Symplectic invariants and existence of periodic orbits of Hamiltonian dynamical systems

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Outline

- Symplectic manifolds and Hamiltonian dynamics
- Hypersurfaces of contact type and the Weinstein Conjecture
- Contact structures
- J-holomorphic curves and Reeb dynamics

Basics

Symplectic manifolds

A symplectic structure ω on a manifold M is a closed ($d\omega = 0$), nondegenerate ($\underbrace{\omega \wedge \ldots \wedge \omega}_{n} \neq 0$) smooth 2-form.

Examples:

- $\mathbb{R}^{2n}$, $\omega_0 = \sum_{i=1}^{n} dp_i \wedge dq_i$: locally, all symplectic manifolds look like this! (Darboux's theorem)

- $\mathbb{S}^2$, $\mathbb{T}^2$ and in fact all oriented closed surfaces

- complex projective space $\mathbb{CP}^n$

- cotangent bundles T^*P, where P is a Riemannian manifold.

Almost complex structures

An almost complex structure on a manifold M is an endomorphism $J : TM \to TM$ of the tangent bundle such that $J^2 = -1$.

Examples:

- the standard almost complex structure $J_0 = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$ on $\mathbb{R}^{2n}$ (under the identification with $\mathbb{C}^n$, this is just multiplication by i)

- every symplectic manifold admits an almost complex structure.

If ω is a symplectic form on M, the almost complex structure J is said to be compatible with ω if

$$g_J(v, w) := \omega(v, Jw)$$

defines a Riemannian metric on M.

Example: consider $M = \mathbb{R}^{2n}$ with the standard symplectic and almost complex structure. Then

$$g_{J_0}(v, w) = \omega_0((v_1, v_2), J_0(w_1, w_2)) = \\ dp \wedge dq((v_1, v_2), (-w_2, w_1)) = v_1 w_1 + v_2 w_2 = \langle v, w \rangle$$

and we recover the usual euclidean metric.

Hamiltonian dynamics

Let (M, ω) be a symplectic manifold and consider a smooth function $H : M \to \mathbb{R}$, the Hamiltonian function.

The Hamiltonian vector field X_H is defined by $i_{X_H}\omega = -dH$ and we are interested in integral curves of this vector field, i.e. solutions of:

$$\dot{x}(t) = X_H(x(t))$$

Example: if $M = \mathbb{R}^{2n}$ with coordinates $x = (p, q)$ and standard symplectic form $\omega_0 = \sum dp_i \wedge dq_i$, the Hamiltonian vector field is given by $J_0 \nabla H = (-\partial_q H, \partial_p H)$ and we recover the classical system of Hamiltonian equations

$$\frac{dp}{dt} = -\frac{\partial H}{\partial q}, \qquad \frac{dq}{dt} = \frac{\partial H}{\partial p}$$

Existence of closed orbits

A major driving force behind the development of symplectic topology in the last decades has been the question of existence of *periodic* solutions of $\dot{x} = X_H(x)$.

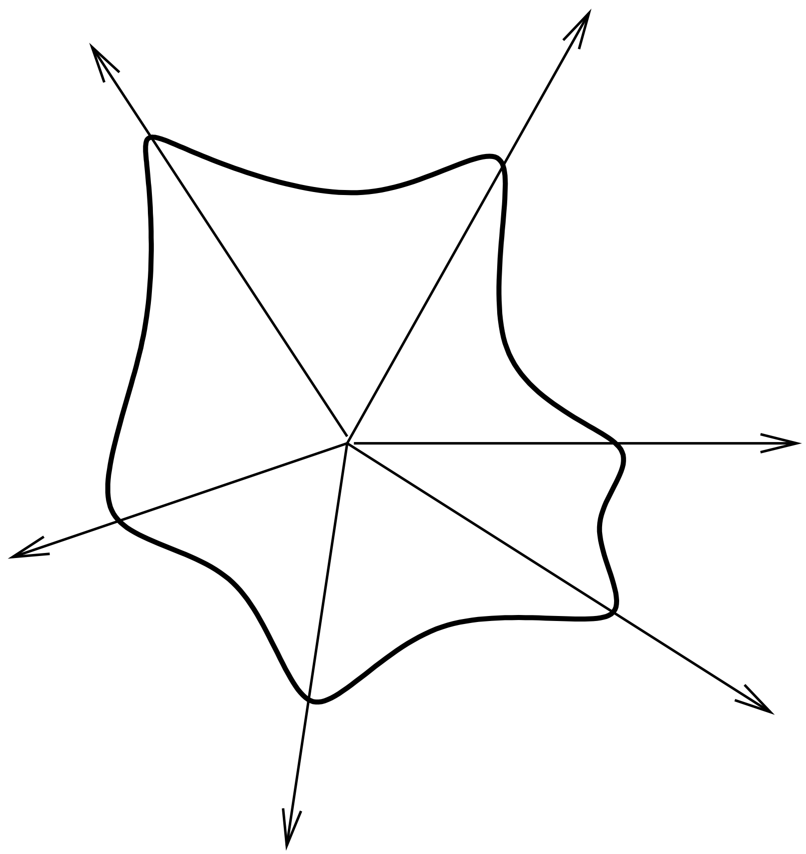
Existence of such solutions on a given energy level S (a level set of the Hamiltonian H) is completely determined by the underlying hypersurface and the symplectic form.

