Some kinetic models of swarming

J. A. Carrillo,

in collaboration with J. A. Cañizo and J. Rosado (UAB), to appear in M3AS.

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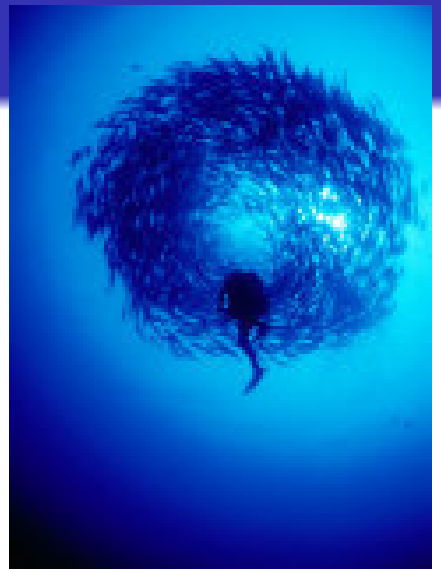
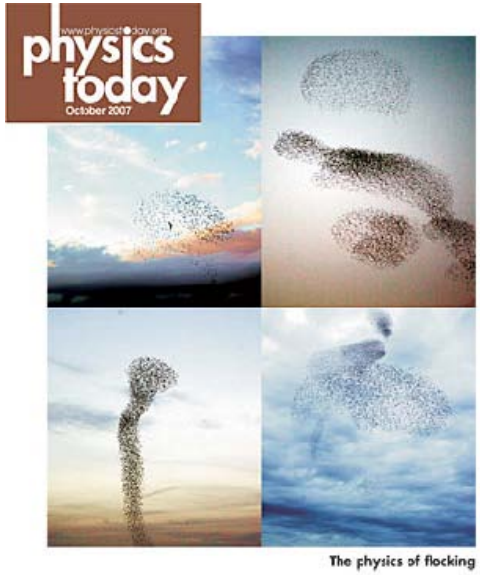
Eindhoven, The Netherlands, 2010

Outline

1. Some Swarming Models
   - Particle models.
   - Kinetic Models and measure solutions.
2. Qualitative Properties
   - Model with asymptotic velocity
   - Cucker-Smale Kinetic model
   - Variations
3. Conclusions

Swarming





Fish schools and Birds flocks.

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Types of interaction

Swarming = Aggregation of agents of similar size and body type generally moving in a coordinated way.

Highly developed social organization: insects (locusts, ants, bees ...), fishes, birds, micro-organisms (myxo-bacteria, ...) and artificial robots for unmanned vehicle operation.

Interaction regions between individuals[a]

[a] Barbaro, Birnir et al. (2008).

- Repulsion Region: R_k.
- Attraction Region: A_k.
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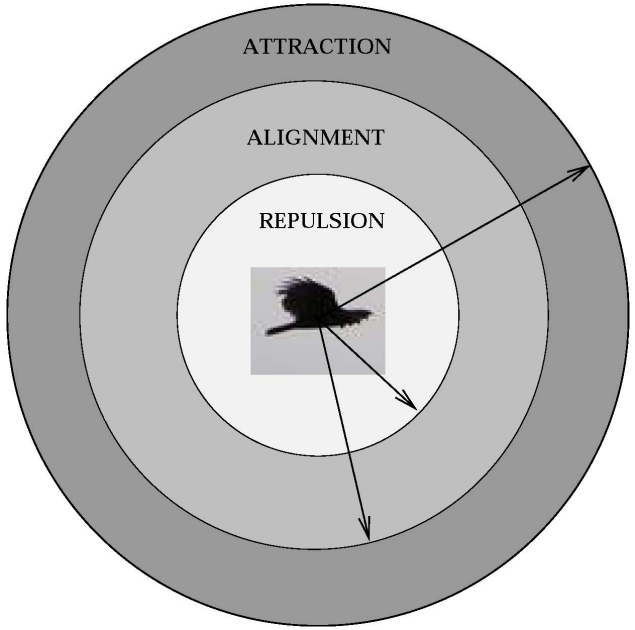

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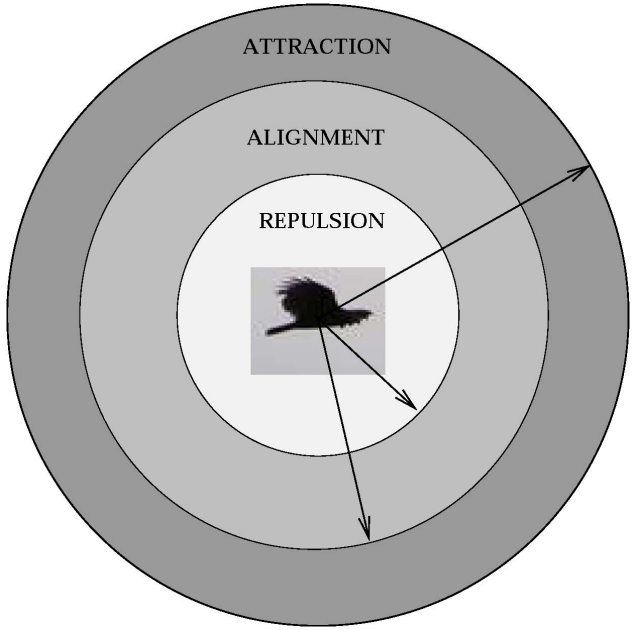

