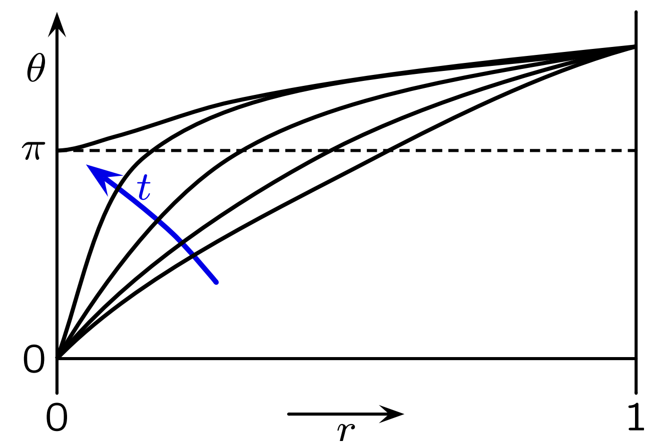
Matched asymptotics for the harmonic map heat flow

- Nematic liquid crystals
- The harmonic map heat flow
- Singularity formation (bubbling)
- Symmetric setting
- Matched asymptotic expansions
- Stability



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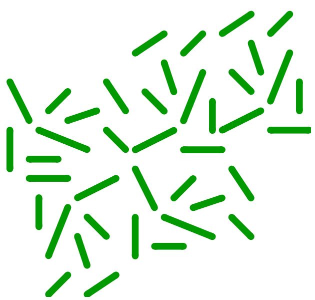
Joost Hulshof (VU Amsterdam)

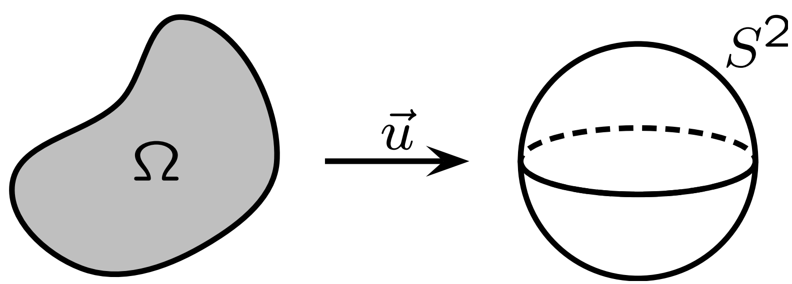
Nematic liquid crystals

For example: LCD screens, polymer fibres
A nematic liquid crystals consists of molecules that are elongated, i.e. like little rods or arrows.
In particular, they have a direction.



The molecules are pointing in direction $\vec{u}(x)$.
Normalise to length 1: $|\vec{u}(x)| = 1$

$$\Rightarrow \quad \vec{u}(x) \in S^2$$



Time dependent: $\vec{u}(x,t) \in S^2 \subset \mathbb{R}^3$

Energy

The (simplest) energy of a configuration $\vec{u}(x)$ is

$$E(\vec{u}) = \frac{1}{2}\int_\Omega |\nabla \vec{u}|^2 dx \qquad \text{where } |\nabla \vec{u}|^2 = \sum_{i,j} \left(\frac{\partial u_i}{\partial x_j}\right)^2$$

Energy is minimal when all molecules are parallel.

The stationary points are called harmonic maps.

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general maps $u : M \to N$ (Riemannian manifolds)

Dynamics: decrease the free energy as fast a possible:

$$\text{gradient flow } \vec{u}_t = -\nabla E(\vec{u})$$

This leads to the harmonic map heat equation.

Mathematical context

$$E(u) = \frac{1}{2}\int |\nabla \vec{u}|^2$$

$u : M \longrightarrow N$ Riemannian manifolds (with a metric)

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$u : M \longrightarrow N$ Riemannian manifolds (with a metric)

- $u : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}$
  Critical points: $\nabla^2 u = \Delta u = 0$ are harmonic functions
  Gradient dynamics: heat equation $u_t = \Delta u$.

- $u : \mathbb{R} \longrightarrow N$ parametrised curves
  Critical points: geodesics.

- $u : \mathbb{R}^2 \longrightarrow S^1$ difficulty in choosing function spaces
  Ginzburg-Landau functional

$$E(u) = \frac{1}{2}\int |\nabla \vec{u}|^2 + \frac{1}{4\varepsilon^2}(1 - |\vec{u}|^2)^2$$

Harmonic map heat flow

$$
\begin{aligned}
\vec{u}_t &= -dE(\vec{u}) \\
&= \Delta\vec{u} - (\Delta\vec{u}, \vec{u})\vec{u}
\end{aligned}
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Harmonic map heat flow

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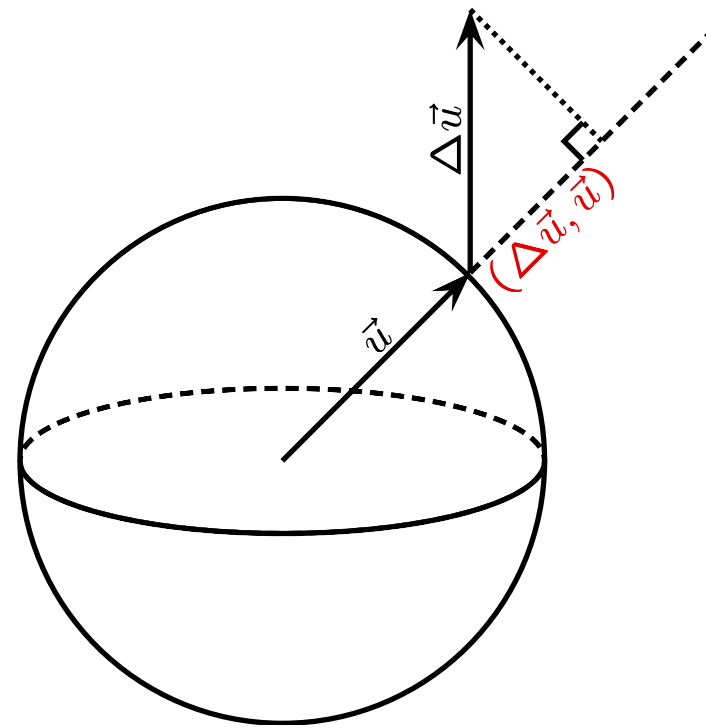
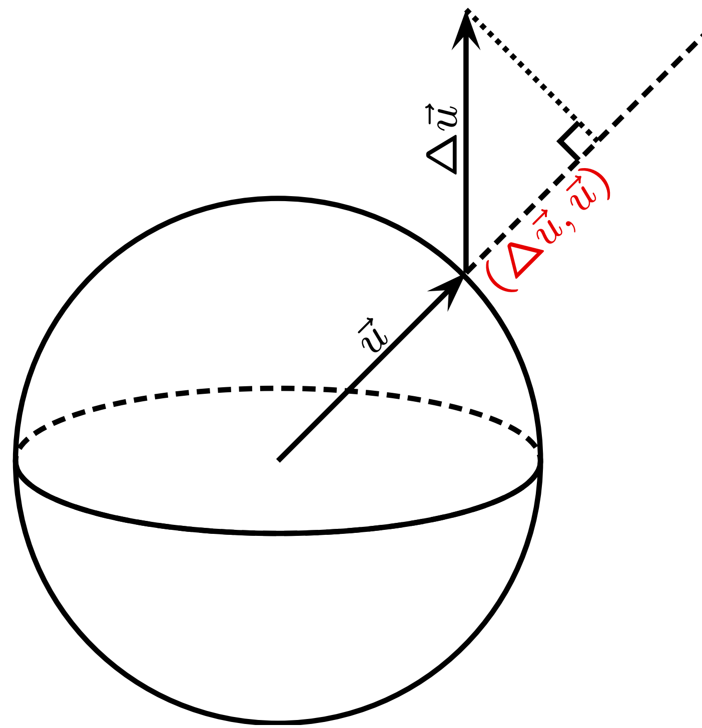


Harmonic map heat flow

$$\begin{aligned}\vec{u}_t &= -dE(\vec{u}) \\ &= \Delta\vec{u} - (\Delta\vec{u},\vec{u})\vec{u} \\ &= \Delta\vec{u} + |\nabla\vec{u}|^2\vec{u}\end{aligned}$$



$$0 = \nabla\cdot\nabla(\vec{u},\vec{u}) = \nabla\cdot 2(\nabla\vec{u},\vec{u}) = 2[(\Delta\vec{u},\vec{u}) + (\nabla\vec{u},\nabla\vec{u})]$$