If H and G are two Hamiltonian functions having S as a (regular) level set, X_H and X_G coincide up to reparametrization, since both are sections of the <u>characteristic line bundle</u>

$$\mathcal{L}_S = \{v \in TS : \omega(v, w) = 0 \text{ for all } w \in TS\} = \ker(\omega|_{TS})$$

Question: Given a symplectic manifold (M, ω) and a smooth hypersurface S, does S have any closed orbits (i.e., closed integral curves of the characteristic line bundle)?

Theorem (Rabinowitz, 1978): Every star-shaped hypersurface in the standard symplectic $\mathbb{R}^{2n}$ admits a closed orbit.



A starshaped hypersurface bounds a starshaped domain.

Remark: if $S \subset (\mathbb{R}^{2n}, \omega_0)$ is starshaped with respect to the origin, then it is everywhere transverse to the radial vector field

$$Y = \frac{1}{2} \sum_{i=1}^{n} \left(p_i \frac{\partial}{\partial p_i} + q_i \frac{\partial}{\partial q_i} \right).$$

Moreover, Y satisfies $\mathcal{L}_Y \omega_0 = \omega_0$ (<u>Liouville vector field</u>).

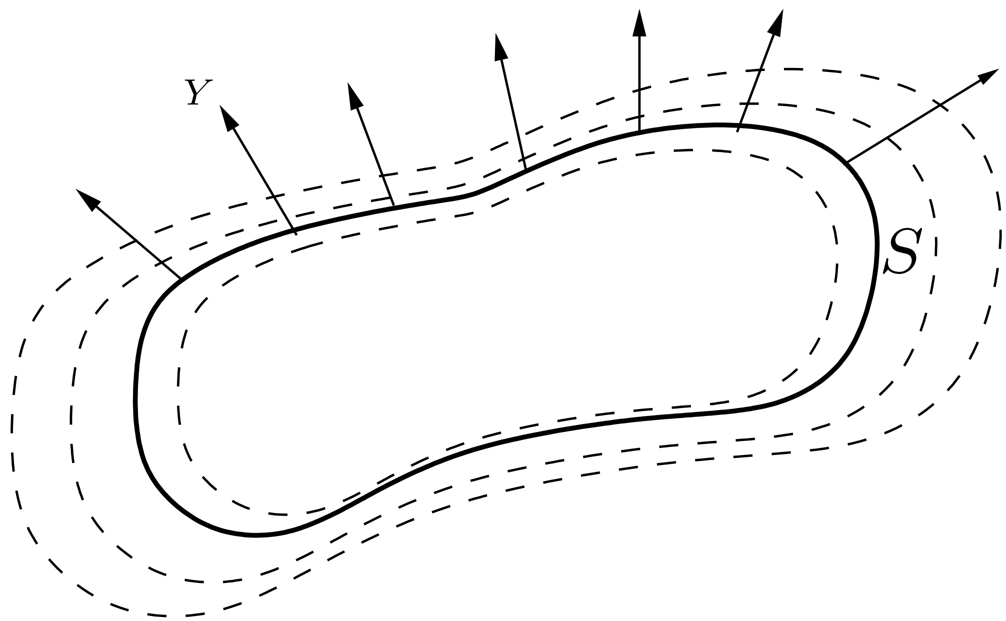
Hypersurfaces of contact type

A compact hypersurface S in a symplectic manifold (M, ω) is called of <u>contact type</u> if there exists a vector field Y, defined in a neighbourhood of S, satisfying:

(i) $\mathcal{L}_Y \omega = \omega$

(ii) $Y(x) \notin T_x S$ if $x \in S$.