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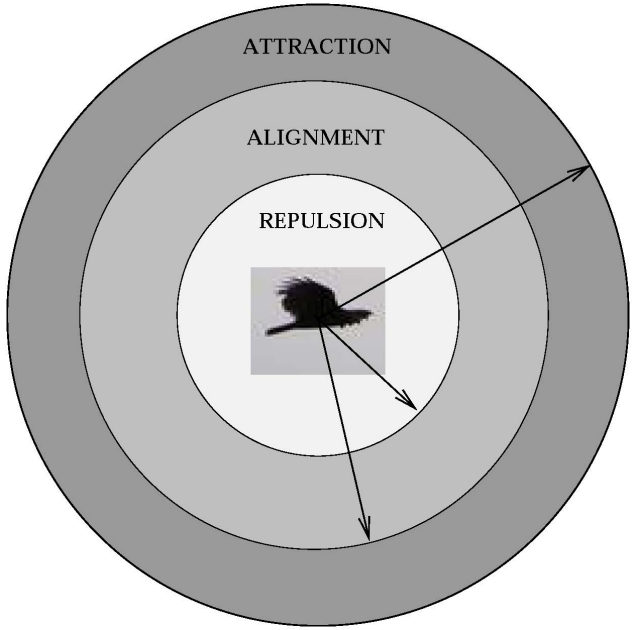

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Model with an asymptotic velocity

D'Orsogna, Bertozzi et al. model (PRL 2006):

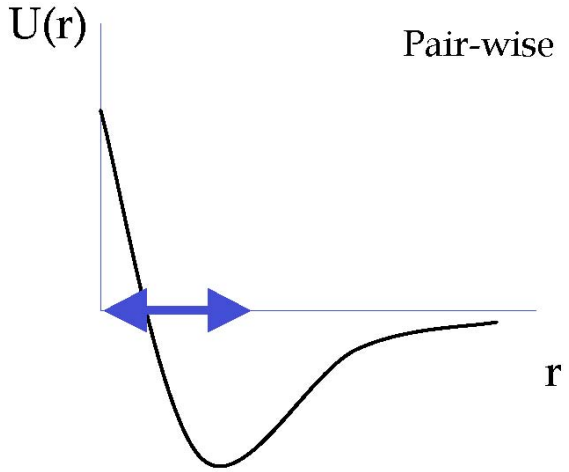
$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \\ \dfrac{dv_i}{dt} = (\alpha - \beta\,|v_i|^2)v_i - \displaystyle\sum_{j\neq i} \nabla U(|x_i - x_j|). \end{cases}$$

Model assumptions:

- Self-propulsion and friction terms determines an asymptotic speed of $\sqrt{\alpha/\beta}$.
- Attraction/Repulsion modeled by an effective pairwise potential $U(x)$.

$$U(r) = -C_A e^{-r/\ell_A} + C_R e^{-r/\ell_R}.$$

$C = C_R/C_A > 1$, $\ell = \ell_R/\ell_A < 1$ and $C\ell^2 < 1$:



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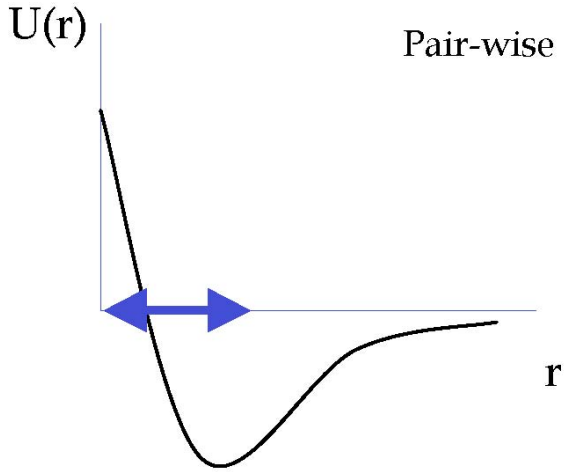

