Kinetic chemotaxis model on a space of measures

Daniël Worm
(in collaboration with
Sander Hille)

Mathematical Institute
Leiden University

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Outline

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$f : \mathbb{R}^n \times V \to \mathbb{R}$. $f(x, v)$ is density of bacteria (e.g. E. Coli) at position x with velocity v. $V \subset \mathbb{R}^n$ compact.

$S : \mathbb{R}^n \to \mathbb{R}$. $S(x)$ is density of chemical signal at position x.

(1) $\partial_t f(t) = -v \cdot \nabla_x f(t) + \mathcal{T}[S(t)]f(t)$,

(2) $\tau\partial_t S(t) = D\Delta S(t) + \alpha\rho(t) - \beta S(t) \qquad (\alpha, \beta \geq 0,\ \tau, D > 0)$,

$\rho(x, t) = \int_V f(x, v, t)\, dv$.

Turning kernel $T[S](x, v', v)$ is probability density of changing velocity from v to v' at position x, given the global signal S

$$\mathcal{T}[S]f(x, v) = -\int_V T[S](x, v', v)\, dv' \cdot f(x, v) + \int_V T[S](x, v, v')f(x, v')dv'.$$

Functional analytic approach

Consider the *integral form* of (1) – (2) (Variation of Constants-formula):

•$f(t) = T_\Phi(t)f_0 + \int_0^t T_\Phi(t-s)\mathcal{T}[S(s)]f(s)\,ds$.

•$S(t) = T_d(t)S_0 + \ldots$

•$(T_\Phi(t))_{t\geq 0}$ (strongly continuous) *transport semigroup* with generator $-v\cdot\nabla_x$. $T_\Phi(t)f(x,v) = f(x-vt,v)$.

•$(T_d(t))_{t\geq 0}$ (strongly continuous) *diffusion semigroup* with generator $D\Delta$.

$f \in L^1(\mathbb{R}^n \times V) \cap L^p(\mathbb{R}^n \times V)$, $S \in W^{k,q}(\mathbb{R}^n) \cap W^{k,\infty}(\mathbb{R}^n)$.

Some achieved results: **global existence**, **uniqueness** and **positivity** of mild solutions to (1) $-$ (2):

- Local Well-posedness of Kinetic Chemotaxis Models (by Hille), *J. Evol. Equ.* **8**, 2008.
- Global existence of positive mild solutions to a class of kinetic chemotaxis equations (by Hille, W), MI-Leiden Report 2007-47

f represents density of the bacteria. $L^1(\mathbb{R}^n \times V) \subset \mathcal{M}(\mathbb{R}^n \times V)$.

Question

(How) can we consider solutions of (1) $-$ (2) in a space of measures?

- Flow on state space $\mathbb{R}^n \times V$: $\Phi_t(x, v) = (x + vt, v)$.
- Transport semigroup $T_\Phi(t)f(x, v) = f(\Phi_t^{-1}(x, v)) = f(x - vt, v)$.
- How to formulate this in a space of measures?

- $P_\Phi(t)\mu(E) := \mu \circ \Phi_t^{-1}(E)$.
- Dirac measure δ_x: $P_\Phi(t)\delta_x = \delta_{\Phi_t(x)}$.
- $\mu(t) = P_\Phi(t)\mu_0 + \int_0^t P_\Phi(t - s)\mathcal{T}[S(s)]\mu(s)\, ds$

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- State space (S, d) complete separable metric space
- $\mathcal{M}(S)$ Banach space of finite Borel measures
- with *total variation norm* $\|\mu\|_{\mathrm{TV}} = \mu^+(S) + \mu^-(S)$.

But: topology is *too strong*!

- If $x \neq y$, then $\|\delta_x - \delta_y\|_{\mathrm{TV}} = 2$.
- So $t \mapsto P_\Phi(t)\delta_x = \delta_{\Phi_t(x)}$ only continuous if $\Phi_t(x) = x$.
- In general $t \mapsto P_\Phi(t)\delta_x$ not even strongly measurable.