Weinstein Conjecture (1978): Every hypersurface of contact type admits a closed orbit.

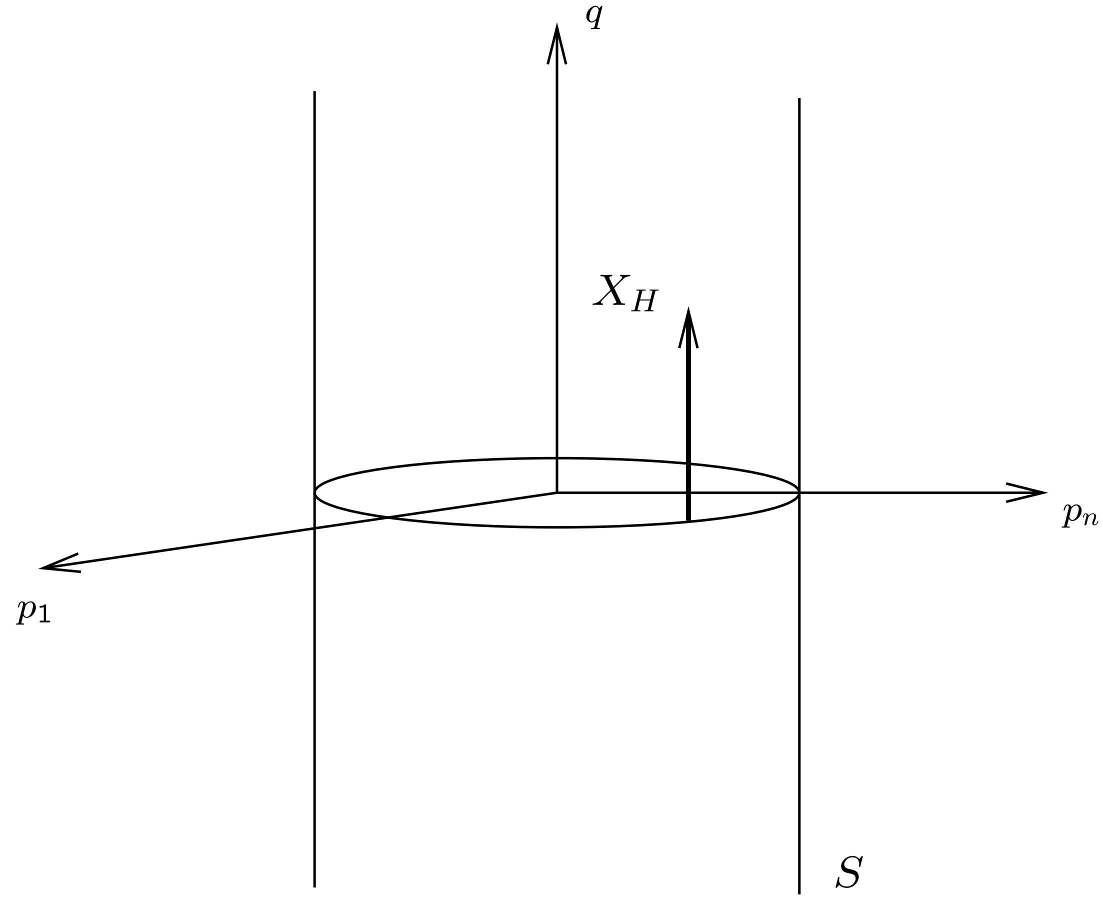


The flow of the Liouville vector field gives a foliation of a neighbourhood of S by smooth hypersurfaces diffeomorphic to S: the dynamics is the same on each hypersurface (almost existence $\Rightarrow$ existence).

Theorem (Viterbo, 1987): Every compact contact type hypersurface $S \subset (\mathbb{R}^{2n}, \omega_0)$ admits a closed orbit.

proof: find periodic orbits of a suitable Hamiltonian system, these correspond to critical points of an action functional on the loop space.

Remark: While non-compact hypersurfaces occur very naturally as energy levels (higher order Lagrangian problems, singular potentials, Lorenzian geoedesic problem...), Viterbo's result does not hold any more if we remove the compactness assumption.