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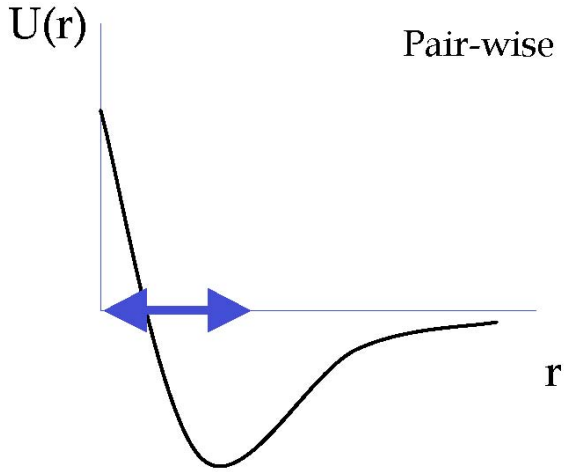
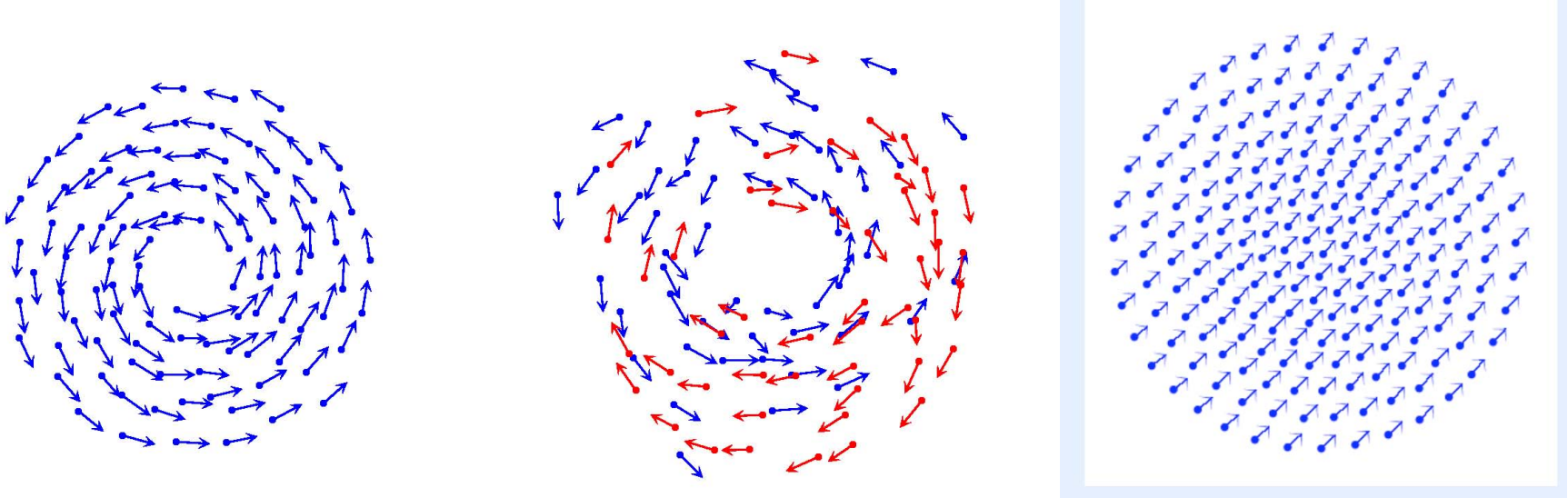


Model with an asymptotic velocity

Typical patterns: milling, double milling or flocking.



Double milling patterns: Carrillo, D'Orsogna, Panferov (2009).

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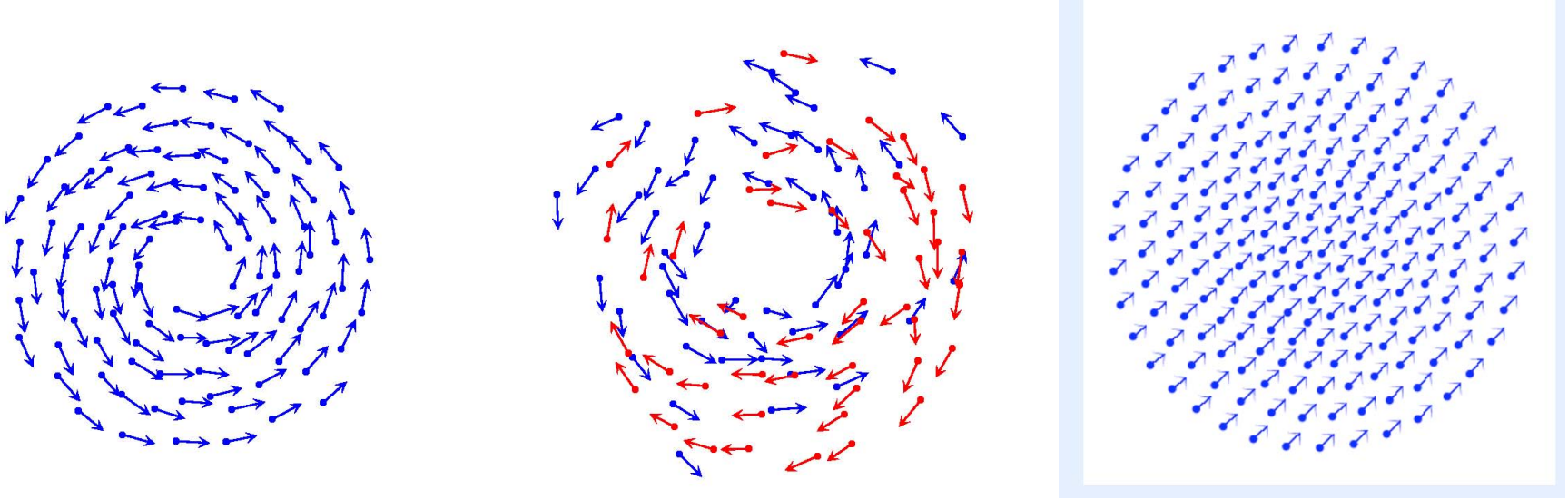

