BRW theory.

2. Show that $\phi^{>}$ satisfies the assumptions of Nerman's theorem, that is,

$$\sup_{t\in\mathbb{R}} \frac{e^{-\alpha t}\phi(t)}{h(t)} \in \mathscr{L}^1$$

   with $h(t) \sim t^{-1}\log^{-(1+\varepsilon)} t$.

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In the given situation it turns out that

$$\sup_{t \in \mathbb{R}} \frac{e^{-\alpha t}\phi(t)}{h(t)} \in \mathscr{L}^1$$

is equivalent to

$$\infty > \mathbb{E} \sum_{v \prec \mathscr{G}_1^>} e^{-\alpha S(v)} \frac{1}{h(S(v))}$$

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$$\sup_{t\in\mathbb{R}} \frac{e^{-\alpha t}\phi(t)}{h(t)} \in \mathscr{L}^1$$

is equivalent to

$$\begin{aligned} \infty &> \mathbb{E}\sum_{v\prec\mathscr{G}_1^>} e^{-\alpha S(v)}\frac{1}{h(S(v))} \\ &= \int \frac{1}{h(s)}\,V^>(\mathrm{d}s) \end{aligned}$$

for the pre-$\sigma^>$-occupation measure of an associated random walk $(S_n)_{n\geq 0}$.

Proposition

Let $f : [0,\infty) \to [0,\infty)$ denote an increasing convex function regularly varying at ∞ of order $p \geq 1$. Denote by $\sigma^<$ the first strictly descending ladder index of the random walk $(S_n)_{n\geq 0}$, i.e., $\sigma^< = \inf\{n \geq 1 : S_n < 0\}$. Then the following assertions are equivalent:

(a) $\mathbb{E} S_1^- f(S_1^-) < \infty$,

(b) $\mathbb{E} f(|S_{\sigma^<}|) \mathbb{1}_{\{\sigma^< < \infty\}} < \infty$.

(c) $\mathbb{E} f(\min_{n\geq 0} S_n) < \infty$,

(d) $\int f(|s|) V^>(ds) < \infty$.

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Thank you for your attention.