Correlation inequalities in IPS

April 9, 2010

Random walk

1. S finite (or countably infinite) set, $\{p(x,y) : x, y \in S\}$ symmetric, irreducible Markov transition matrix, $p(x,x) = 0$.
2. Continuous time random X_t walk based on S: start from x wait exponential time with parameter 1 (α), then jump to y with probability $p(x,y)$. Generator

$$\begin{aligned} Lf(x) &= \lim_{t \to 0} \frac{\mathbb{E}_x f(X_t) - f(x)}{t} \\ &= \alpha \sum_y p(x,y)(f(y) - f(x)) \end{aligned}$$

Semigroup

$$S_t f(x) = e^{tL} f(x) = \mathbb{E}_x f(X_t)$$

Independent random walks

n independent copies of the process X_t; $(X_1(t), \ldots, X_n(t))$.
Generator

$$Lf(x_1, \ldots, x_n) = \sum_{i=1}^{n} \sum_{y} p(x_i, y)(f(x^{x_i,y}) - f(x))$$

Configuration space notation $\eta \in \Omega = \mathbb{N}^S$:
$\eta_t(x) = \sum_{i=1}^{n} I(X_i(t) = x)$

$$Lf(\eta) = \sum_{x} \eta(x) p(x, y)(f(\eta^{xy}) - f(\eta))$$

Duality and local equilibrium measures

Define for $\xi \in \Omega$, $\sum_x \xi_x < \infty$, $\eta \in \Omega$:

$$D(\xi, \eta) = \prod_{x \in S} d(\xi_x, \eta_x)$$

with

$$d(n, k) = \frac{n!}{(n-k)!}$$

Then we have **self-duality** (Spitzer, Doob, De Masi-Presutti)

$$\mathbb{E}_\eta D(\xi, \eta_t) = \mathbb{E}_\xi D(\xi_t, \eta)$$

Stationary (equilibrium) measures

$$\nu_\rho = \otimes_{x \in S} \mathrm{Poisson}(\rho)(d\eta_x)$$

Local stationary measures: for $\overline{\rho} : S \to [0, \infty)$, define

$$\nu_{\overline{\rho}} = \otimes_{x \in S} \mathrm{Poisson}(\rho_x)(d\eta_x)$$

This is not stationary but if $\xi = \delta_{x_1} + \ldots + \delta_{x_n}$, then

$$\int D(\xi, \eta)\nu_{\overline{\rho}}(d\eta) = \prod_{i=1}^{n} \rho(x_i)$$

By self-duality such local stationary measures are propagated in time, i.e.,

$$\nu_{\overline{\rho}} S_t = \nu_{\overline{\rho}_t}$$

with

$$\overline{\rho}_t(x) = \mathbb{E}_x \rho(X_t)$$

$$
\begin{aligned}
\int D(\xi,\eta)\nu_{\overline{\rho}}S_t(d\eta) &= \int \mathbb{E}_\eta(D(\xi,\eta_t))\nu_{\overline{\rho}}(d\eta)\\
&= \int \mathbb{E}_\xi(D(\xi_t,\eta))\nu_{\overline{\rho}}(d\eta)\\
&= \mathbb{E}_{x_1,\ldots,x_n}\left(\prod_{i=1}^n \overline{\rho}(X_i(t))\right)\\
&= \prod_{i=1}^n \mathbb{E}_{x_i}\overline{\rho}(X_i(t))\\
&= \int D(\xi,\eta)\nu_{\overline{\rho}_t}(d\eta)
\end{aligned}
$$

with

$$
\overline{\rho}_t(x) = \mathbb{E}_x\overline{\rho}(X_t) = \sum_y p_t(x,y)\overline{\rho}(y)
$$

Symmetric Exclusion Process (SEP)

Random walkers restricted by the fact that there can be at most 1 particle per site $x \in S$. Generator

$$\begin{aligned} Lf(x_1, \ldots, x_n) &= \sum_{i=1}^{n} \sum_{y \notin \{x_1, \ldots x_n\}} p(x_i, y)(f(x^{x_i,y}) - f(x)) \\ &= L_{ind} f(x) - \mathcal{L} f(x) \end{aligned}$$

with

$$\mathcal{L} f(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} p(x_i, x_j)(f(x^{x_i,x_j}) - f(x))$$