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Velocity consensus model

Cucker-Smale Model (IEEE Automatic Control 2007):

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \\ \dfrac{dv_i}{dt} = \displaystyle\sum_{j=1}^{N} a_{ij}\left(v_j - v_i\right), \end{cases}$$

with the communication rate, $\gamma \geq 0$:

$$a_{ij} = a(|x_i - x_j|) = \frac{1}{(1 + |x_i - x_j|^2)^{\gamma}}.$$

Unconditional flocking: $\gamma \leq 1/2$; Ha-Tadmor, Ha-Liu, Carrillo-Fornasier-Toscani-Rosado.
Conditional flocking: $\gamma > 1/2$.

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Mesoscopic models

Model with asymptotic velocity + Attraction/Repulsion:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + \mathrm{div}_v[(\alpha - \beta|v|^2)vf] - \mathrm{div}_v\left[(\nabla_x U \star \rho)f\right] = 0.$$

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$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \nabla_v \cdot \Big[\underbrace{\left(\int_{\mathbb{R}^{2d}} \frac{v - w}{(1 + |x - y|^2)^\gamma} f(y, w, t)\, dy\, dw \right)}_{:=\xi(f)(x,v,t)} f(x, v, t) \Big]$$

Orientation, Attraction and Repulsion:

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Definition of the distance[1]

Transporting measures:

Given $T : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ mesurable, we say that $\nu = T\#\mu$, if $\nu[K] := \mu[T^{-1}(K)]$ for all mesurable sets $K \subset \mathbb{R}^d$, equivalently

$$\int_{\mathbb{R}^d} \varphi \, d\nu = \int_{\mathbb{R}^d} (\varphi \circ T) \, d\mu$$

for all $\varphi \in C_o(\mathbb{R}^d)$.

Random variables:

Say that X is a random variable with law given by μ, is to say $X : (\Omega, \mathcal{A}, P) \longrightarrow (\mathbb{R}^d, \mathcal{B}_d)$ is a mesurable map such that $X\#P = \mu$, i.e.,

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Definition of the distance

Kantorovich-Rubinstein-Wasserstein Distance:

$$W_1(\mu,\nu) = \inf_\pi \left\{ \iint_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|\, d\pi(x,y) \right\} = \inf_{(X,Y)} \left\{ \mathbb{E}\left[|X-Y|\right] \right\}$$

where the transference plan π runs over the set of joint probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ with marginals f and $g \in \mathcal{P}_1(\mathbb{R}^d)$ and (X,Y) are all possible couples of random variables with μ and ν as respective laws.

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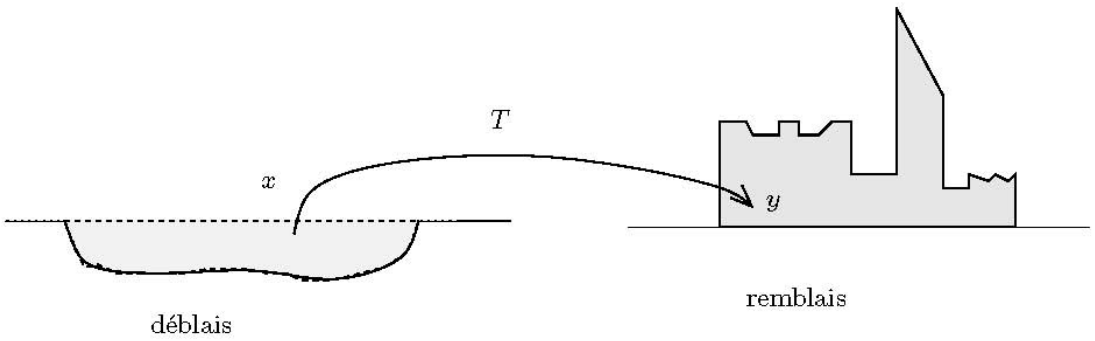
Definition of the distance

Monge's optimal mass transport problem:

Find

$$I := \inf_T \left\{ \int_{\mathbb{R}^d} |x - T(x)|\, d\mu(x);\ \nu = T\#\mu \right\}.$$

Take $\gamma_T = (1_{\mathbb{R}^d} \times T)\#\mu$ as transference plan π.



1-Wasserstein Distance

Basic Properties

1. **Distance to a Dirac:** $f \in \mathcal{P}_1(\mathbb{R}^d)$ and $a \in \mathbb{R}^d$, then

$$W_1(f, \delta_a) = \int_{\mathbb{R}^d} |x - a| df(x).$$

2. **Translation:** $f \in \mathcal{P}_1(\mathbb{R}^d)$ and $a \in \mathbb{R}^d$, let f_a denotes the translated f with vector a, then

$$W_1(f_a, f) = |a|.$$

3. **Convergence of measures:** $W_1(f_n, f) \to 0$ is equivalent to $f_n \rightharpoonup f$ weakly-* as measures and convergence of first moments.