Weak topology

$C_b(S)$ = space of bounded continuous functions $S \to \mathbb{R}$.

- More natural topology on $\mathcal{M}(S)$: *weak topology* $\sigma(\mathcal{M}(S), C_b(S))$.
- Drawbacks: is only a *locally convex topology*, not given by a norm.

$\mathrm{BL}(S)$ = Banach space of bounded Lipschitz functions with norm $\|f\|_{\mathrm{BL}} = |f|_{\mathrm{Lip}} + \|f\|_\infty$.

Results by Dudley (1966):

- $\mathcal{M}(S)$ embeds into $\mathrm{BL}(S)^*$: $\mu(f) = \int_S f\,d\mu$.
- $\mathcal{M}^+(S)$ is complete with respect to norm on $\mathrm{BL}(S)^*$.
- Restriction of norm topology of $\mathrm{BL}(S)^*$ to $\mathcal{M}^+(S)$ *equals* restriction of weak topology to $\mathcal{M}^+(S)$.

Consider $\mathcal{S}_{\mathrm{BL}} := \overline{\{\text{span of Dirac measures}\}}$ in $\mathrm{BL}(S)^*$. Then

- $\mathcal{S}_{\mathrm{BL}}$ is a separable Banach space.
- $\mathcal{M}(S) \subset \mathcal{S}_{\mathrm{BL}}$ dense.
- $\mathcal{S}_{\mathrm{BL}}^* = \mathrm{BL}(S)$
- $\mathcal{M}^+(S)$ is a closed convex cone in $\mathcal{S}_{\mathrm{BL}}$.
- So we can view $\mathcal{S}_{\mathrm{BL}}$ as ordered Banach space with positive cone $\mathcal{M}^+(S)$.
- In general $\mathcal{M}^+(S) - \mathcal{M}^+(S) = \mathcal{M}(S) \subsetneq \mathcal{S}_{\mathrm{BL}}$ (unless S uniformly discrete).

Embedding of semigroups of Lipschitz maps into positive linear semigroups on ordered Banach spaces generated by measures (by Hille, W.), *Integr. Equ. Oper. Theory* **63**, 2009.

Theorem (Hille, W.)

If $(\Phi_t)_{t\geq 0}$ is a semiflow on S such that

1. $\limsup_{t\downarrow 0} |\Phi_t|_{\mathrm{Lip}} < \infty$
2. $t \mapsto \Phi_t(x)$ is continuous for all x,

then $(P_\Phi(t))_{t\geq 0}$ extends to a strongly continuous semigroup on $\mathcal{S}_{\mathrm{BL}}$, leaving $\mathcal{M}^+(S)$ invariant.

Allows for an interpretation of

$$\mu(t) = P_\Phi(t)\mu_0 + \int_0^t P_\Phi(t-s)\mathcal{T}[S(s)]\mu(s)\,ds.$$

+ equation for S.

• Under certain conditions on turning kernel we obtain global existence and uniqueness of (positive) mild solutions.

Remark: we can exploit $\mathcal{S}_{\mathrm{BL}}$ to obtain results of existence, uniqueness and stability of invariant measures for Markov semigroups on spaces of measures.

Thank you for your attention

www.math.leidenuniv.nl/$\sim$dworm

Let (Ω, Σ, μ) be a measure space.

Theorem (Hille, W.)

Let $p : \Omega \to \mathcal{M}^+(S) \subset \mathcal{S}_{\mathrm{BL}}$. Then the following are equivalent:

(i) p is strongly measurable as function from Ω to $\mathcal{S}_{\mathrm{BL}}$

(ii) for every Borel set $E \subset S$, $\omega \mapsto p(\omega)(E)$ is measurable.

Theorem (Hille, W.)

Let $p : \Omega \to \mathcal{M}^+(S) \subset \mathcal{S}_{\mathrm{BL}}$ be Bochner integrable with respect to $\mu \in \mathcal{M}^+(S)$. Then

$$\int_\Omega p(\omega) d\mu(\omega)(E) = \int_\Omega p(\omega)(E) d\mu(\omega).$$