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&= -\vec{u} \times (\vec{u} \times \Delta\vec{u})
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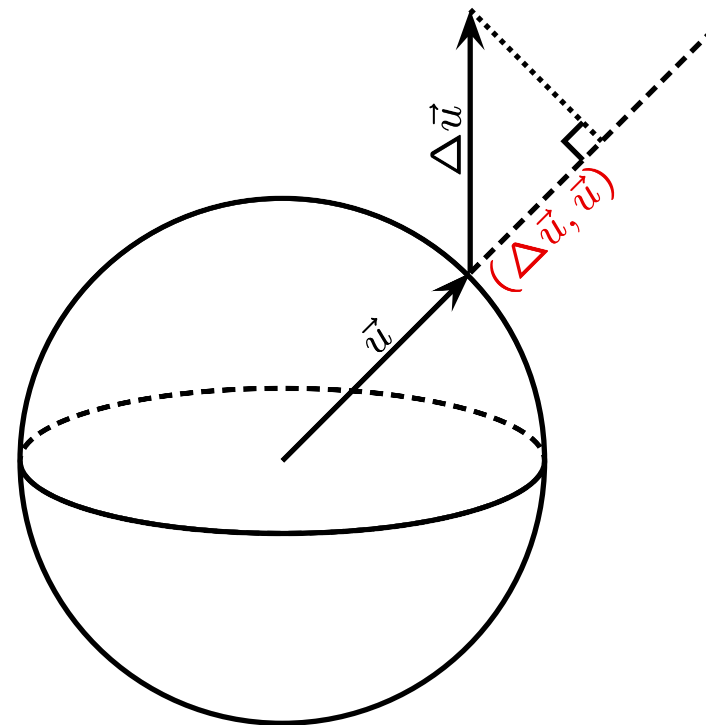
$$



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In ferromagnetism (Landau-Lifshitz equation):

$$\vec{u}_t = \alpha\,\vec{u} \times \Delta\vec{u} - \beta\,\vec{u} \times (\vec{u} \times \Delta\vec{u})$$

PDE Properties

$$\begin{cases} \vec{u}_t = \Delta \vec{u} + |\nabla \vec{u}|^2 \vec{u} & x \in \Omega, t > 0 \\ \vec{u}(x,t) = \vec{u}_1(x) & x \in \partial\Omega \text{ (boundary conditions)} \\ \vec{u}(x,0) = \vec{u}_0(x) & \text{initial conditions} \end{cases}$$

- $|\vec{u}_0(x)| = 1 \;\Rightarrow\; |\vec{u}(x,t)| = 1$ for all t
- $\frac{d}{dt} E(\vec{u}(t)) \leq 0$
- Classical solution on some maximal interval $[0, T)$
- If $T < \infty$, then $|\nabla \vec{u}| \to \infty$ as $t \uparrow T$.
- How to continue after $t = T$?

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- If $T < \infty$, then $|\nabla \vec{u}| \to \infty$ as $t \uparrow T$.
- Weak solutions exist for all time
- Unique if you require $E(t)$ non-increasing ($\Omega \subset \mathbb{R}^2$)
- $\Omega \subset \mathbb{R}^3$ is much harder: 1. too many solutions
  2. singularities have finite energy

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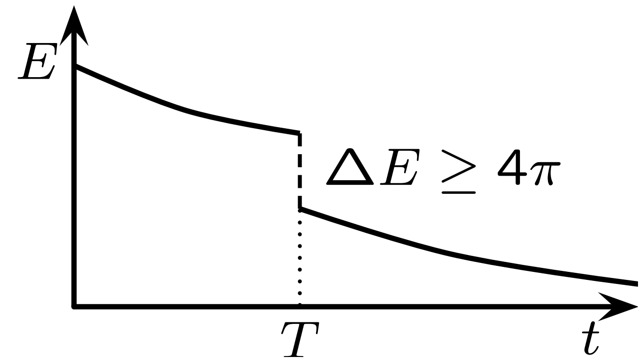
- Weak solutions exist for all time
- Unique if you require $E(t)$ non-increasing ($\Omega \subset \mathbb{R}^2$)
- Smooth except at a finite number of points (x_0, T)

[Struwe]

Singularity/blowup/bubbling

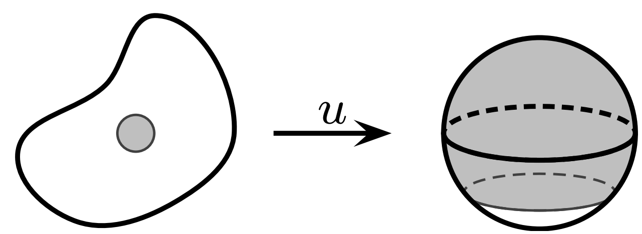
Near a singular point (x_0, T) there is a scaling factor $R(t)$

1. $R(t) \to 0$ as $t \to T$
2. $\vec{u}\left(\frac{x-x_0}{R(t)}, t\right) \to \bar{u}(x)$ as $t \to T$



where $\bar{u}$ solves $\Delta \bar{u} + |\nabla \bar{u}|^2 \bar{u} = 0$, a non-constant harmonic map.

A sphere "bubbles off"



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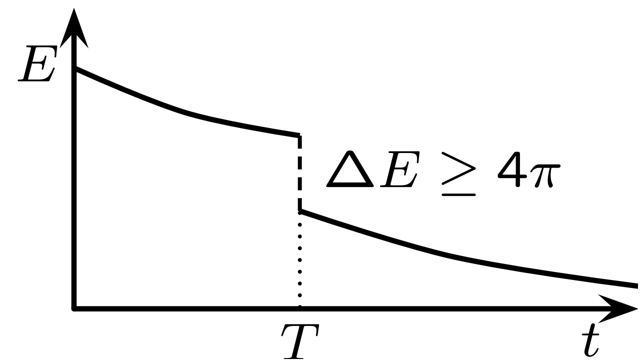

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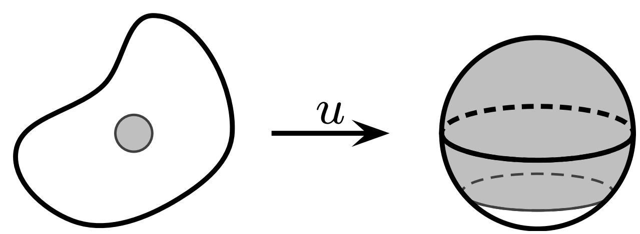
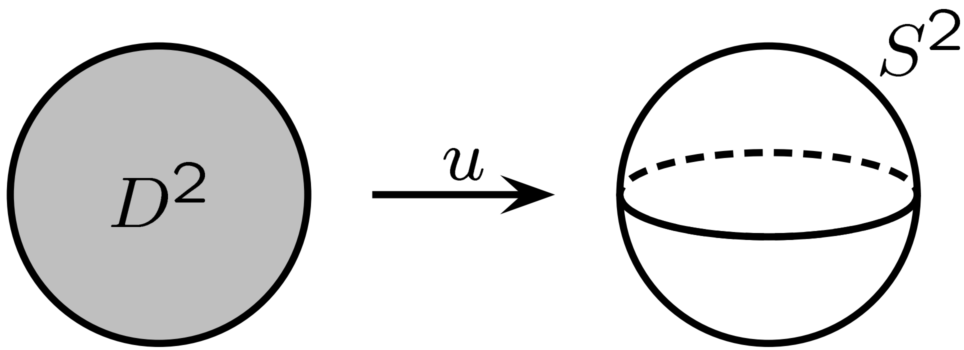


[Chang,Ding,Ye] Example where singularity occurs in finite time

Goals:
- analyse the unknown scaling factor $R(t)$.
- analyse the stability of bubbling.