The hypersurface $S = \{H = \sum_{i=1}^{n} p_i^2 = 1\} \simeq S^{n-1} \times \mathbb{R}^n$ satisfies the contact type condition, but the vector field X_H doesn't have any closed orbit on S.

Non-compact hypersurfaces in $\mathbb{R}^{2n}$

Compactness needs to be replaced by other assumptions!

Theorem (van den Berg-P.-Vandervorst, 2009): A mechanical hypersurface $S \subset \mathbb{R}^{2n}$ satisfying

(i) S is *asymptotically regular*;

(ii) $H_i(S) \neq 0$ for some $i \in \{n, \ldots, 2n-1\}$

always admits a closed orbit.

proof: variational, linking argument.

Remark: Topological characterization of S is essential to the proof. Extension to more general hypersurfaces needs different approach.

Contact manifolds

A contact form on a $(2n-1)$-dimensional manifold N is a smooth 1-form λ such that $d\lambda$ is everywhere nondegenerate on $\xi = \ker \lambda$. The hyperplane distribution ξ is then called a contact structure.

Example: the unit sphere $S^{2n+1} \subset \mathbb{R}^{2n+2}$ with the form

$$\alpha_0 = \sum_{i=1}^{n+1} (p_i \, dq_i - q_i \, dp_i).$$

The kernel of $d\lambda$ is one-dimensional, so there exists a unique vector field X_λ (called the Reeb vector field) satisfying $i_{X_\lambda} d\lambda \cong 0$ and $\lambda(X_\lambda) = 1$.

Intrinsic version of the Weinstein Conjecture: For every closed odd-dimensional manifold N with contact form λ, the Reeb vector field X_λ admits a closed orbit.

Remark: If S is a hypersurface of contact type in a symplectic manifold (M,ω) and Y is a transverse Liouville vector field for S, then S is a contact manifold with contact form $\lambda = i_Y\omega|_S$ and the induced Hamiltonian flow is conjugated to the Reeb flow.

Theorem (Hofer, 1993): the Weinstein Conjecture holds for the 3-sphere S^3.

Theorem (Taubes, 2007): the Weinstein Conjecture holds for any closed 3-dimensional manifold.

The proof of these results is based on (invariants constructed using) *J-holomorphic curves*.

J-holomorphic curves

Let (Σ, j) be a Riemannian surface with its conformal structure and J an almost complex structure on the manifold M.

A map $u : \Sigma \to M$ is called a J–holomorphic curve if its differential is complex linear: $du \circ j = J \circ du$.

Equivalently,

$$\bar{\partial}_J u = \frac{1}{2}(du + J \circ du \circ j) = 0$$

If J is compatible with a symplectic structure on M, the space of such curves has very nice (compactness) properties (**Gromov**) and can be used to define very powerful invariants .