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Well-posedness in probability measures[2]

Existence, uniqueness and stability

Take a potential $U \in \mathcal{C}_b^2(\mathbb{R}^d)$, and f_0 a measure on $\mathbb{R}^d \times \mathbb{R}^d$ with compact support. There exists a solution $f \in \mathcal{C}([0,+\infty);\mathcal{P}_1(\mathbb{R}^d))$ in the sense of solving the equation through the characteristics: $f_t := P^t \# f_0$ with P^t the flow map associated to the equation.

Moreover, the solutions remains compactly supported for all time with a possibly growing in time support.

Moreover, given any two solutions f and g with initial data f_0 and g_0, there is an increasing function depending on the size of the support of the solutions and the parameters, such that

$$W_1(f_t, g_t) \leq \alpha(t)\, W_1(f_0, g_0)$$

[2] Dobrushin-Hepp-Neunzert, 1977-79 for the Vlasov.

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[2] Dobrushin-Hepp-Neunzert, 1977-79 for the Vlasov.

Convergence of the particle method

- Empirical measures: if $x_i, v_i : [0,T) \to \mathbb{R}^d$, for $i = 1, \dots, N$, is a solution to the ODE system,

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \\ \dfrac{dv_i}{dt} = \overbrace{(\alpha - \beta\,|v_i|^2)v_i}^{\text{propulsion-friction}} - \overbrace{\sum_{j\neq i} m_j \nabla U(|x_i - x_j|)}^{\text{attraction-repulsion}} + \overbrace{\sum_{j=1}^{N} m_j a_{ij}\,(v_j - v_i)}^{\text{orientation}}. \end{cases}$$

  then the $f : [0,T) \to \mathcal{P}_1(\mathbb{R}^d)$ given by

$$f_N(t) := \sum_{i=1}^{N} m_i \delta_{(x_i(t), v_i(t))}$$

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1. Some Swarming Models
   - Particle models.
   - Kinetic Models and measure solutions.
2. Qualitative Properties
   - Model with asymptotic velocity
   - Cucker-Smale Kinetic model
   - Variations
3. Conclusions

Macroscopic equations

Monokinetic Solutions

Assuming that there is a deterministic velocity for each position and time, $f(x,v,t) = \rho(x,t)\,\delta(v - u(x,t))$ is a distributional solution if and only if,

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \mathrm{div}_x(\rho u) = 0, \\ \rho\,\dfrac{\partial u}{\partial t} + \rho\,(u\cdot\nabla_x)u = \rho\,(\alpha - \beta|u|^2)u - \rho\,(\nabla_x U \star \rho). \end{cases}$$

Superposition of Monokinetic Solutions

$f(x,v,t) = \rho_1(x,t)\,\delta(v - u_1(x,t)) + \rho_2(x,t)\,\delta(v - u_2(x,t))$ is a distributional solution if and only if

$$\begin{cases} \dfrac{\partial(\rho_1+\rho_2)}{\partial t} + \mathrm{div}_x(\rho_1 u_1 + \rho_2 u_2) = 0. \\ \displaystyle\sum_{i=1}^{2} \rho_i \left[\dfrac{\partial u_i}{\partial t} + (u_i\cdot\nabla_x)u_i - (\alpha - \beta|u_i|^2)u_i\right] = -(\nabla_x U \star \rho)\,\rho. \end{cases}$$

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Particular solutions

Let us look for stationary solutions with an asymptotic speed value $\beta|u(x,t)|^2 = \alpha$.

Flocking

Traveling wave case, $u = \text{const}$ such that $\beta|\mathbf{u}(\mathbf{x},t)|^2 = \alpha$, then $\rho(x,t) = \tilde{\rho}(x - ut)$, and the density is determined by

$$\tilde{\rho}\,(\nabla_{\mathbf{x}} U \star \tilde{\rho}) = 0,$$

from which

$$U \star \tilde{\rho} = C, \quad \tilde{\rho} \neq 0,$$

in the support of $\tilde{\rho}$ if the support has not empty interior.

Complete set of solutions depending on regularity of the potential and stability are open problems.

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Milling

we set **u** in a rotatory state,

$$u = \pm\sqrt{\frac{\alpha}{\beta}}\,\frac{x^\perp}{|x|},$$

where $x = (x_1, x_2)$, $x^\perp = (-x_2, x_1)$, and look for $\rho = \rho(|x|)$ radial, then

$$U \star \rho = D + \frac{\alpha}{\beta}\ln|x|, \quad \text{whenever } \rho \neq 0.$$

A special family of singular solutions are given by $\rho(r) = c\,\delta(r - r_0)$.