Choosing coordinates

$\vec{u}_t = \Delta\vec{u} + |\nabla\vec{u}|^2\vec{u}$ harmonic map heat flow (gradient)
$\Omega = D^2 =$ unit disk (or cylinder uniform in z).



polar coordinates on D^2
spherical coordinates on S^2

$$\vec{u}(\cdot, t) : (r, \phi) \to \begin{pmatrix} \sin\theta\ \cos\psi \\ \sin\theta\ \sin\psi \\ \cos\theta \end{pmatrix}$$

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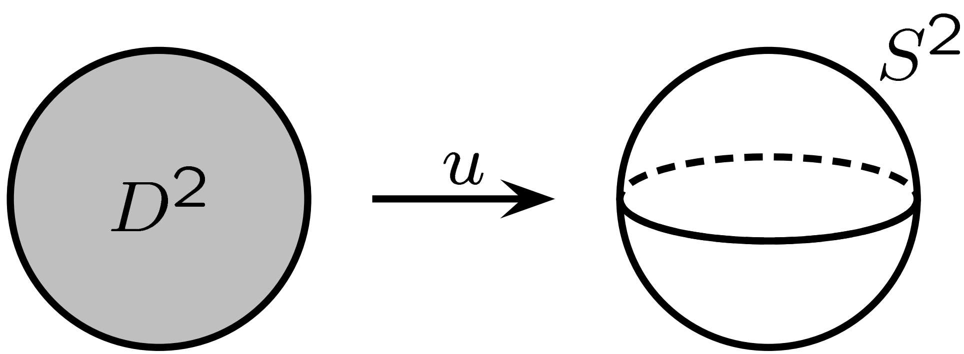

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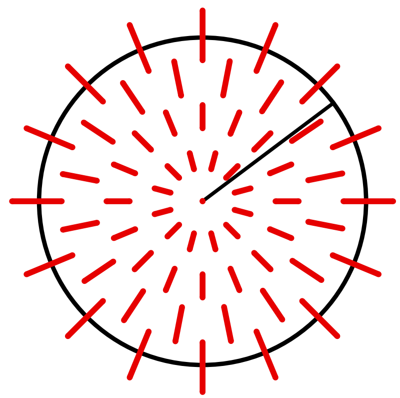
$$\vec{u}(\cdot,t) : (r,\phi) \rightarrow \begin{pmatrix} \sin\theta\,\cos\psi \\ \sin\theta\,\sin\psi \\ \cos\theta \end{pmatrix}$$

$$\begin{cases} \theta_t = \theta_{rr} + \frac{1}{r}\theta_r + \frac{1}{r^2}\theta_{\phi\phi} - \frac{\sin 2\theta}{2}(\psi_r^2 + \frac{1}{r^2}\psi_\phi^2) \\ \psi_t = \psi_{rr} + \frac{1}{r}\psi_r + \frac{1}{r^2}\psi_{\phi\phi} + \frac{\sin 2\theta}{(\sin\theta)^2}(\psi_r\theta_r + \frac{1}{r^2}\psi_\phi\theta_\phi) \end{cases}$$

Radially symmetric situation

All molecules are directed in the radial direction.

top view  side view 

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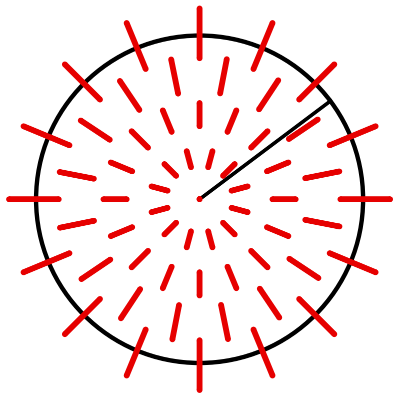
top view  side view 

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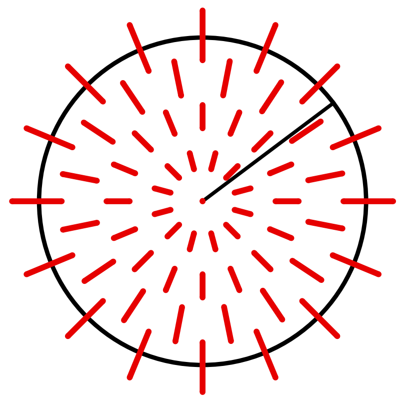
top view  side view 

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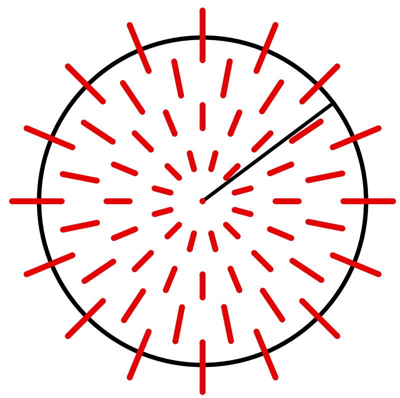
top view  side view 

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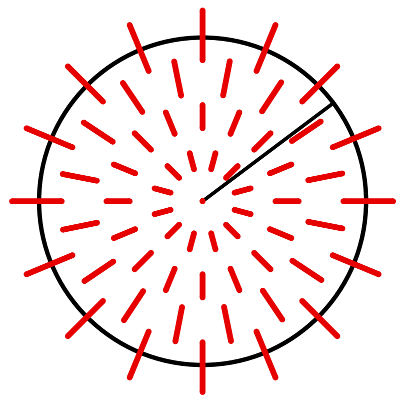
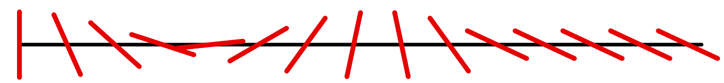
top view  side view 

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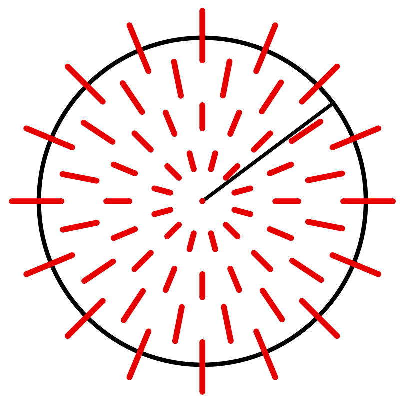
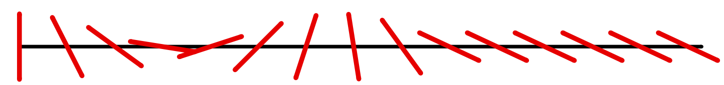
top view  side view 

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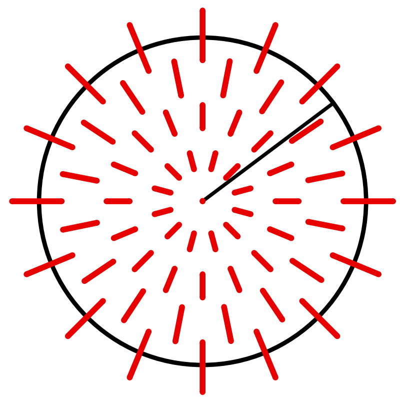
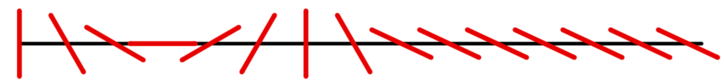
top view  side view 

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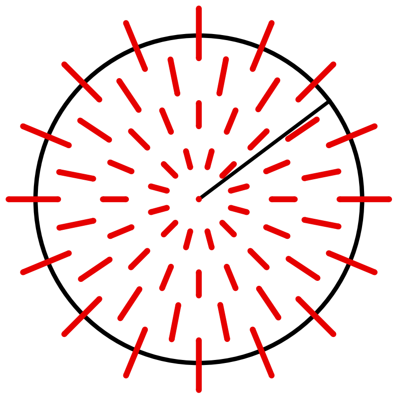
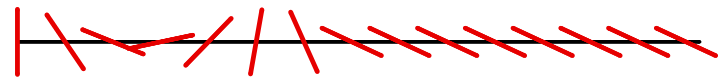
top view  side view 

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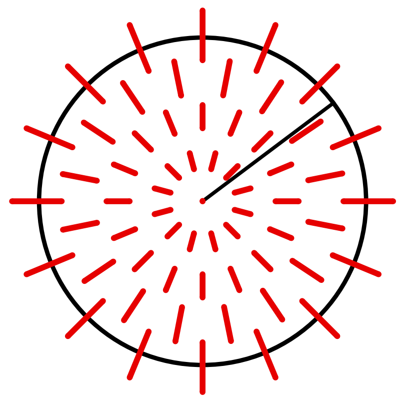
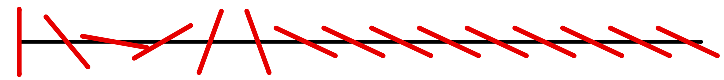
top view  side view 