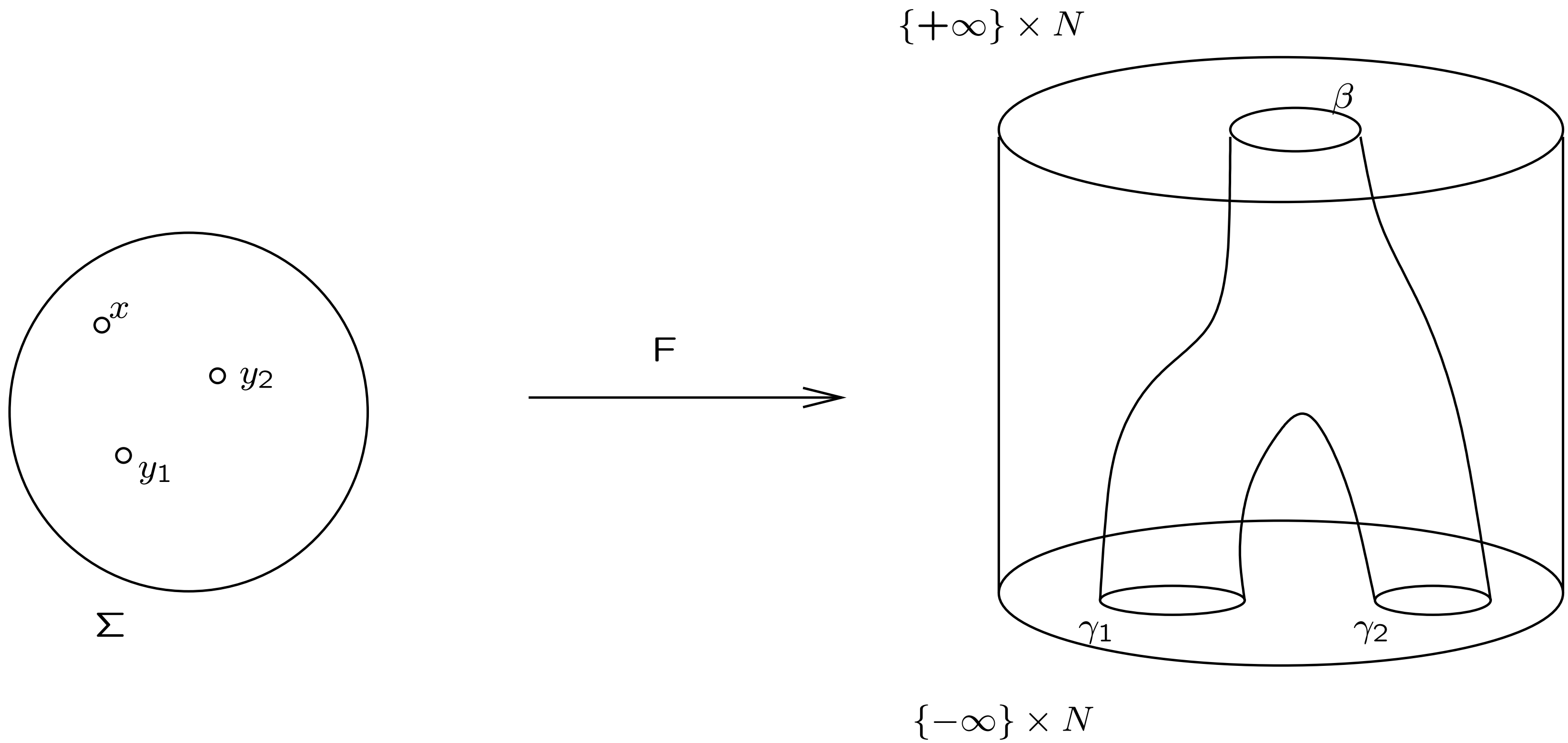
J-holomorphic curves and Reeb dynamics

Given a contact manifold (N, λ), the symplectization of N is the manifold $M = \mathbb{R} \times N$ with symplectic form $\omega = d(e^t \lambda)$, where t is the coordinate on $\mathbb{R}$.

The idea is to look at J-holomorphic curves in $(\mathbb{R} \times N, d(e^t \lambda))$ for a suitable class of almost complex structures. One considers J-holomorphic maps

$$F : (\Sigma, j) \longrightarrow (\mathbb{R} \times N, J)$$

where Σ now denotes a closed Riemann surface with finitely many punctures.



Example: a holomorphic sphere with 3 punctures in the symplectization of the contact manifold N.

Under suitable conditions (*finite energy, generic* λ), the image of such a map near each puncture converges at $t = \pm\infty$ to a cylinder of the form $\mathbb{R} \times \gamma$, where γ is a closed Reeb orbit of X_λ.

$$\text{Existence of punctured } J\text{-holomorphic curves in } \mathbb{R} \times N$$
$$\Downarrow$$
$$\text{Weinstein Conjecture on } N.$$

First existence results obtained with adhoc method. Later a homology theory appeared in the background: **Contact Homology** (Eliashberg, Hofer)

Roughly speaking, the construction of contact homology for the contact manifold (N, λ) goes as follows:

- consider a graded algebra $\mathcal{R}$ and the graded $\mathcal{R}$-module C_* freely generated by all (good) closed Reeb orbits of X_λ;

- the boundary homomorphisms 'count' J-holomorphic curves in the symplectization of N (this is possible because the curves we consider form spaces that are *compact*).

Think of Morse trajectories and Morse homology!

Non-compact Contact Homology

Joint with Rob Vandervorst, work in progress: define a homology theory for non-compact contact manifolds that encodes information about the Reeb dynamics.

Main issue: additional assumptions needed to guarantee compactness of the spaces of J-holomorphic curves in spite of the non-compactness of the contact manifolds.