"clumping part of generator" In configuration notation: $\eta \in \{0,1\}^S$ and

$$Lf(\eta) = \sum_{x} \eta(x)(1 - \eta(y)) p(x,y)(f(\eta^{xy}) - f(\eta))$$

Duality and invariant measures

For $\xi \in \Omega$, $\sum_x \xi_x < \infty$, $\eta \in \Omega$

$$D(\xi, \eta) = \prod_{x:\xi_x = 1} \eta_x$$

then

$$\mathbb{E}_\eta^{SEP} D(\xi, \eta_t) = \mathbb{E}_\xi^{SEP} D(\xi_t, \eta)$$

For $\overline{\rho} : S \to [0,1]$ a "density profile, define the product measure

$$\nu_{\overline{\rho}} = \otimes_x Bin(1, \rho(x))(d\eta_x)$$

Let $\xi = \sum_{i=1}^n \delta_{x_i}$ be a configuration. The polynomials $D(\xi, \eta)$ satisfy

$$\int D(\xi, \eta) \nu_{\overline{\rho}}(d\eta) = \prod_{i=1}^n \overline{\rho}(x_i)$$

By self-duality

$$\int D(\xi,\eta)\nu_{\overline{\rho}}S_t(d\eta) = \mathbb{E}^{SEP}_{x_1,\dots,x_n}\left(\prod_{i=1}^{n}\overline{\rho}(X_i(t))\right)$$

But now these are not independent anymore, so no further simplification, unless $\overline{\rho} = \rho$ is **constant**, then we see ν_ρ is **invariant**. However, we have **the inequality**

$$\mathbb{E}^{SEP}_{x_1,\dots,x_n}\left(\prod_{i=1}^{n}\overline{\rho}(x_i)\right) \le \prod_{i=1}^{n}\mathbb{E}_{x_i}\overline{\rho}(X_i(t))$$

as we will see soon.

2J- Symmetric Exclusion Process

Same idea, but this time at most $2J$ particles per site. In configuration notation $\eta \in \Omega = \{0, 1, \ldots, 2J\}^S$,

$$Lf(\eta) = \sum_x \eta_x (2J - \eta_y) p(x, y)(f(\eta^{xy}) - f(\eta))$$

In particle notation

$$L = 2JL_{ind} f - \mathcal{L} f$$

Self-duality polynomials

$$D(\xi, \eta) = \prod_x d(\xi_x, \eta_x)$$

$$d(k, n) = \frac{\binom{n}{k}}{\binom{2J}{k}}$$

Self-duality relation

$$\mathbb{E}_\eta^{SEP(2J)}\left(D(\xi, \eta_t)\right) = \mathbb{E}_\xi^{SEP(2J)}\left(D(\xi_t, \eta)\right)$$

Local equilibrium measures $\rho : S \to [0,1]$

$$\nu_{\overline{\rho}} = \otimes_{x \in S} Bin(2J, \rho_x)$$

Relation between local equilibrium measures and the polynomials is as before: for $\xi = \sum_{i=1}^{n} \delta_{x_i}$

$$\int D(\xi, \eta) \nu_{\overline{\rho}}(d\eta) = \prod_{i=1}^{n} \rho(x_i)$$

As we will see later, for this process we will have once more the inequality

$$\mathbb{E}_{x_1,\ldots,x_n}^{SEP(2J)} \left(\prod_{i=1}^{n} \overline{\rho}(x_i) \right) \leq \prod_{i=1}^{n} \mathbb{E}_{x_i}^{SEP(2J)} \left(\overline{\rho}(X_i(t)) \right)$$

Liggett's comparison inequality

The function

$$f : (x_1, \ldots, x_n) \mapsto \prod_{i=1}^{n} \overline{\rho}(x_i)$$

is **positive definite**, i.e., for $\beta : S \to \mathbb{R}$ with $\sum_x |\beta(x)| < \infty$ we have

$$\sum_{x_i, x_j} f(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n)\beta(x_i)\beta(x_j) \geq 0$$

Moreover f is clearly symmetric. We then have the following inequality; if $f : S^n \to \mathbb{R}$ is positive definite and symmetric, then

$$\mathbb{E}^{SEP}_{x_1,\ldots,x_n} f(X_1(t), \ldots, X_n(t)) \leq \mathbb{E}^{IRW}_{x_1,\ldots,x_n} f(X_1(t), \ldots, X_n(t))$$

Proof

Remember

$$L_{SEP} = L_{IRW} - \mathcal{L}$$

now we show that for f positive definite and symmetric $\mathcal{L}f \leq 0$

$$\begin{aligned}
\mathcal{L}f &= \sum_{i=1}^{n}\sum_{j=1}^{n} p(x_i, x_j)(f(x^{x_i,x_j}) - f(x)) \\
&= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} p(x_i, x_j)(f(x^{x_i,x_j}) + f(x^{x_j,x_i}) - 2f(x)) \\
&= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} p(x_i, x_j) f(x_1, \ldots, z, \ldots, y, \ldots, x_n)(\delta_{x_i,z} - \delta_{x_j,z})(\delta_{x_i,y} - \delta_{x_j} \\
&\leq 0
\end{aligned}$$