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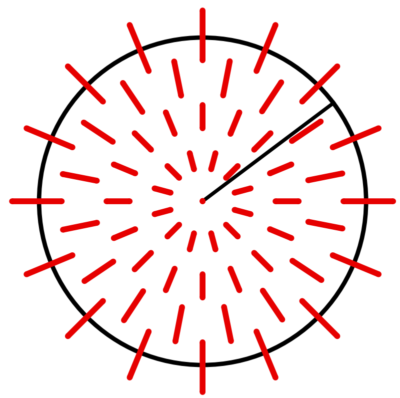
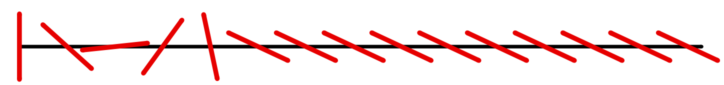
top view  side view 

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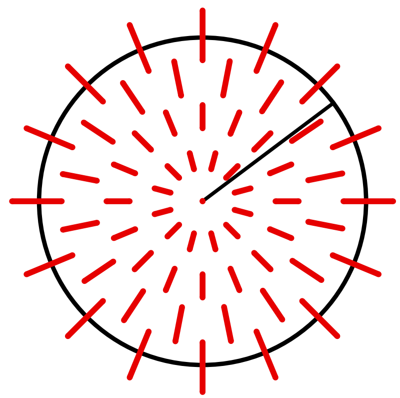
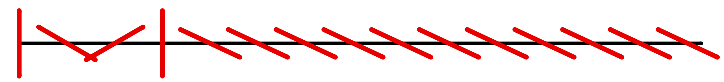
top view  side view 

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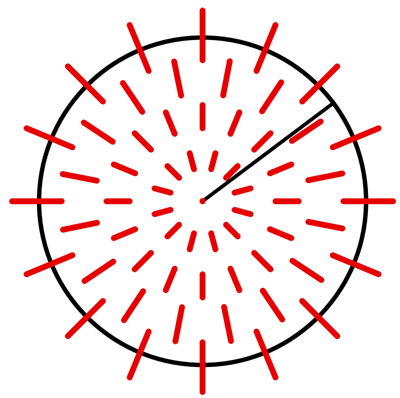
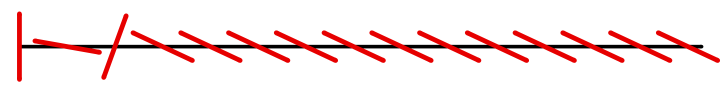
top view 

side view 

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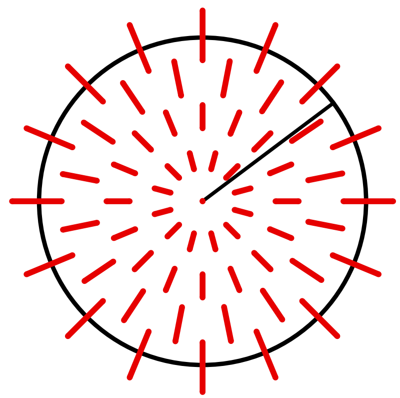
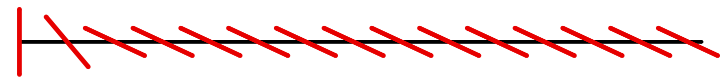
top view  side view 

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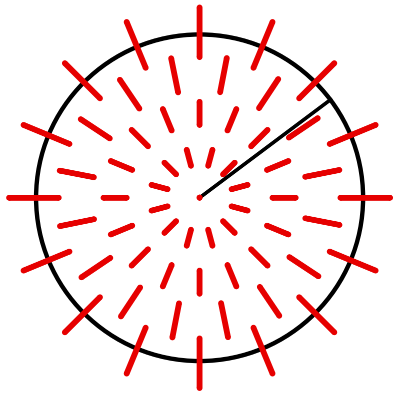
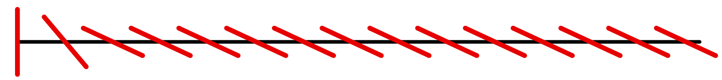
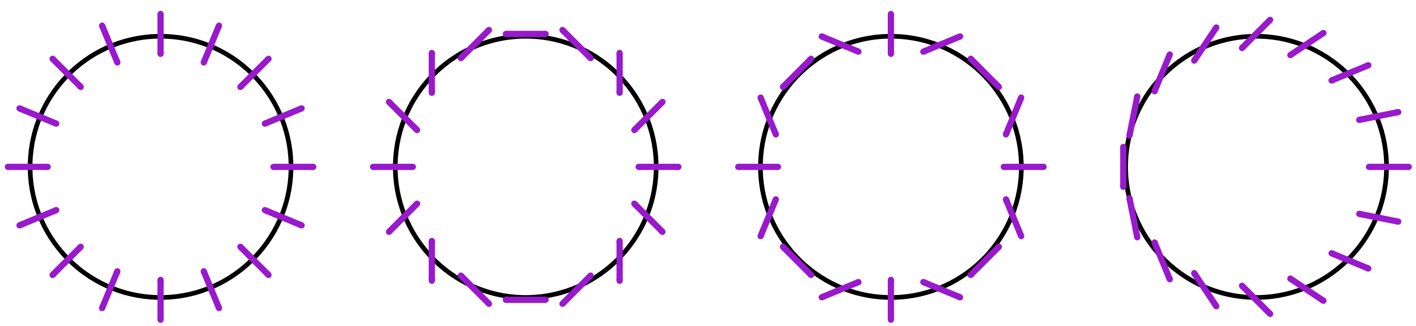
top view  side view 

Other symmetries:



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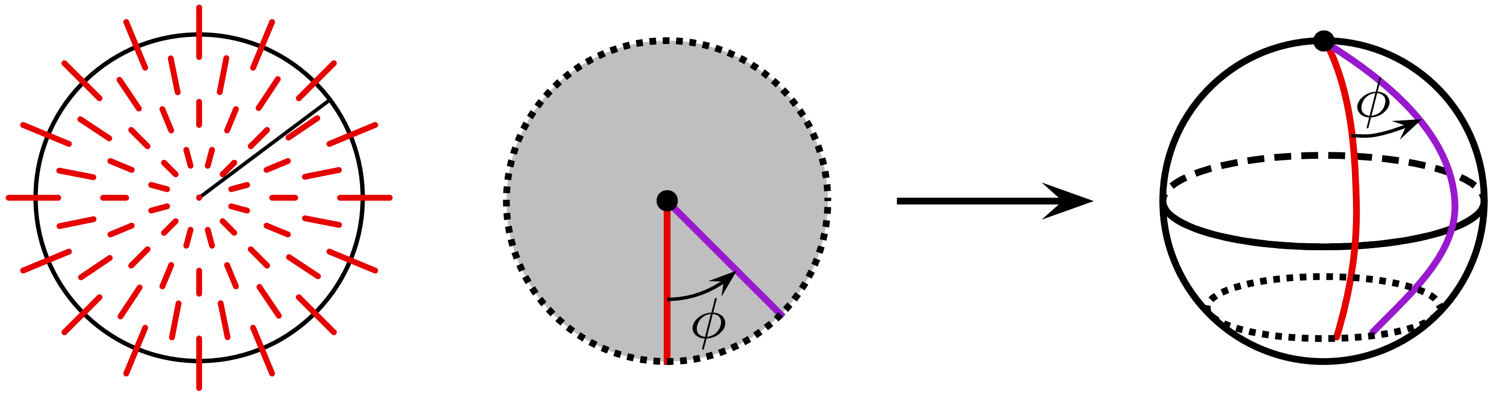