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Asymptotic Flocking

Let us consider the N_p-particle system:

$$\begin{cases} \dfrac{dx_i}{dt} = v_i & , x_i(0) = x_i^0 \\ \dfrac{dv_i}{dt} = \displaystyle\sum_{j=1}^{N_p} m_j a(|x_i - x_j|)\,(v_j - v_i) & , v_i(0) = v_i^0, \end{cases} .$$

Due to translation invariancy, w.l.o.g. the mean velocity is zero and thus the center of mass is preserved along the evolution, i.e.,

$$\sum_{i=1}^{N_p} m_i v_i(t) = 0 \qquad \text{and} \qquad \sum_{i=1}^{N_p} m_i x_i(t) = x_c$$

for all $t \geq 0$ and $x_c \in \mathbb{R}^d$.

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Let us fix any $R_0^x > 0$ and $R_0^v > 0$, such that all the initial velocities lie inside the ball $B(0, R_0^v)$ and all positions inside $B(x_c, R_0^x)$.

Let us define the function $R^v(t)$ to be

$$R^v(t) := \max\left\{|v_i(t)|\,,\ i = 1, \ldots, N_p\right\}.$$

Choosing the label i to be the one achieving the maximum, we get

$$\frac{d}{dt}R^v(t)^2 = \frac{d}{dt}|v_i|^2 = -2\sum_{j\neq i} m_j\left[(v_i - v_j)\cdot v_i\right]\, a(|x_i - x_j|)\,.$$

Because of the choice of the label i, we have that $(v_i - v_j)\cdot v_i \geq 0$ for all $j \neq i$ that together with $a \geq 0$ imply $R^v(t) \leq R_0^v$ for all $t \geq 0$.

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Coming back to the equation for the positions,

$$|x_i(t) - x_i^0| \leq R_0^v t \quad \text{for all } t \geq 0 \text{ and all } i = 1, \ldots, N_p.$$

$$a(|x_i - x_j|) \geq \frac{1}{[1 + 4R_0^2(1+t)^2]^\gamma} \quad \text{for all } t \geq 0 \text{ and all } i, j = 1, \ldots, N_p,$$

with $R_0 = \min(R_0^x, R_0^v)$.

Coming back to the equation for the maximal velocity

$$\begin{aligned}
\frac{d}{dt} R^v(t)^2 &= -2 \sum_{j \neq i} m_j \left[(v_i - v_j) \cdot v_i\right] a(|x_i - x_j|) \\
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Gronwall's lemma:

$$R^v(t) \leq R_0^v \exp\left\{-\frac{1}{2}\int_0^t f(s)\,ds\right\}.$$

For $\gamma \leq 1/2$, the function $f(t)$ is not integrable at ∞ and therefore

$$\lim_{t\to\infty} \int_0^t f(s)\,ds = +\infty$$

and $R^v(t) \to 0$ as $t \to \infty$ giving the convergence to a single point, its mean velocity, of the support for the velocity.
Again for the position variables, we get

$$\begin{cases} \int_0^t |v_i(s)|\,ds \leq C_1 \int_0^t (1+s)^{-1-\epsilon}\,ds & \gamma < 1/2 \\ \int_0^t |v_i(s)|\,ds \leq C \int_0^t \frac{1}{1+s}\,ds = C\ln(1+t) & \gamma = 1/2, \end{cases}.$$

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Asymptotic Flocking

Unconditional Non-universal Flocking Result for Particles

The unique measure-valued solution for the CS kinetic model with $\gamma \leq 1/2$, with a finite number of particles given by

$$\tilde{\mu}(t) = \sum_{i=1}^{N_p} m_i \, \delta(x - x_i(t)) \, \delta(v - v_i(t)),$$

satisfies that

$$\lim_{t\to\infty} W_1 \left(\tilde{\mu}(t), \tilde{\mu}^{\infty}\right) = 0$$

with

$$\tilde{\mu}^{\infty} = \sum_{i=1}^{N_p} m_i \, \delta(x - x_i^{\infty} - mt) \, \delta(v - m)$$

with m the initial mean velocity of the particles.

Asymptotic Flocking

Unconditional Non-universal Flocking Result for general measures

Given $\mu_0 \in \mathcal{M}(\mathbb{R}^{2d})$ compactly supported, then the unique measure-valued solution to the CS kinetic model with $\gamma \leq 1/2$, satisfies the following bounds on their supports:

$$\text{supp}\ \mu(t) \subset B(x_c(0) + mt, R^x(t)) \times B(m, R^v(t))$$

for all $t \geq 0$, with $R^x(t) \leq \bar{R}$ and $R^v(t) \leq R_0\, e^{-\lambda t}$ with $\bar{R}^x$ depending only on the initial support radius.

Moreover,

$$\lim_{t\to\infty} W_1(\mu_x^{mt}(t), L_\infty(\mu_0)) = 0,$$

where the measure $L_\infty(\mu_0)$ is defined as

$$\int_{\mathbb{R}^d} \zeta(x)\, dL_\infty(\mu_0)(x) = \int_{\mathbb{R}^{2d}} \zeta\left(x + \int_0^\infty [V(s;x,v) - m]\, ds\right) d\mu_0(x,v),$$

for all $\zeta \in C_b^0(\mathbb{R}^d)$.