Use

$$e^{tL_{SEP}}f = e^{tL_{IRW}}f + \int_0^t e^{(t-s)L_{SEP}}(-\mathcal{L})e^{sL_{IRW}}f$$

Now for f positive definite and symmetric, also

$$e^{sL_{IRW}}f(x_1,\ldots,x_n) = \sum_{y_1,\ldots,y_n}\prod_{i=1}^{n} p_s(x_i,y_i)f(y_1,\ldots,y_n)$$

is symmetric and positive definite, hence

$$e^{(t-s)L_{SEP}}(-\mathcal{L})e^{sL_{IRW}}f \leq 0$$

which is exactly what we wanted. With exactly the same argument we have

$$\mathbb{E}^{SEP(2J)}_{x_1,\ldots,x_n} f(X_1(t),\ldots,X_n(t)) \leq \mathbb{E}^{IRW(2J)}_{x_1,\ldots,x_n} f(X_1(t),\ldots,X_n(t))$$

Consequences for SEP(2J)

For the SEP: if we start with a local equilibrium measure $\nu_{\overline{\rho}}$, then $\nu_{\overline{\rho}} S_t$ satisfies

$$\int \prod_{i=1}^{n} \eta_{x_i} \nu_{\overline{\rho}} S_t(d\eta) \leq \prod_{i=1}^{n} \int \eta_{x_i} \nu_{\overline{\rho}} S_t(d\eta)$$

i.e., $\eta_t(x)$ are negatively correlated. The same holds for the $SEP(2J)$. Remark however that in that case not every product measure is a local equilibrium measure ! The more general inequality for the $SEP(2J)$ that we obtain is

$$\int \nu_{\overline{\rho}}(d\eta) \mathbb{E}_{\eta}^{SEP(2J)} D\left(\sum_{i=1}^{n} \delta_{x_i}, \eta_t\right) \leq \prod_{i=1}^{n} \int \nu_{\overline{\rho}}(d\eta) \mathbb{E}_{\eta}^{SEP(2J)} D\left(\delta_{x_i}, \eta_t\right)$$

E.g. for $SEP(2)$:

$$\mathbb{E}_{\nu_{\overline{\rho}}}^{SEP(2J)}\left(\frac{1}{2}\eta_t(x)(1-\eta_t(x))\right) \leq \left(\mathbb{E}_{\nu_{\overline{\rho}}}^{SEP(2J)}(\eta_t(x))\right)^2$$

The Symmetric Inclusion Process (SIP)

Instead of **excluding** particles to jump to the same site, we now want to **favour** particles to jump to the same site (inclusion). For $m > 0$ we define the $SIP(m)$ as the process with generator

$$L^{SIP(m)} = mL_{IRW} + 2\mathcal{L}$$

i.e.,

$$\begin{aligned} L^{SIP(m)} f(x_1, \ldots, x_n) &= m \sum_{i=1}^{n} \sum_{y} p(x_i, y) \left(f(x^{x_i, y}) - f(x) \right) \\ &+ 2 \sum_{i=1}^{n} \sum_{j=1}^{n} p(x_i, x_j) \left(f(x^{x_i, x_j}) - f(x) \right) \end{aligned}$$

In configuration space notation $\eta \in \Omega = \mathbb{N}^S$:

$$Lf(\eta) = \sum_{x,y} \eta_x (m + 2\eta_y) p(x, y) (f(\eta^{xy}) - f(\eta))$$

Invariant measures and self-duality

The invariant measure for the $SIP(m)$ are product measures of "discrete" Gamma-distributions

$$\nu_\lambda^m(k) = (1-\lambda)^{m/2} \frac{\lambda^k}{k!} \frac{\Gamma\left(\frac{m}{2}+k\right)}{\Gamma\left(\frac{m}{2}\right)}$$

with $0 \le \lambda < 1$. For $m = 2$: "geometric", for $m = 2K$, negative binomial. The polynomials for self-duality are given by

$$D_m(\xi, \eta) = \prod_x d(\xi_x, \eta_x)$$

with

$$d(k, n) = \frac{n!}{(n-k)!} \frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m}{2}+k\right)}$$

i.e,

$$\mathbb{E}_\eta^{SIP(m)} D(\xi, \eta_t) = \mathbb{E}_\xi^{SIP(m)} D(\xi_t, \eta)$$