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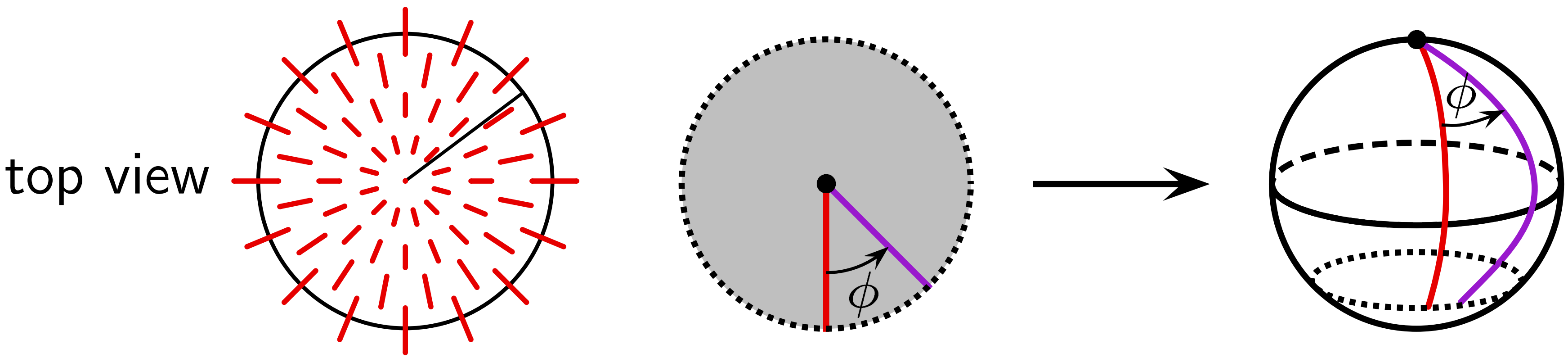
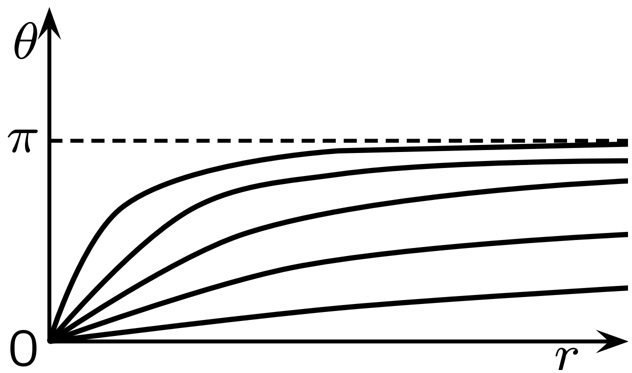


This means: $\psi(r,\phi,t)=\phi$ and $\theta(r,\phi,t)=\theta(r,t)$

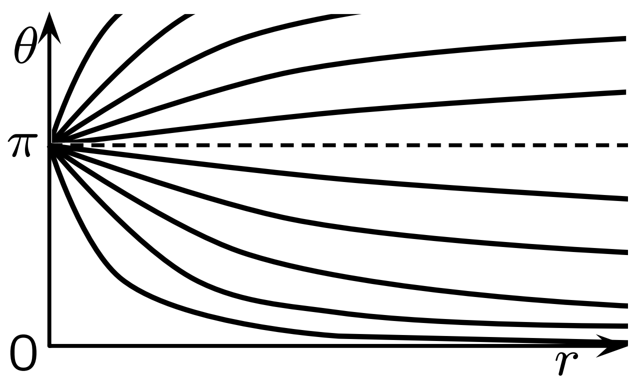
$$\begin{cases} \theta_t = \theta_{rr} + \dfrac{1}{r}\theta_r - \dfrac{\sin 2\theta}{2r^2}, \\ \theta(1,t) = \theta_1, \\ \theta(0,t) \in \pi\mathbb{Z} \qquad \text{finite energy } E = \pi \displaystyle\int_0^1 \Big(\theta_r^2 + \frac{\sin^2\theta}{r^2}\Big) r\,dr. \end{cases}$$

Equilibria (harmonic maps)

$\theta(r) = 2\arctan qr$ with $q \in \mathbb{R}$



and $m\pi + 2\arctan qr$



Scaling invariance/symmetry:

$$\theta(r,t) \;\Rightarrow\; \theta(\lambda r, \lambda^2 t)$$

Equilibria (harmonic maps) and blowup

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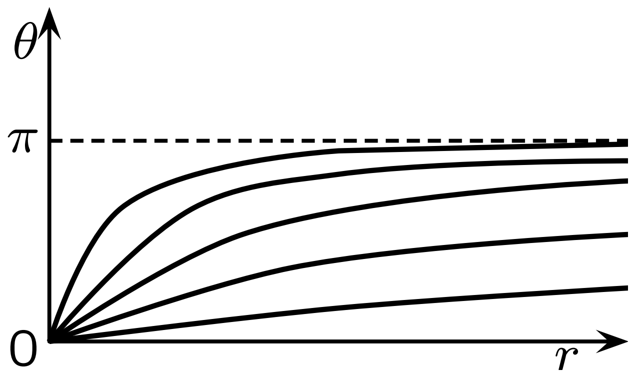

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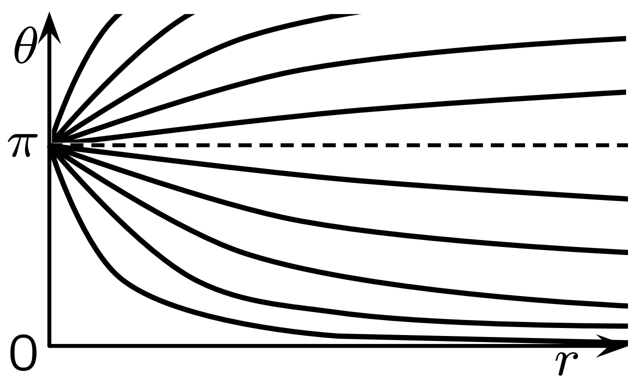

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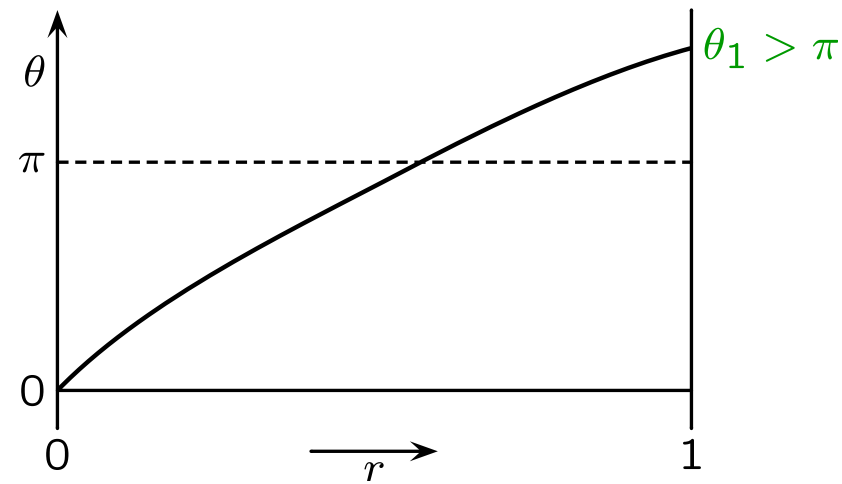
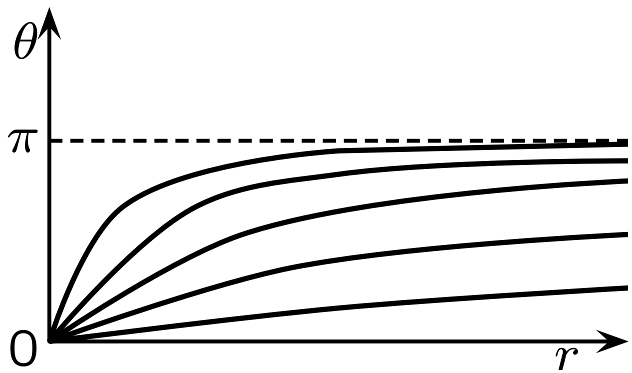