Asymptotic Flocking

Unconditional Non-universal Flocking Result for general measures

Given $\mu_0 \in \mathcal{M}(\mathbb{R}^{2d})$ compactly supported, then the unique measure-valued solution to the CS kinetic model with $\gamma \leq 1/2$, satisfies the following bounds on their supports:

$$\text{supp } \mu(t) \subset B(x_c(0) + mt, R^x(t)) \times B(m, R^v(t))$$

for all $t \geq 0$, with $R^x(t) \leq \bar{R}$ and $R^v(t) \leq R_0\, e^{-\lambda t}$ with $\bar{R}^x$ depending only on the initial support radius.
Moreover,

$$\lim_{t\to\infty} W_1(\mu_x^{mt}(t), L_\infty(\mu_0)) = 0,$$

where the measure $L_\infty(\mu_0)$ is defined as

$$\int_{\mathbb{R}^d} \zeta(x)\, dL_\infty(\mu_0)(x) = \int_{\mathbb{R}^{2d}} \zeta\left(x + \int_0^\infty [V(s;x,v) - m]\, ds\right) d\mu_0(x,v),$$

for all $\zeta \in \mathcal{C}_b^0(\mathbb{R}^d)$.

Outline

1. Some Swarming Models
   - Particle models.
   - Kinetic Models and measure solutions.
2. Qualitative Properties
   - Model with asymptotic velocity
   - Cucker-Smale Kinetic model
   - Variations
3. Conclusions

Leadership, Geometrical Constraints, and Cone of Influence

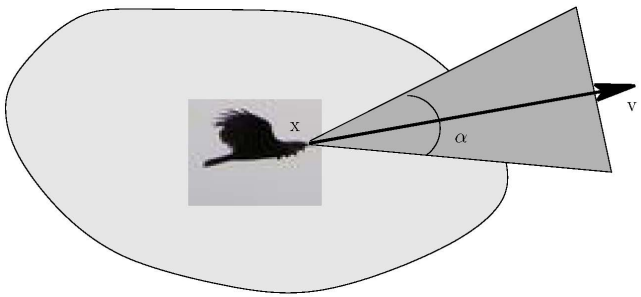
Cucker-Smale with local influence regions:

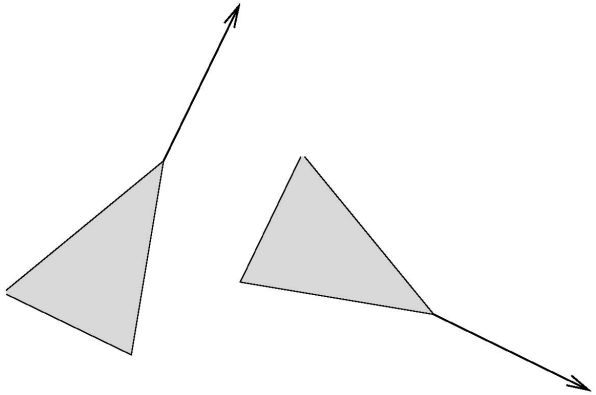
$$\begin{cases} \dfrac{dx_i}{dt} = v_i \ , \\ \dfrac{dv_i}{dt} = \displaystyle\sum_{j \in \Sigma_i(t)} a(|x_i - x_j|)(v_j - v_i) \ , \end{cases}$$

where $\Sigma_i(t) \subset \{1, \ldots, N\}$ is the set of dependence, given by

$$\Sigma_i(t) := \left\{ 1 \leq \ell \leq N : \frac{(x_\ell - x_i) \cdot v_i}{|x_\ell - x_i||v_i|} \geq \alpha \right\} .$$

Cone of Vision:





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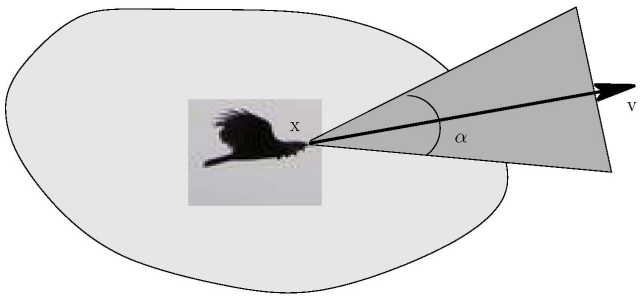
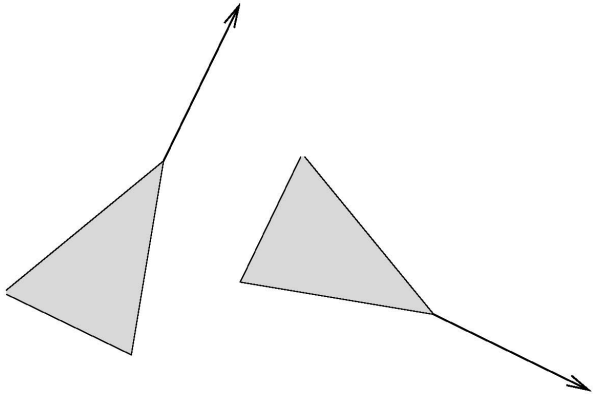