Local equilibrium measures

For $\overline{\lambda} : S \to [0,1)$ we define, as usual

$$\nu^m_{\overline{\lambda}}(d\eta) = \otimes_{x\in S} \nu^m_{\lambda(x)}(d\eta_x)$$

Then the relation between the polynomials and these measures is, for $\xi = \sum_{i=1} \delta_{x_i}$

$$\int D(\xi,\eta)\nu^m_{\overline{\lambda}}(d\eta) = \prod_{i=1}^{n} \overline{\rho}(x_i)$$

with

$$\overline{\rho} = \frac{\overline{\lambda}}{1-\overline{\lambda}}$$

Reversed Liggett inequality

We have for all $f : S^n \to \mathbb{R}$ symmetric and positive definite,

$$\mathbb{E}^{SIP(m)}_{x_1,\ldots,x_n} f(X_1(t),\ldots,X_n(t)) \geq \mathbb{E}^{IRW(m)}_{x_1,\ldots,x_n} f(X_1(t),\ldots,X_n(t))$$

As a consequence, for a local equilibrium measure $\nu_{\overline{\lambda}}$ we have the inequality

$$\int \nu_{\overline{\rho}}(d\eta)\mathbb{E}^{SIP(m)}_{\eta} D\left(\sum_{i=1}^{n}\delta_{x_i},\eta_t\right) \geq \prod_{i=1}^{n}\int \nu_{\overline{\rho}}(d\eta)\mathbb{E}^{SIP(m)}_{\eta} D\left(\delta_{x_i},\eta_t\right)$$

in particular, $\eta_t(x)$ are **positively correlated** when started from $\nu_{\overline{\lambda}}$ initially.

This is surprising because the $SIP(m)$ is not **a monotone process**, i.e., starting at two configurations $\eta \leq \xi$ we cannot find a coupling (joint movie) where the configurations stay ordered. So the propagation of positive correlations is **not related to FKG-property** such as in ferromagnetic spin ystems (Harris' theorem).

Brownian Momentum Process

We show here one example where the correlation inequalities for the $SIP(1)$ can be transported to a diffusion process. The BMP is a diffusion process on $\mathbb{R}^S$ with generator

$$Lf(\eta) = \sum_{x,y\in S} 2p(x,y)\left(\eta_x \frac{\partial}{\partial \eta_y} - \eta_y \frac{\partial}{\partial \eta_x}\right)^2 f$$

This is a diffusion process that conserves the "energy" $\sum_x \eta_x^2$ (like the inclusion process conserves the total number of particles). The stationary measures are Gaussian product measures

$$\otimes_{x\in S} \frac{e^{-\eta_x^2/2\rho}}{\sqrt{2\pi\rho}} d\eta_x$$

Duality of BMP and SIP(1)

Consider for $\eta \in \mathbb{R}^S$, $\xi \in \mathbb{N}^S$ with $\sum_x \xi_x < \infty$

$$D(\xi, \eta) = \prod_{i \in S} d(\xi_i, \eta_i)$$

with

$$d(n, x) = \frac{x^{2n}}{(2n-1)!!}$$

then we have **duality**

$$\mathbb{E}_\eta^{BMP} D(\xi, \eta_t) = \mathbb{E}_\xi^{SIP(1)} D(\xi_t, \eta)$$

as a consequence we obtain that, starting from a local equilibrium measure,

$$\mu_{\overline{\rho}} := \otimes_{x \in S} \frac{e^{-\eta_x^2/2\overline{\rho}(x)}}{\sqrt{2\pi\overline{\rho}(x)}} d\eta_x$$

at time t we have the inequality

$$\int \mu_{\overline{\rho}}(d\eta)\mathbb{E}_\eta^{BMP} D\left(\sum_{i=1}^n \delta_{x_i}, \eta_t\right) \geq \prod_{i=1}^n \int \mu_{\overline{\rho}}(d\eta)\mathbb{E}_\eta^{BMP} D\left(\delta_{x_i}, \eta_t\right)$$

Summary

So far we have the following picture

1. SEP: local equilibrium go to measures with **negative** correlations (repulsion, fermions, $SU(2)$-symmetry).
2. SIP: local equilibrium go to measures with **positive** correlations (attraction, bosons, $SU(1,1)$-symmetry).
3. Processes via duality related to SIP: local equilibrium go to measures with **positive** correlations.
4. Also in non-equilibrium context: SIP or SEP coupled to boundary reservoirs (where particles are created and annihilated at specific rates) this holds.

We expect that this can be generalized to systems with particles of different types, exclusion/inclusion processes with birth and deaths, and slight modifications of exclusion/inclusion.