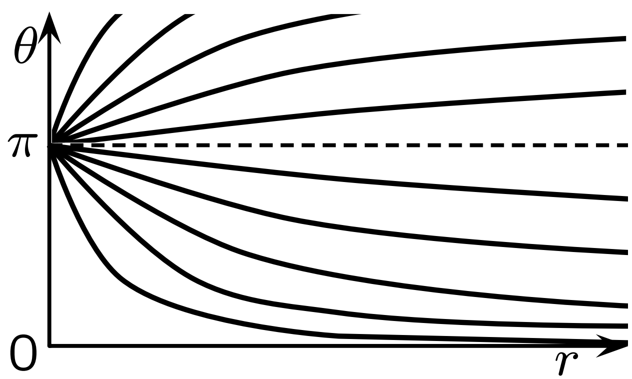
No suitable equilibrium

Equilibria (harmonic maps) and blowup

$\theta(r) = 2\arctan qr$ with $q \in \mathbb{R}$



and $m\pi + 2\arctan qr$



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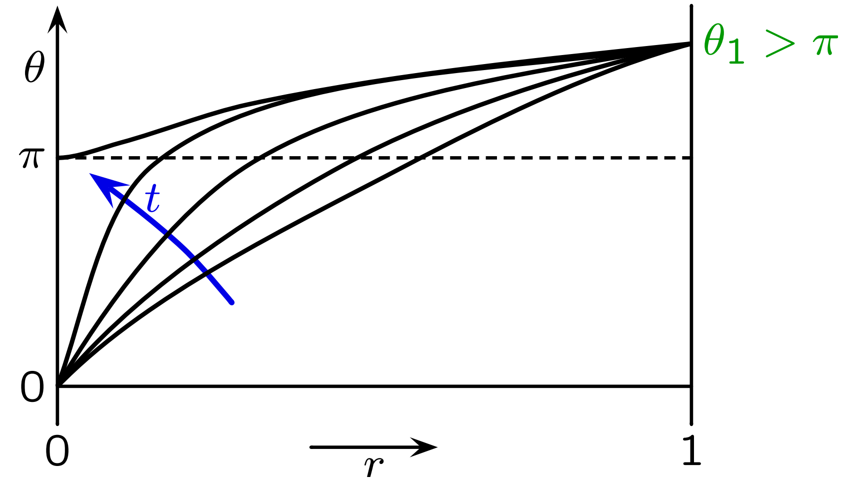


No suitable equilibrium

$\Rightarrow$ jump/singularity in $r = 0$

Scaling $R(t) \stackrel{\text{def}}{=} \frac{2}{\theta_r(0,t)} \to 0$

Scaled variables $\xi = \frac{r}{R(t)}$

Then $\theta(\xi, t) \to 2\arctan\xi$

Blowup rate not self-similar

Blowup in other equations

$u' = u^2$

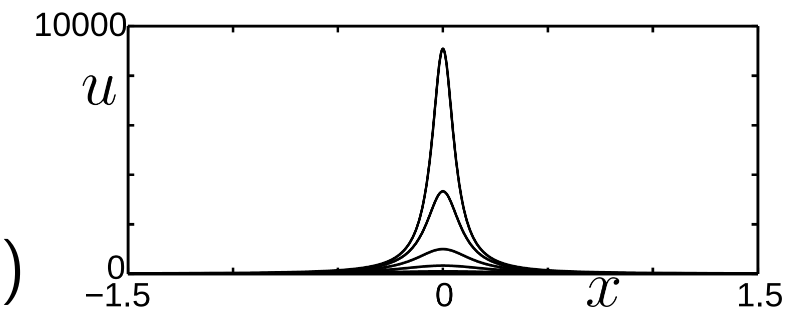
symmetry: $u(t)$ solution $\Rightarrow$ $\lambda u(\lambda t)$ solution

predicts blowup speed $u(t) = \frac{1}{T-t}$

$u_t = u_{xx} + u^3$

symmetry: $u(x,t) \longrightarrow \lambda u(\lambda x, \lambda^2 t)$

predicts blowup scales (approximately)



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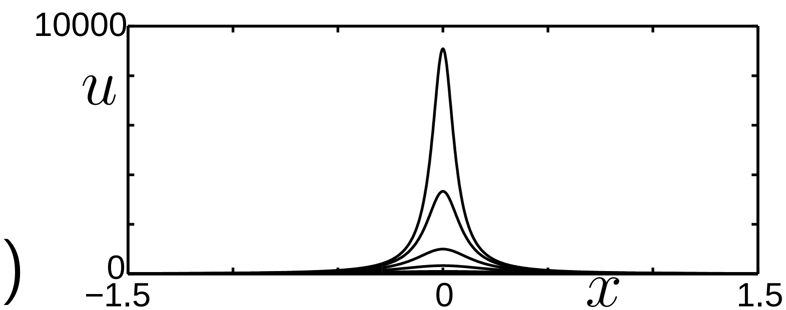


Symmetry-prediction does not work for harmonic map:

1. $|\nabla u|$ blows up
2. behaviour is "quasi-stationary"

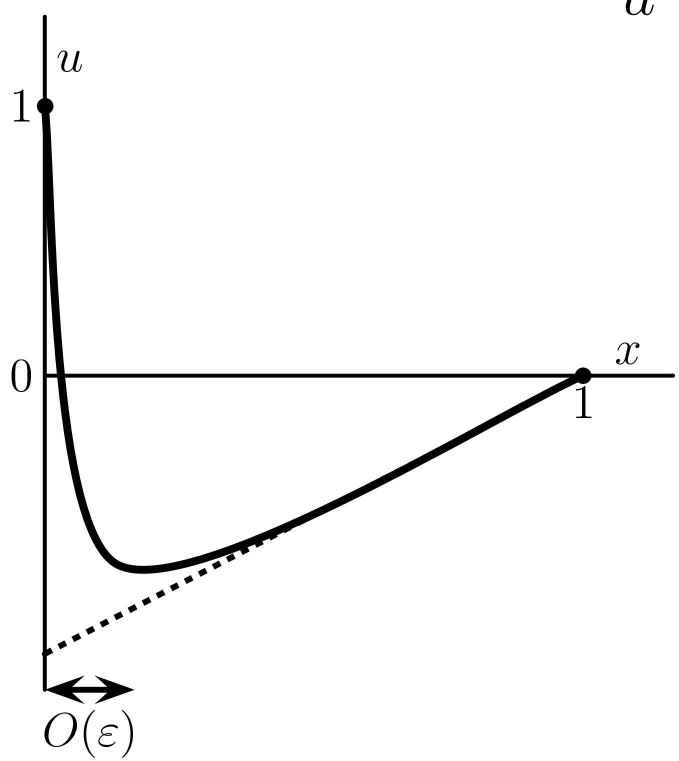
Matched asymptotics: simpler example

$$\begin{cases} \varepsilon u'' + u' = 1 \\ u(0) = 1, u(1) = 0 \end{cases} \qquad 0 < \varepsilon \ll 1$$

Outer scale $x = O(1)$: $u' \approx 1 \;\Rightarrow\; u(x) \approx x - 1$

Inner scale $x = O(\varepsilon)$: $y = x/\varepsilon$, $\hat{u}(y) = u(x)$

$$\hat{u}'' + \hat{u}' = \varepsilon \;\Rightarrow\; \hat{u}(y) \approx C + (1 - C)e^{-y}$$



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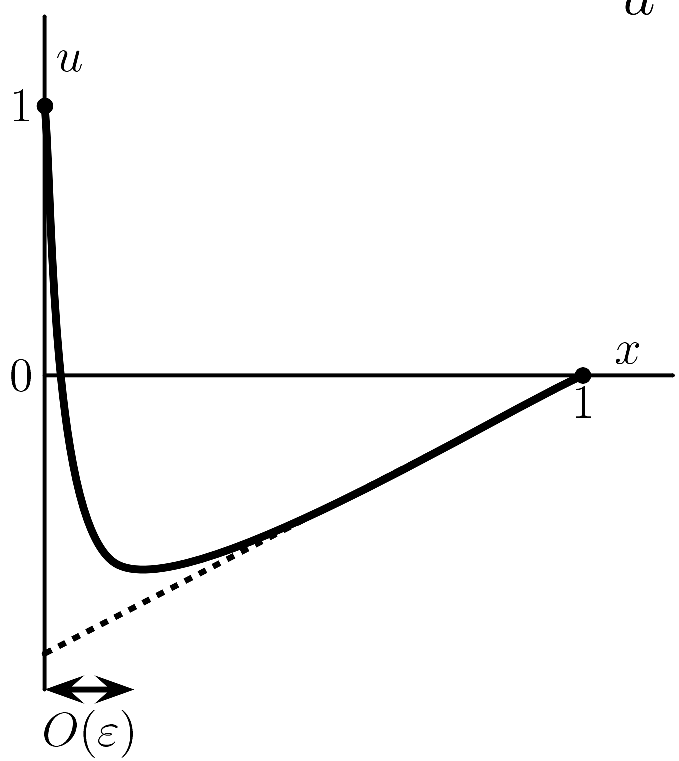