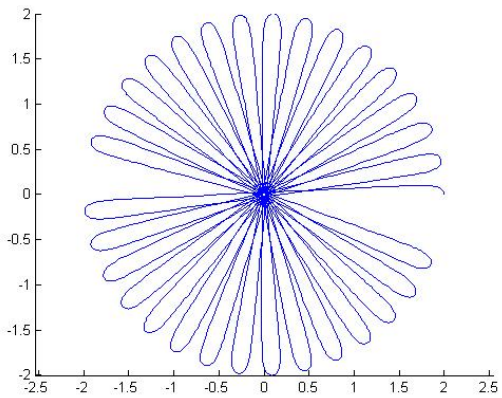
Roosting Forces

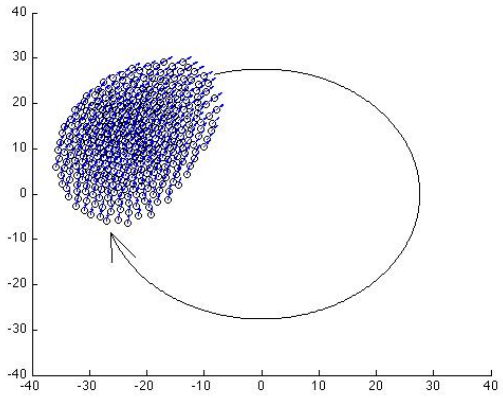
Adding a roosting area to the model:

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \\ \dfrac{dv_i}{dt} = (\alpha - \beta\,|v_i|^2)v_i - \displaystyle\sum_{j\neq i} \nabla U(|x_i - x_j|) - v_i^{\perp}\,\nabla_{x_i}\left[\phi(x_i)\cdot v_i^{\perp}\right]. \end{cases}$$

with the roosting potential ϕ given by $\phi(x) := \dfrac{b}{4}\left(\dfrac{|x|}{R_{\text{Roost}}}\right)^4$.

Roosting effect: milling flocks $N = 400, R_{\text{roost}} = 20$.





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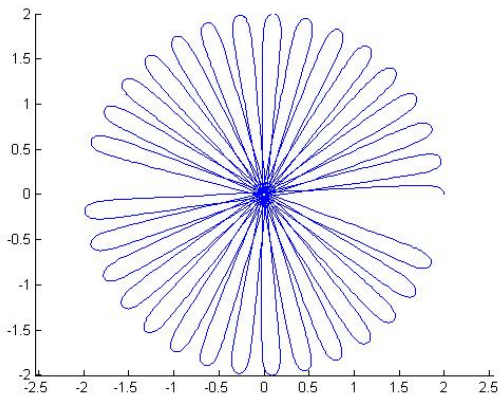

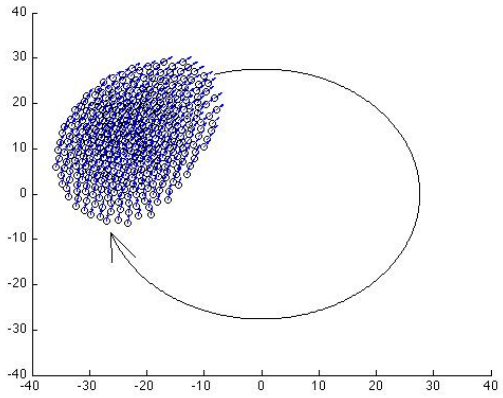

to appear in M3AS, in collaboration with A. Klar, S. Martin, S. Tiwari.

Adding Noise

Self-Propelling/Friction/Interaction with Noise Particle Model:

$$\begin{cases} \dot{x}_i = v_i, \\ dv_i = \left[(\alpha - \beta\,|v_i|^2)v_i - \nabla_{x_i} \sum_{j \neq i} U(|x_i - x_j|) \right] dt + \sqrt{2\sigma}\, d\Gamma_i(t)\ , \end{cases}$$

where $\Gamma_i(t)$ are N independent copies of standard Wiener processes with values in $\mathbb{R}^d$ and $\sigma > 0$ is the noise strength.

The Cucker–Smale Particle Model with Noise:

$$\begin{cases} dx_i = v_i dt\ , \\ dv_i = \sum_{j=1}^{N} a(|x_j - x_i|)(v_j - v_i)\, dt + \sqrt{2\sigma \sum_{j=1}^{m} a(|x_j - x_i|)}\, d\Gamma_i(t)\ . \end{cases}$$

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Conclusions

- Simple modelling of the three main phenomena: attraction, repulsion, and orientation; lead to complicated patterns which are observed in nature and not completely understood.
- Stability of these patterns will certainly be quite useful to understand more complicated situations.
- The results are also interested from the point of view of the control of electronic robots.
- More information from particular species should be included to make more realistic models: visual sectors for birds, more adapted interactions, pheromone trails for ants, ...

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