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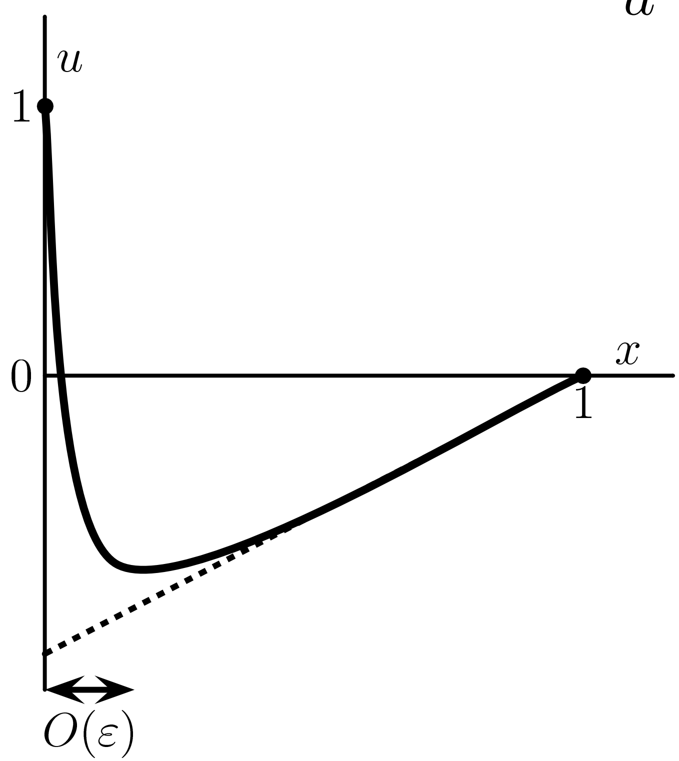

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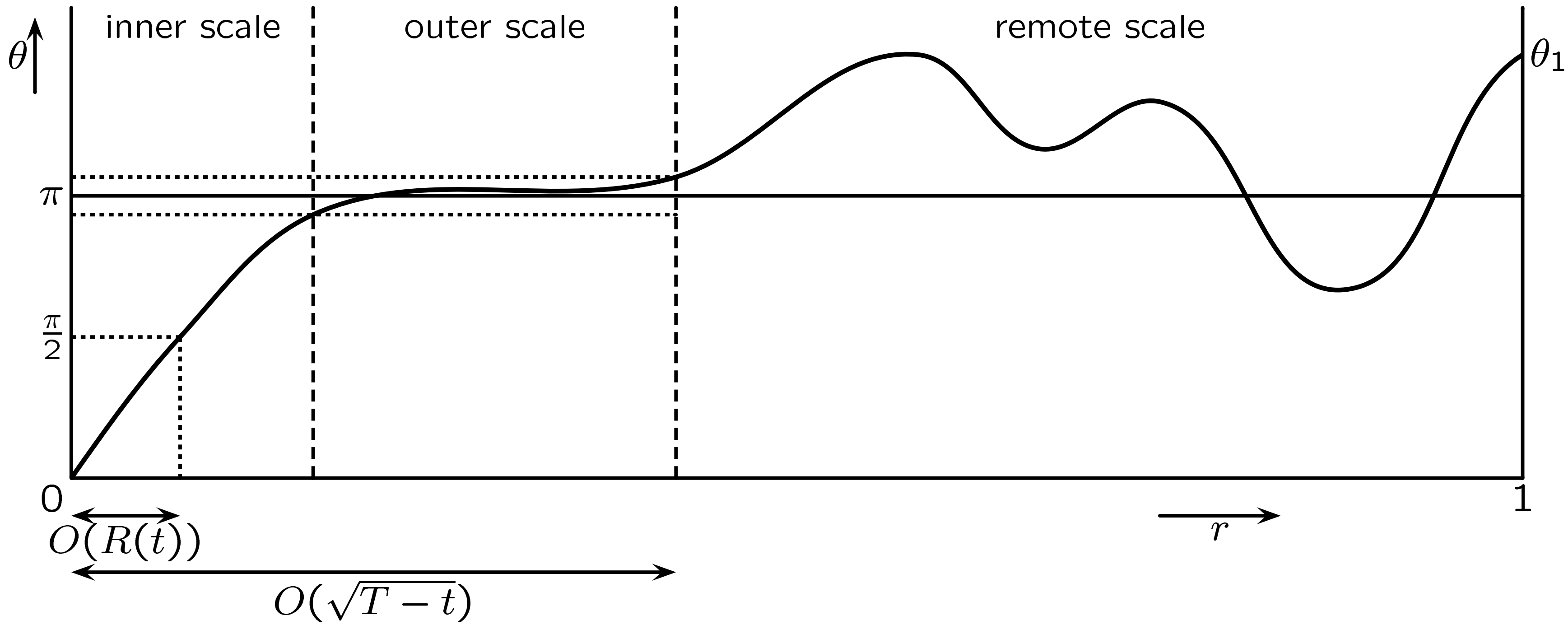
$$\theta_t = \theta_{rr} + \frac{1}{r}\theta_r - \frac{\sin 2\theta}{2r^2}$$

Matched asymptotics

We need three scales to calculate $R(t)$



inner:	$r = O(R(t))$	$\xi = \frac{r}{R(t)}$	$\theta \sim 2 \arctan \xi + \ldots$
outer:	$r = O(\sqrt{T-t})$	$y = \frac{r}{\sqrt{T-t}}$	$\theta \sim \pi + \ldots$
remote:	$r = O(1)$		$\theta \sim \theta(r, T) + \ldots.$

Inner approximation

New equation in $\xi = \frac{r}{R(t)}$ with $R(t) \to 0$:

$$R^2\theta_t - R'R\xi\theta_\xi = \theta_{\xi\xi} + \frac{1}{\xi}\theta_\xi - \frac{\sin 2\theta}{2\xi^2}$$

Expand $\theta = \theta_0 + RR'\,\theta_1 + (RR')^2\,\theta_2 + \dots$

θ_0 solves $\theta_{0\xi\xi} + \frac{1}{\xi}\theta_{0\xi} - \frac{\sin 2\theta_0}{2\xi^2} = 0.$

θ_1 solves $\theta_{1\xi\xi} + \frac{1}{\xi}\theta_{1\xi} - \frac{\cos 2\theta_0}{\xi^2}\theta_1 = -\xi\theta_{0\xi}.$

$$\theta(\xi,t) \sim \pi - 2\xi^{-1} + R'(t)R(t)(-\xi\ln\xi + \xi) + \dots$$

as $\xi \to \infty$ and $t \uparrow T$.

Matching

outer variables: $y = \frac{r}{\sqrt{T-t}}$, $\quad \tau = -\ln(T-t) \to \infty$

$$\theta_{\mathsf{outer}}(y,\tau) \sim \pi + e^{-\tau/2}[\sigma(\tau)y + \sigma'(\tau)(4y^{-1} - 2y\ln y) + \ldots] + \ldots$$

with $\sigma(\tau)$ unknown.

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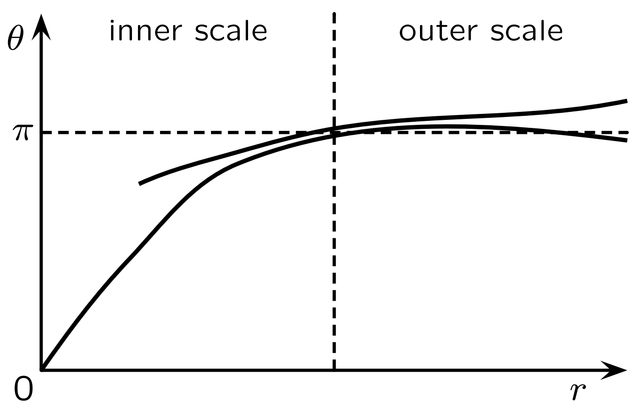
$$\theta_{\text{inner}}(y,\tau) \sim \pi - 2e^{\tau/2}Ry^{-1} + e^{\tau/2}R'y(-\ln y + \ln R + \tau/2 + 1) + \ldots$$



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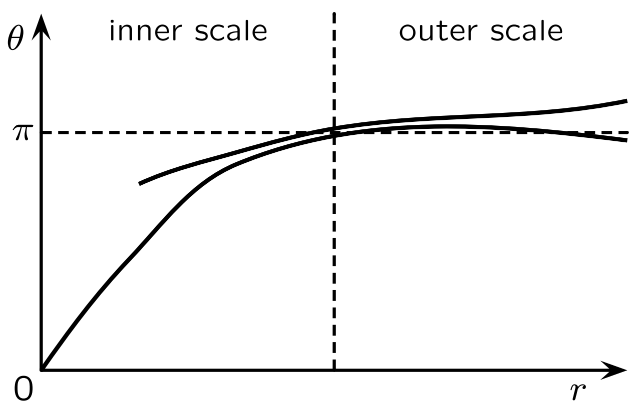

Comparison of the coefficients gives:

$$y^{-1} \;:\; 4e^{-\tau/2}\sigma' \sim -2Re^{\tau/2}$$

$$y \;:\; e^{-\tau/2}\sigma \sim R'e^{\tau/2}(\ln R + \tau/2 + 1)$$

Result

Zoom/scaling factor $R(t)$:

$$R(t) \sim \kappa \frac{T-t}{|\ln(T-t)|^2} \qquad \text{as } t \uparrow T. \qquad \ll \sqrt{T-t}$$

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Proofs [Angenent, Hulshof, Matano]

- $R(t) = o(T-t)$ as $t \uparrow T$.

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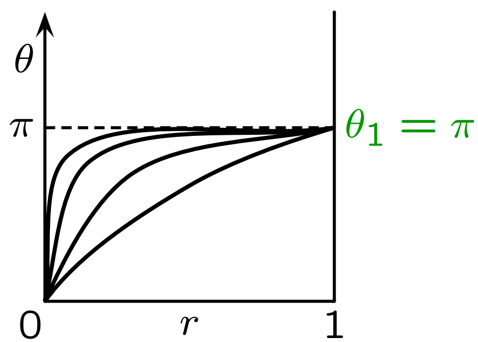
- 

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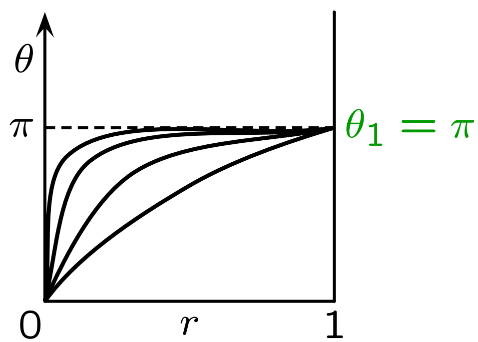
- 

  $R(t) = e^{-2\sqrt{t}+o(\sqrt{t})}$ as $t \to \infty$.
- Partial results: general case is open.

$$\theta_t = \theta_{rr} + \tfrac{1}{r}\theta_r - \tfrac{\sin 2\theta}{2r^2}$$

Stability : Equivariance

High symmetry $\Rightarrow$ Topological obstruction $\Rightarrow$ Blowup

How about non-symmetric perturbations?

$$\begin{cases} \theta_t &= \theta_{rr} + \frac{1}{r}\theta_r + \frac{1}{r^2}\theta_{\phi\phi} - \frac{\sin 2\theta}{2}(\psi_r^2 + \frac{1}{r^2}\psi_\phi^2) \\ \psi_t &= \psi_{rr} + \frac{1}{r}\psi_r + \frac{1}{r^2}\psi_{\phi\phi} + \frac{\sin 2\theta}{(\sin\theta)^2}(\psi_r\theta_r + \frac{1}{r^2}\psi_\phi\theta_\phi) \end{cases}$$

Stability : Equivariance

High symmetry $\Rightarrow$ Topological obstruction $\Rightarrow$ Blowup

Equivariant: $\theta = \theta(r,t)$ and $\psi = \phi + \chi(r,t)$

$$\begin{cases} \theta_t &= \theta_{rr} + \frac{1}{r}\theta_r - \frac{\sin 2\theta}{2}(\chi_r^2 + \frac{1}{r^2}) \\ \chi_t &= \chi_{rr} + \frac{1}{r}\chi_r + \frac{\sin 2\theta}{(\sin\theta)^2}\chi_r\theta_r \end{cases}$$

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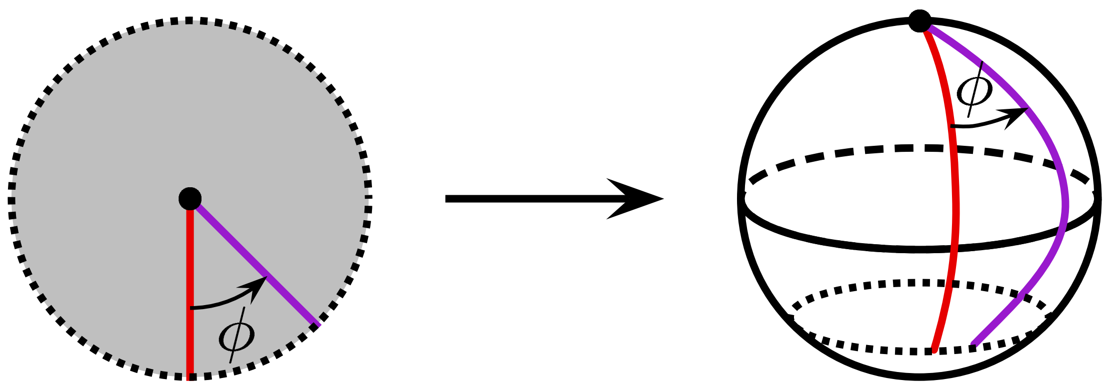

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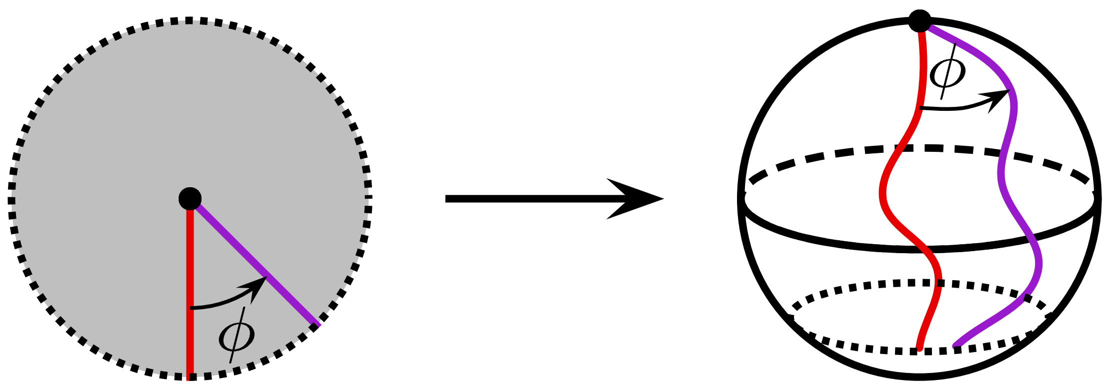

We only need to consider one radius

Matched asymptotics again

inner: $$\begin{cases} \theta = 2\arctan\xi + R'(t)R(t)\ldots \\ \chi = \zeta(t) + R(t)^2\zeta'(t)\ldots \end{cases}$$

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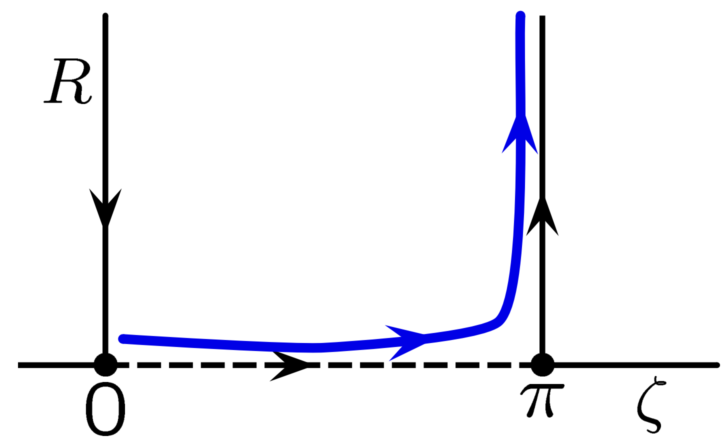
The singularity is a saddle point



Numerics

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Conclusion: instability

Radially symmetric blowup in the harmonic map heat flow is co-dimension 1 unstable under equivariant perturbations

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Other results

- Instability dynamics: fast rotation of sphere over 180°
- Same result for Landau-Lifshitz

  $$\vec{u}_t = \alpha\,\vec{u} \times \Delta\vec{u} - \beta\,\vec{u} \times (\vec{u} \times \Delta\vec{u})$$

  No radially symmetric case, but an equivariant one
- Hints for continuation after bubbling:
  immediately reattach sphere rotated by 180°